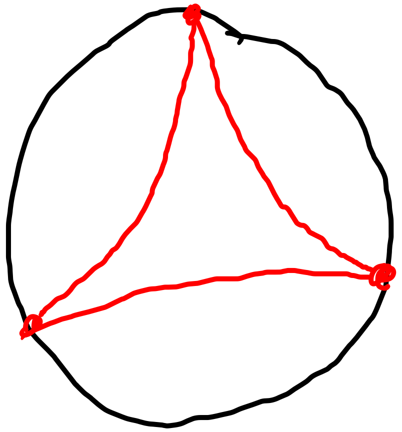
Neg. curved spaces 7. 10. 2020

Generalized triangles: $A, B, C \in \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ + geodesic segments, rays or lines with endpoints A and B, B and C, C and A.



$A, B, C \in \partial_\infty \mathbb{H}^2$

ideal triangle

Prop. If Δ, Δ' are ideal triangles in $\mathbb{H}^2$, then $\exists\ \gamma \in \mathrm{Isom}\,\mathbb{H}^2$ s.t. $\gamma(\Delta) = \Delta'$.

Proof Exercise next week

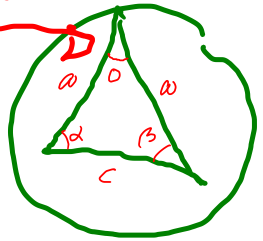
2nd law of cosines
Prop. 4.26

$$\cosh c = \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

$\gamma = 0$ // 1

Prop. 5.22 (2nd law of cosines for generalized triangles)

① the angle at a vertex at ∞ is 0.



$$\cosh c = \frac{1 + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

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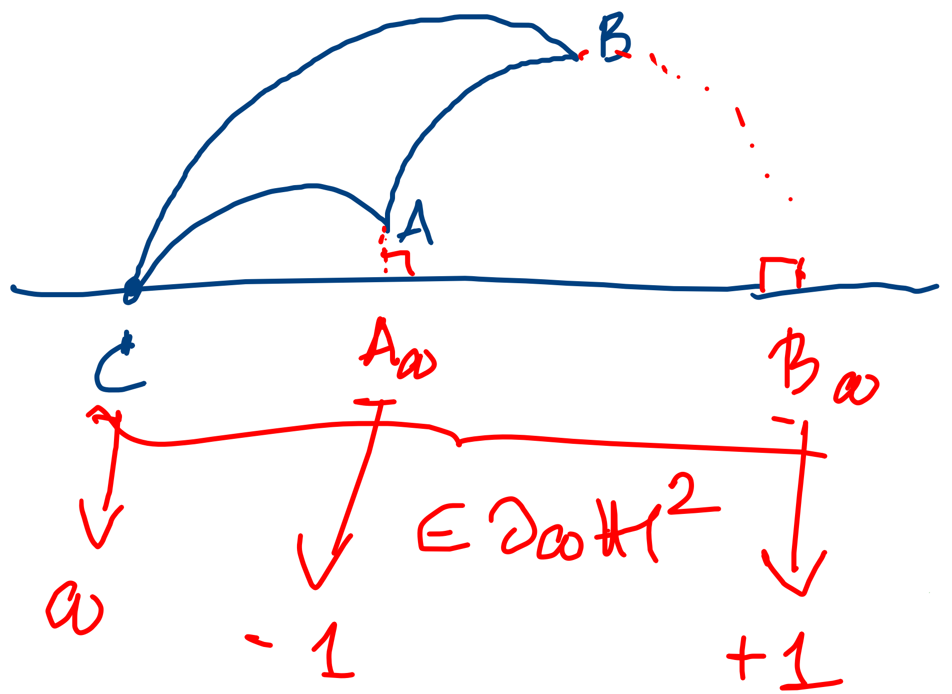
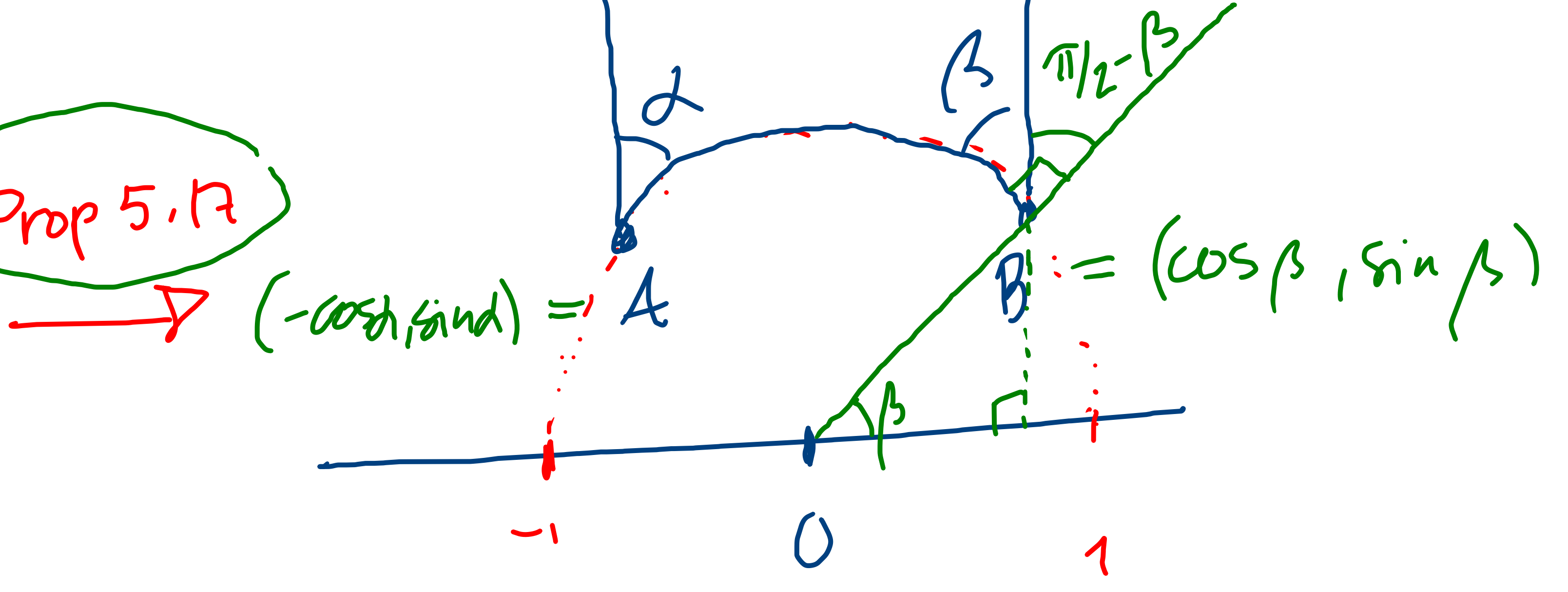
Proof. Let us move the vertex C in $\partial_\infty \mathbb{H}$ to ∞ and the vertices A and B to the Euclidean unit circle.





