Growth estimates through scaling for quasilinear partial differential equations *

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(to appear in Ann. Acad. Sci. Fenn. Ser. A I Math.)

Abstract

In this note we use a scaling or blow up argument to obtain estimates to solutions of equations of p-Laplacian type.

1. Introduction

Weak solutions of equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0\,, \quad 1 < p < \infty\,,$$

are called p-harmonic. It is known that p-harmonic functions are in $C^{1,\alpha}$ for some $\alpha > 0$, where for $p \neq 2$ one cannot have $\alpha \geq 1$ in general; see [3] for sharp regularity in the planar case. In this note we present a blow up argument and show that if $0 < \alpha \leq 1$ is such that the class of p-harmonic functions are continuously embedded into $C^{1,\alpha}$, then the only entire p-harmonic functions that grow at infinity slower than $|x|^{1+\alpha}$ are the linear ones.

We formulate the proof and the growth rate result only in the p-Laplacian setting, but the argument is more general. The only ingredients required are the following: there is a class $\mathcal{F}$ of functions so that $\mathcal{F}$ contains certainly rescaled versions of

* *2000 Mathematics Subject Classification.* Primary 35J60; Secondary 35J15
Keywords. p-Laplace, scaling.

† T.K. (partially) and X.Z. supported by the Academy of Finland, H.S. partially supported by Swedish Research Council. The present work is part of the ESF programme *GLOBAL*.

functions and $\mathcal{F}$ can be embedded into $C^{1,\alpha}$. Then nonlinear functions in $\mathcal{F}$ grow at least as fast as $|x|^{1+\alpha}$.

As an application of the growth rate result we show that a nonnegative p-harmonic function in a half space is actually linear if it vanishes on the boundary of the half space. This gives an affirmative answer to a query of Mario Bonk, who also found independently a different proof for this fact.

2. Growth of entire solutions

We prove the following two theorems:

2.1. Theorem. *Let u be p-harmonic in $\mathbf{R}^n$. There is a number $\beta > 0$ depending only on p and n so that if*

$$|u(x)| = o(|x|^{1+\beta}) \quad \text{as} \quad |x| \to \infty,$$

then u is (affine) linear.

The second is an immediate consequence of the first one.

2.2. Theorem. *Let u be p-harmonic in $\mathbf{R}^n$. If*

$$|u(x)| = o(|x|) \quad \text{as} \quad |x| \to \infty,$$

then u is constant.

2.3. Remark. It is known that there are no entire harmonic functions (i.e. $p = 2$) with noninteger growth rate. That is, if u is harmonic (i.e. 2-harmonic) in $\mathbf{R}^n$ with

$$\limsup_{|x|\to\infty} \frac{\log|u(x)|}{\log|x|} = \gamma \in]0, \infty[\,,$$

then γ is an integer. If $p \neq 2$, the situation is different. Let

$$\gamma = \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right).$$

Observe that $\gamma \in]\frac{4}{3}, 2[$ for $p > 2$; for $1 < p < 2$ the constant $\gamma > 2$ but it has noninteger value for most of p's. Then there are entire p-harmonic functions whose growth rate is $= \gamma$. Constructions for such solutions are done by Krol' [4], Tolksdorff [9], Aronsson [2], and Iwaniec and Manfredi in [3]. Basically all these examples are quasiradial functions in the plane (higher dimensional examples are obtained by adding dummy variables.)

Hence Theorem 2.1 is optimal as stated. In the plane case one can choose $\beta \geq \frac{4}{3}$ for all p (see Theorem 2.5 below), but in the higher dimensions we do not know if β needs to be closer to 0.

Proof of Theorem 2.1. Choose a sequence $R_j \to \infty$ and write

$$S_j = \sup_{B(0,R_j)} |u| \,.$$

Then the scaled functions

$$u_j(x) = \frac{u(R_j x)}{S_j}$$

are p-harmonic and $|u_j| \leq 1$ in $B(0,1)$.

By a well known regularity estimate (see e.g. Lewis [5]), there is a constant $\beta = \beta(n,p) > 0$ so that the $C^{1,\beta}(B(0,1))$ norms of u_j are bounded, independently of j. Hence the quantities

$$C_j(x) = \frac{|Du_j(x) - Du_j(0)|}{|x|^\beta} = \frac{R_j^{1+\beta}}{S_j} \frac{|Du(R_j x) - Du(0)|}{|R_j x|^\beta}$$

are uniformly bounded in $B(0,\frac{1}{2})$. Since the growth condition $|u(x)| = o(|x|^{1+\beta})$ implies

$$\lim_{j\to\infty} \frac{R_j^{1+\beta}}{S_j} = \infty \,,$$

we conclude that

$$\sup_{y \in B(0,\frac{R_j}{2})} \frac{|Du(y) - Du(0)|}{|y|^\beta} = \sup_{x\in B(0,\frac{1}{2})} \frac{|Du(R_j x) - Du(0)|}{|R_j x|^\beta} \to 0 \qquad \text{as } j \to \infty \,.$$

