Number theory 2 2024

Exercises 6

A number theoretical function f is a *multiplicative function*, if $f(1) = 1$ and

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbb{N}$, with $\gcd(m, n) = 1$.

1. Let f be a multiplicative function. Let

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

where $p_1 < \cdots < p_k$ are prime numbers and $e_1, \ldots e_k \in \mathbb{N} - \{0\}$ be the prime factorization of n. Prove that

$$f(n) = f(p_1^{e_1}) \cdots f(p_k^{e_k}).$$

2. Let $r, m, n \in \mathbb{Z}$, $n \geq 2$. Prove that any two different elements of

$$\{km + r : 0 \leq k \leq n - 1\}$$

are not congruent $\mod n$.

3. Let p be a prime. Prove that $\phi(p^k) = p^{k-1}(p - 1)$ for all $k \in \mathbb{N}^*$.

4. Determine $\phi(1000)$ and $\phi(2343)$.

5. What are the two last decimals of 3^{400}?

6. Let $m, n \in \mathbb{N}$ and let $s = \prod_{p|m \text{ and } p|n} p$. Prove that

$$\phi(mn) = s \, \frac{\phi(m)\phi(n)}{\phi(s)}.$$

Let $k \in \mathbb{N}$. The kth *divisor function* is $\sigma_k \colon \mathbb{N}^* \to \mathbb{N}$,

$$\sigma_k(n) = \sum_{d|n} d^k.$$

A positive natural number $n \in \mathbb{N}^*$ is *abundant*, if $\sigma_1(n) > 2n$.

7. Give an example of an odd abundant number.[1]

[1]See figures 0.1 and 0.2.

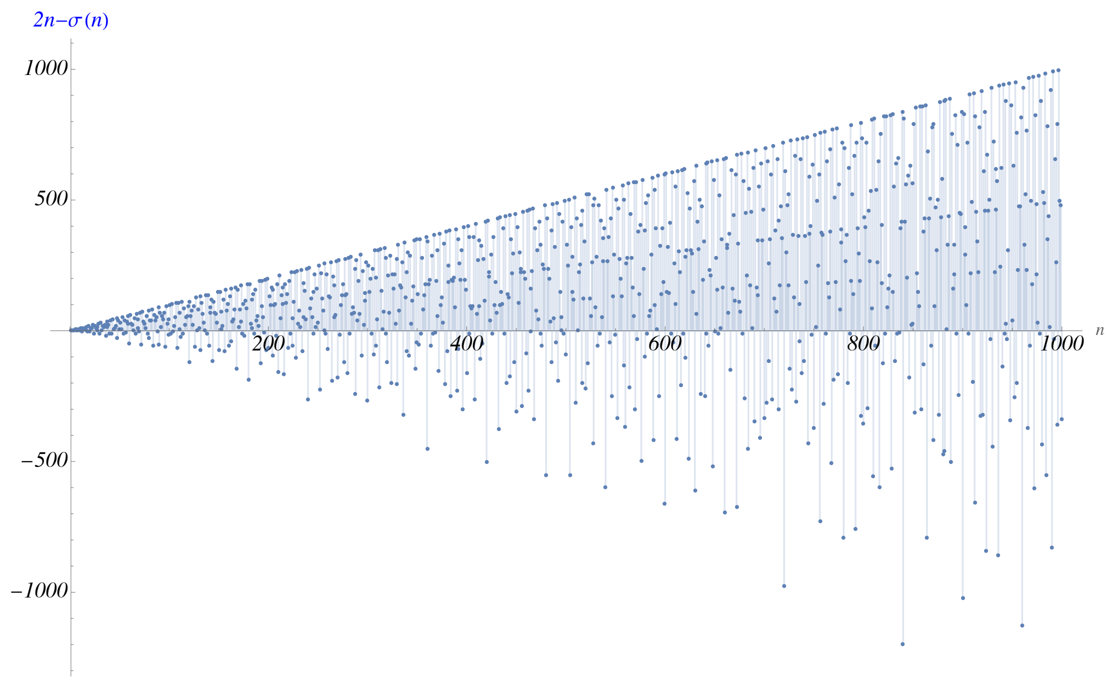

Figure 0.1: The values of $2n - \sigma_1(n)$ for $1 \leq n \leq 999$.

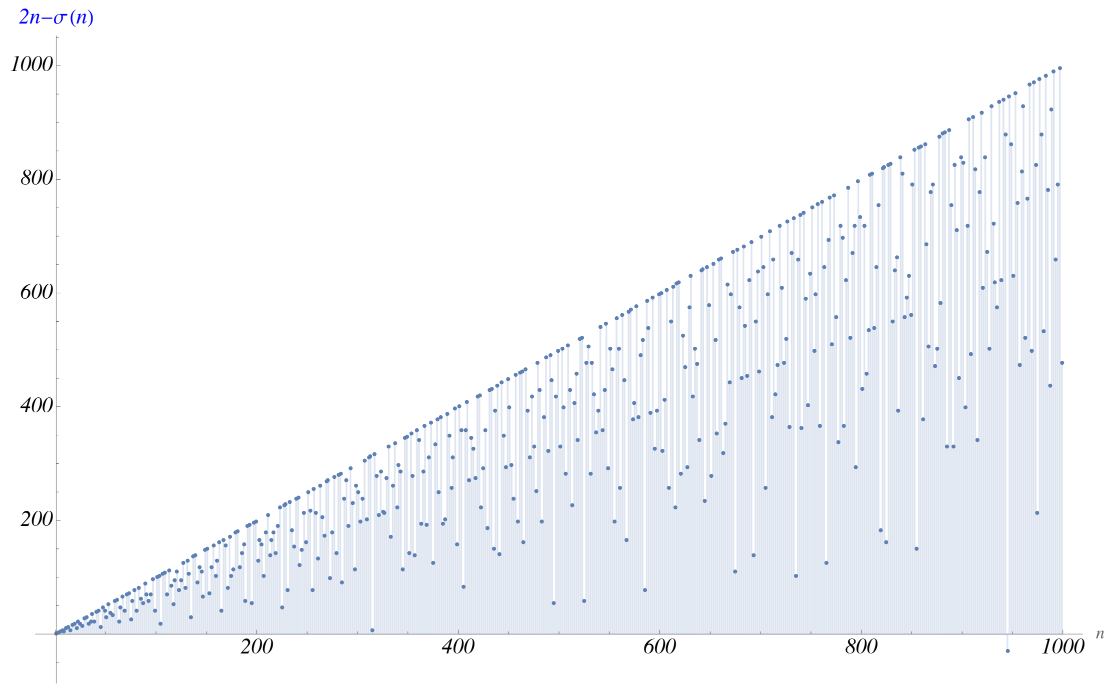

Figure 0.2: The values of $2n - \sigma_1(n)$ for odd numbers $1 \leq n \leq 999$.