$$\cosh d(A,B) = 1 + \frac{\|A-B\|^2}{2A_2B_2} = \dots = \frac{1+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

Prop. 5.17 If x_1, x_2, x_3 and y_1, y_2, y_3 $\in \partial_\infty \mathbb{H}^n$ are triples of distinct points, then

$\exists\ \gamma \in \mathrm{Isom}\,\mathbb{H}^n$ s.t. $\gamma(x_1) = y_1$, $\gamma(x_2) = y_2$, $\gamma(x_3) = y_3$.

that are the restrictions to $\mathbb{H}^n$ of homeos of $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$

Proof. If $x_1 \neq \infty$ then any inversion γ_1 with pole at x_1 maps x_1 to ∞ (and γ_1 is an isometry). If $\gamma_1(x_2) \neq 0$, then take $\gamma_2 = T_{-\gamma_1(x_2)}$

$$x_1,\ x_2,\ x_3 \xrightarrow{\gamma_1} \overset{\infty}{\cdot}\quad \gamma_1(x_2),\ \gamma_1(x_3) \xrightarrow{\gamma_2} \overset{\infty}{\cdot}\quad 0,\ \gamma_2(\gamma_1(x_3)) \xrightarrow{\gamma_4 \circ \gamma_3} \overset{\infty}{\cdot}\quad 0,\ e_1$$

Let $\lambda = \dfrac{1}{\|\gamma_2(\gamma_1(x_3))\|}$ and take $\gamma_3 = L_\lambda$. $(x \mapsto \lambda x)$

Now $\|\gamma_3 \circ \gamma_2 \circ \gamma_1(x_3)\| = 1$. Let γ_4 the extension of an elt of $O(n-1)$ that maps $\gamma_3 \circ \gamma_2 \circ \gamma_1(x_3)$ to e_1

acts trans. on unit sphere

$\to \exists$ isometry $\gamma_x = \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1$ that maps
$$x_1 \xrightarrow{\gamma_x} \infty$$
$$x_2 \xrightarrow{\gamma_x} 0$$
$$x_3 \xrightarrow{\gamma_x} e_1$$

Take a corresponding isom. γ_y for y_1, y_2, y_3
$$y_1 \xrightarrow{\gamma_y} \infty$$
$$y_2 \xrightarrow{\gamma_y} 0$$
$$y_3 \xrightarrow{\gamma_y} e_1$$

$\to \gamma_y^{-1} \circ \gamma_x (x_i) = y_i \quad \forall\, i \in \{1,2,3\}$.

