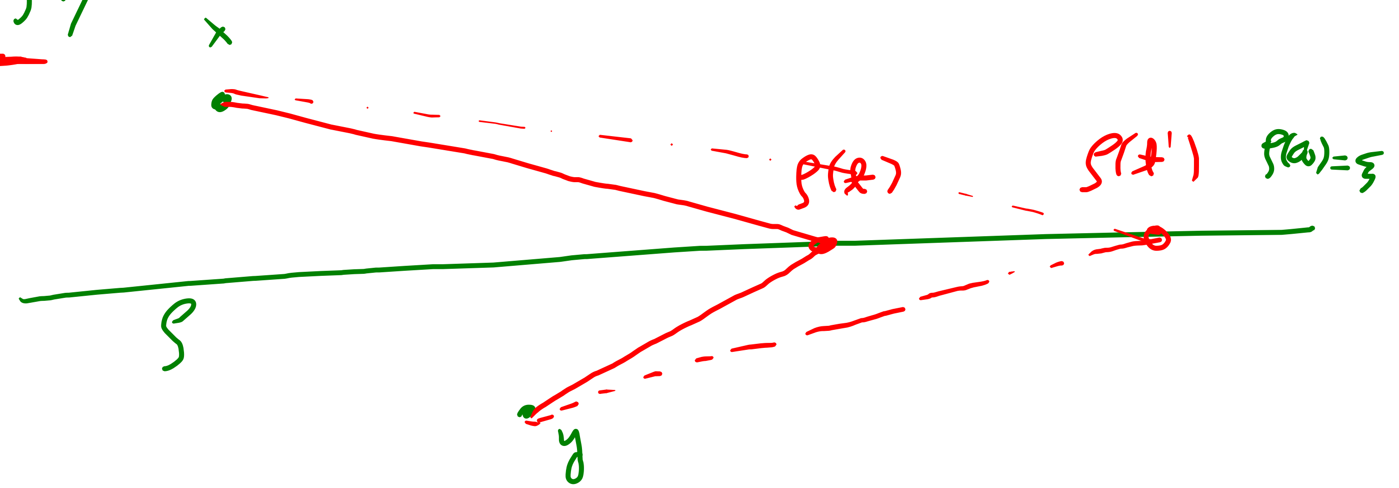
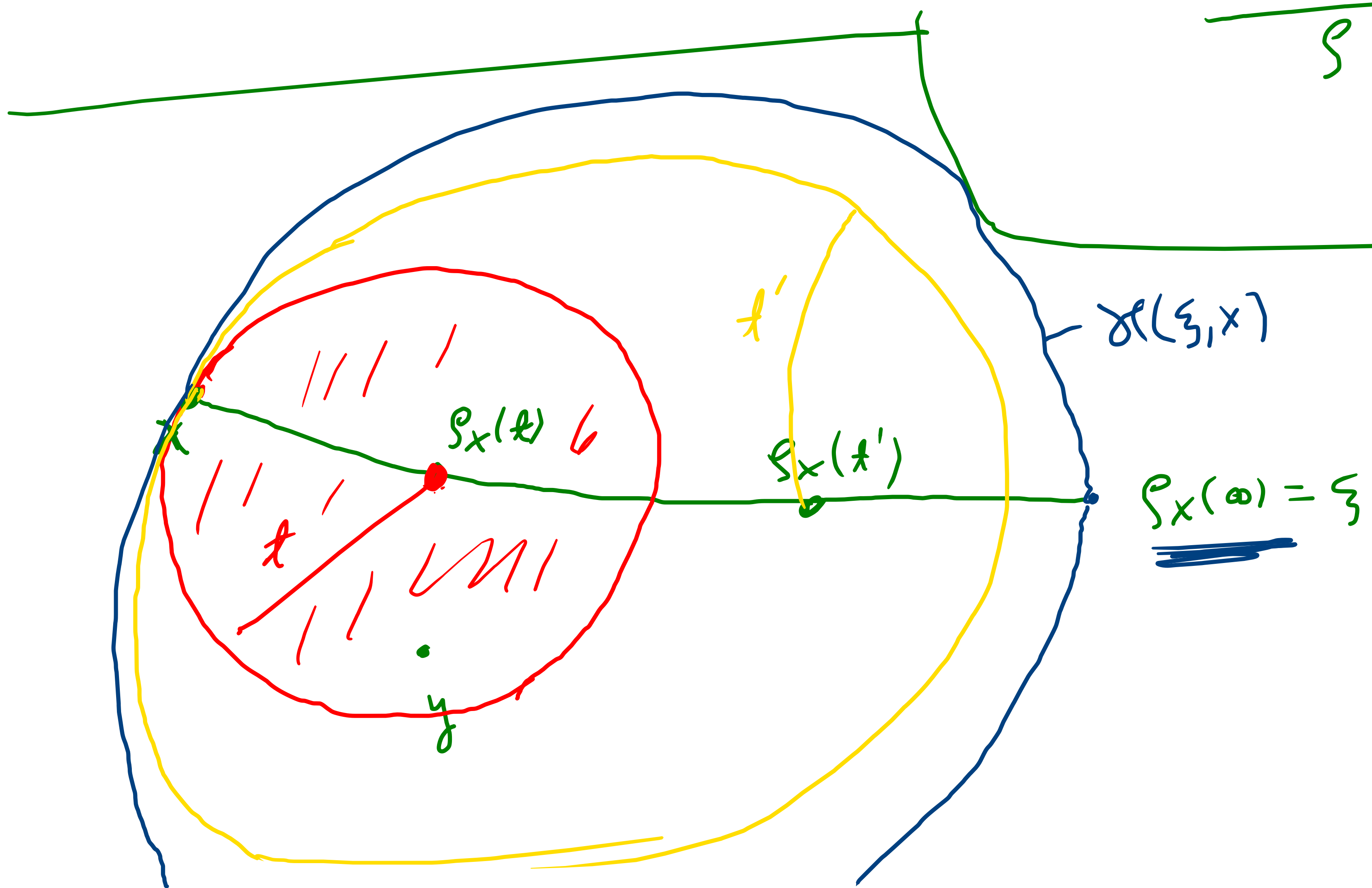
Neg. curved spaces 3.12.2020