But this implies that

$$Du(y) = Du(0) \qquad \text{for all } y \in \mathbf{R}^n \,,$$

and Theorem 2.1 follows. □

2.4. Remark. Another way to prove Theorem 2.1 for the p-Laplacian goes via the estimate

$$\operatorname*{osc}_{B(x_0,r)} |\nabla u| \le C \sup_{B(x_0,R)} |\nabla u| \left(\frac{r}{R}\right)^{\alpha}$$

that can be found e.g. in [7, Theorem 3.44]. For more general operators the oscillation estimate might not be available but one can prove the embedding into $C^{1,\alpha}$ by other means. We would like to emphasize here that our method works also in those cases where one can establish bounded embedding to $C^{1,\alpha}$ even though there is no oscillation estimate for the gradient.

Appealing to the sharp regularity result in [3] our method immediately yields the following result in the planar case:

2.5. Theorem. *Let u be p-harmonic in $\mathbf{R}^2$ so that*

$$|u(x)| = o(|x|^{\gamma}) \qquad as \qquad |x| \to \infty.$$

If

$$\gamma = \begin{cases} 2\,, & \text{if } 1 < p \le 2, \\ \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right) & \text{if } p > 2, \end{cases}$$

then u is (affine) linear.

3. Nonnegative functions in the half space

As an application of Theorem 2.1 we prove the following result.

3.1. Theorem. *If u is a nonnegative p-harmonic function on a half space H, continuous up to the boundary with $u = 0$ on ∂H, then u is (affine) linear.*

Theorem 3.1 follows by combining the following lemma with Theorem 2.1, when we observe that u in Theorem 3.1 can be reflected through the hyperplane ∂H and the resulting function is p-harmonic in the whole space $\mathbf{R}^n$ (this can be easily verified by a direct computation, see [8]).

3.2. Lemma. *Let u be a nonnegative p-harmonic function on a half space half space H, continuous up to the boundary. If $u = 0$ on ∂H, then*

$$|u(x)| = O(|x|) \qquad as \qquad |x| \to \infty.$$

PROOF: We assume, as we clearly may, that the half space H is the upper half space

$$H = \mathbf{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}.$$

We first show that there is a constant $c = c(n,p) > 0$ so that

$$u(Re_n) \le c\, R\, u(e_n) \quad \text{for all } R > 2\,; \tag{3.3}$$

here $e_n = (0, 0, \ldots, 0, 1)$ is the nth unit vector in $\mathbf{R}^n$. For this, we write $x_0 = Re_n = 2re_n$ and observe that by Harnack's inequality

$$u(x) \approx c\, u(x_0) \quad \text{for all } x \in \bar{B}(x_0, r)\,,$$

where $c = c(n,p) > 0$. Now, let v be the p-capacitary potential in $B(x_0, 2r) \setminus \bar{B}(x_0, r)$, i.e.

$$v(x) = \frac{\int_{|x-x_0|}^{2r} t^{(1-n)/(p-1)}\, dt}{\int_r^{2r} t^{(1-n)/(p-1)}\, dt}\,.$$

Then since v is p-harmonic in $B(x_0, 2r) \setminus \bar{B}(x_0, r)$, we have by comparison principle that

$$u(x) \ge c u(x_0) v(x) \quad \text{for all } x \in B(x_0, 2r) \setminus \bar{B}(x_0, r)\,,$$

where $c = c(n,p) > 0$. The claim (3.3) follows from this estimate evaluated at $x = e_n$, for

$$\begin{aligned}
\frac{1}{v(e_n)} &= \frac{\int_r^{2r} t^{(1-n)/(p-1)}\, dt}{\int_{2r-1}^{2r} t^{(1-n)/(p-1)}\, dt} \\
&= 1 + \frac{\int_r^{2r-1} t^{(1-n)/(p-1)}\, dt}{\int_{2r-1}^{2r} t^{(1-n)/(p-1)}\, dt} \\
&\le 1 + \frac{(r-1) r^{(1-n)/(p-1)}}{(2r)^{(1-n)/(p-1)}} \\
&\le 1 + \frac{r-1}{2^{(1-n)/(p-1)}} \\
&\le c2r = cR\,,
\end{aligned}$$

where $c = c(n,p)$. The estimate (3.3) is proved.

To complete the proof the lemma, we employ the boundary Harnack principle (see [1] or [6]) which states that there is a constant c depending on n and p only so that

$$\frac{u(x)}{x_n} \leq c\frac{u(Re_n)}{R} \quad \text{for all } x \in B(0,2R) \cap \mathbf{R}^n_+ \quad \text{and } R > 0\,;$$

here x_n is the nth coordinate of x. Next we combine this with (3.3) and have

$$u(x) \leq c\frac{u(Re_n)}{R}x_n \leq c\,u(e_n)x_n \leq c\,|x|u(e_n)$$

for $x \in B(0,2R)$ and $R > 2$. The lemma follows. □

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