$\square$

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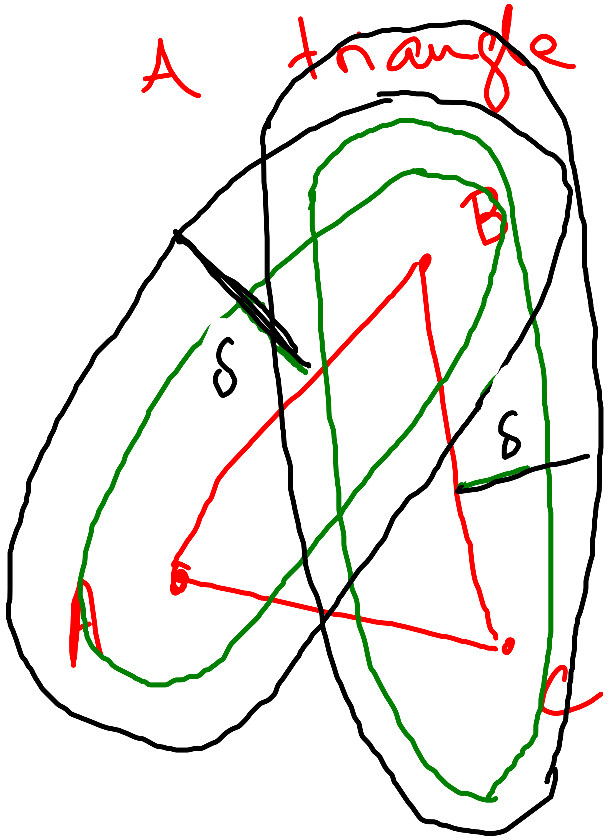
6 Gromov-hyperbolic spaces

take a property / properties of $\mathbb{H}^2$ and use that / them to define a class of geodesic metric spaces.

Defn Let X be a geodesic metric space, let $\delta > 0$.
A triangle Δ satisfies the $\boxed{\text{Rips condition with constant } \delta}$ if any side of Δ is contained in the union of the closed δ-neighbourhoods of the other two.



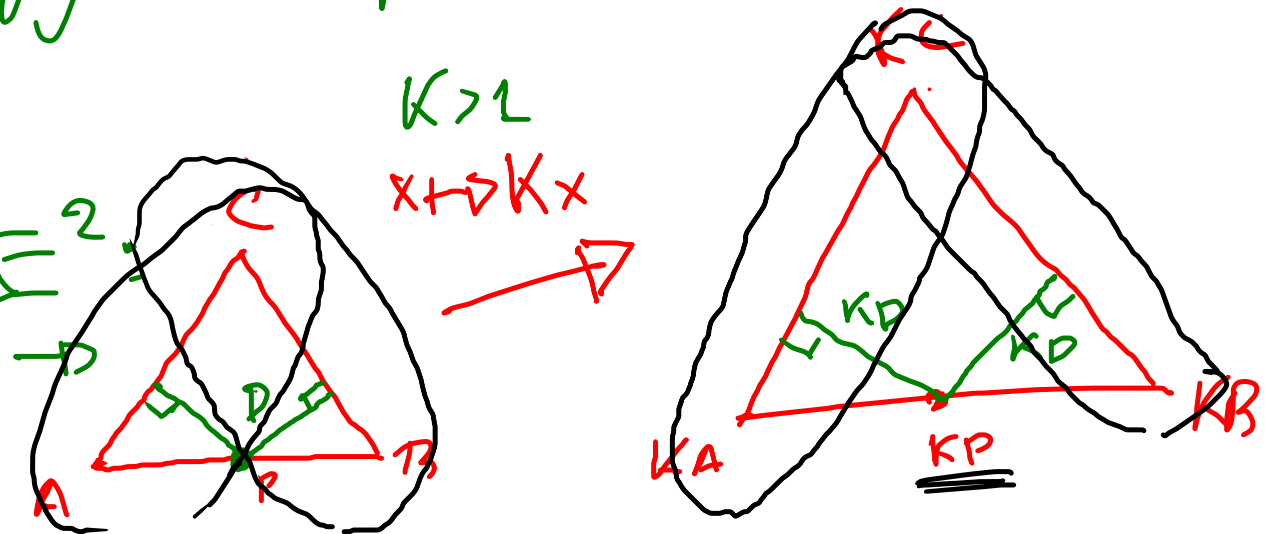
$A \subset X$

$$\overline{N}_\delta(A) = \{x \in X : d(x,A) \le \delta\}$$

Prop. 6.1 All triangles in $\mathbb{H}^n$ satisfy the Rips condition with constant $\log(1+\sqrt{2})$.