$X$ CAT(-1). $\beta : \partial_\infty X \times X \times X \to \mathbb{R}$.

$(\xi, x, y) \mapsto \beta_\xi(x,y) = \lim_{t\to\infty} \big( d(\rho(t), x) - d(\rho(t), y) \big)$

$\rho \in \mathcal{G}_+(X)$ with $\rho(\infty) = \xi$




Horoball centered at $\xi$ through $x$

$\mathcal{H}(\xi, x) = \{ y \in X : \beta_\xi(x,y) \geq 0 \}$

$\partial\mathcal{H}(\xi, x) = \{ y \in X : \beta_\xi(x,y) = 0 \}$

horosphere centered at $\xi$, through $x$

**Prop. 11.7.** The Busemann cocycle is well-defined.

the limit exists and is indep. of the ray.

**Proof.** $\rho \in \mathcal{G}_+(X)$

$$t \mapsto f_x(t) = d(\rho(t), x) - t = d(\rho(t), x) - d(\rho(t), \rho(0)).$$

$t > s$: $$d(x, \rho(t)) \le d(x, \rho(s)) + \underbrace{d(\rho(s), \rho(t))}_{t-s}$$

Δ-ineq.

$$\Rightarrow \underbrace{d(x, \rho(t)) - t}_{f_x(t)} \le d(x, \rho(s)) - s = f_x(s).$$

$\Rightarrow f_x$ decreasing.

$$\beta_\xi(x,y) = \lim_{t\to\infty} \left( f_x(t) - f_y(t) \right)$$

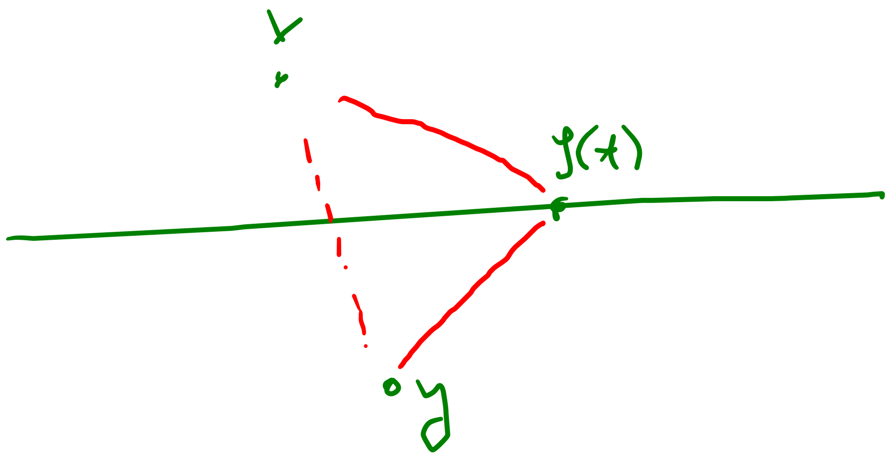
$$t = d(\rho(0), \rho(t)) \le \underbrace{d(\rho(0), x)}_{\text{const}} + d(x, \rho(t))$$

$$\Rightarrow -d(\rho(0), x) \le d(x, \rho(t)) - t = \boxed{f_x(t)}$$ bounded below.

$\Rightarrow$ limit exists!

Prop. 11.6 $\Rightarrow$ limit is indep. of the ray.

**Lemma 11.9.** Properties of the Busemann cocycle:




1) $|\beta_\xi(x,y)| \leq d(x,y)$

2) $\beta_{g\cdot\xi}(g(x), g(y)) = \beta_\xi(x,y)$.