Note. Nothing like this holds in $\mathbb{E}^2$.
equilateral

$$[A,B] \subset \overline{N}_D([A,C]) \cup \overline{N}([B,C])$$



Defn Let X be a geod. metric space and let $\delta > 0$. If all triangles in X satisfy the Rips condition with const. δ, then X is a δ-hyperbolic space.

If X is δ-hyperbolic for some $\delta > 0$, then X is a Gromov-hyperbolic space.

Ex. 1) $\mathbb{H}^n$ is $\log(1+\sqrt{2})$-hyp.

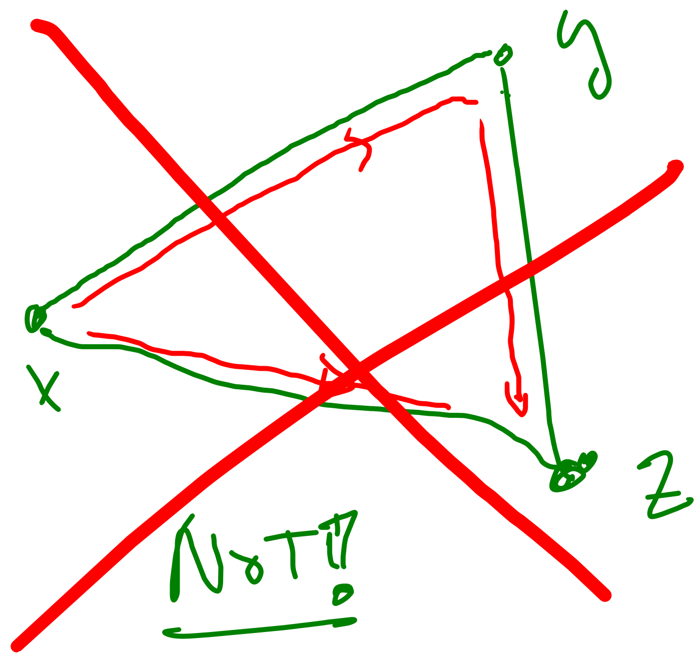
2) $\mathbb{E}^n$ is not Gromov-hyperbolic.

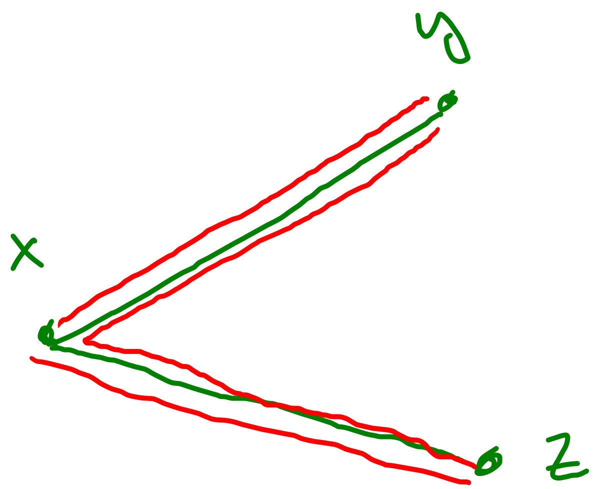
3) Metric trees, $\mathbb{R}$-trees are δ-hyperbolic for all $\delta > 0$ $\leadsto$ 0-hyperbolic, Gromov-hyperbolic.

uniquely arcwise connected geodesic metric space.

4) If X is a geod. metric space s.t. $\operatorname{diam}(X) = K < \infty$ then X is K-hyperbolic. $\to$ Not very interesting spaces in this context.

3) Let $x, y, z \in X$ ($\mathbb{R}$-tree)

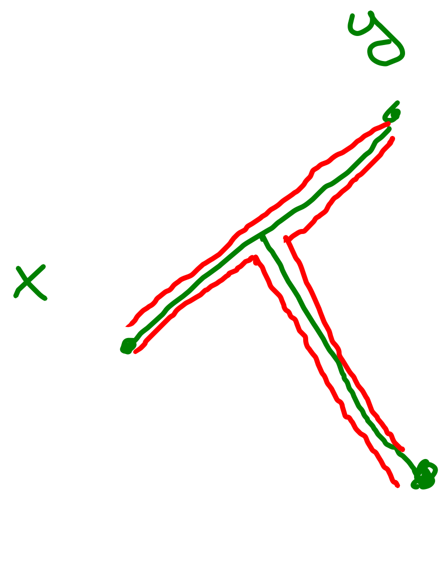




$[x,y] \subset [y,z]$

$[x,z] \subset [y,z]$

$[y,z] = [y,x] \cup [x,z]$



$[x,y] \subset [x,z] \cup [z,y]$

$\vdots$