3) $\beta_\xi(x,y) + \beta_\xi(y,z) = \beta_\xi(x,z)$

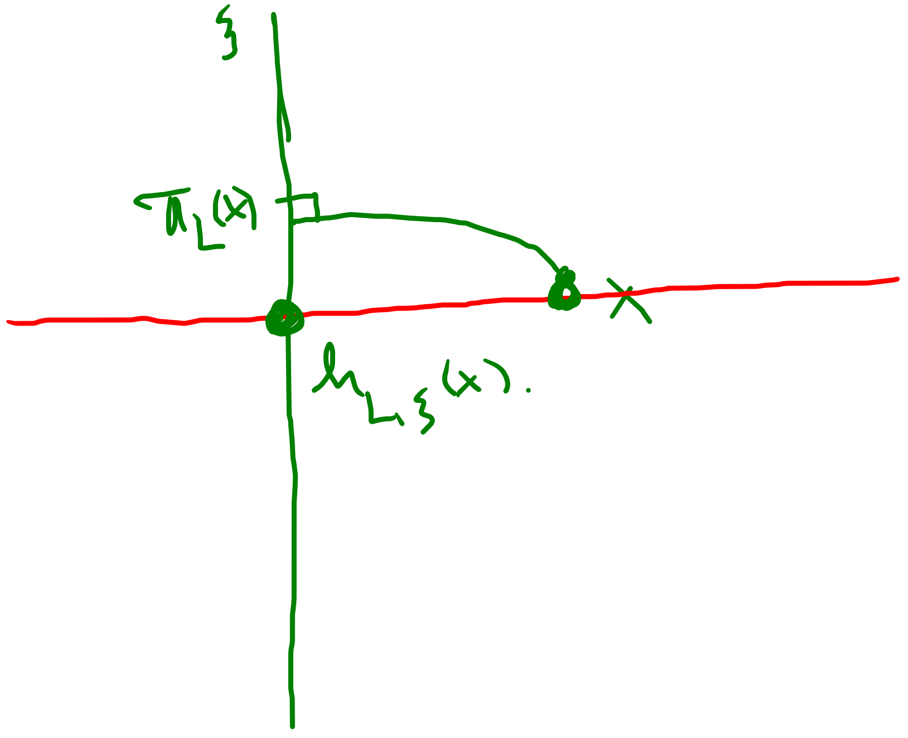
**Proof.** 1) $-d(x,y) \leq d(\gamma(t),x) - d(\gamma(t),y) \leq d(x,y)$ by the triangle ineq.

3)

$$\underbrace{d(\gamma(t),x) - d(\gamma(t),y)}_{\beta_\xi(x,y)} + \underbrace{d(\gamma(t),y) - d(\gamma(t),z)}_{\beta_\xi(y,z)} = \underbrace{d(\gamma(t),x) - d(\gamma(t),z)}_{\beta_\xi(x,z)}. \qquad \square$$

Fix $\xi, x$. $\quad y \longmapsto \beta_\xi(x,y)$ is the Busemann function

$\xi, y \quad x \longmapsto \beta_\xi(x,y)$




$L$ geod. line

$\pi_L : X \to L$ closest point map. 1-Lip.

if $\xi$ is an endpt of $L$.

$h_{L,\xi}(x)$ = the unique pt. in $L$ with

$$\beta_\xi(h_{L,\xi}(x), x) = 0.$$

**Prop. 11.10.** $h_{L,\xi}$ is 1-Lip.

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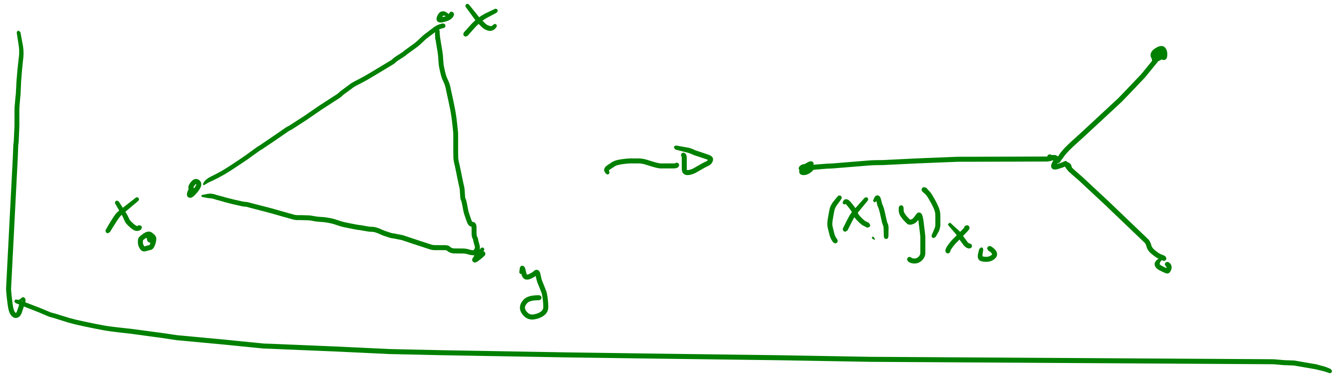
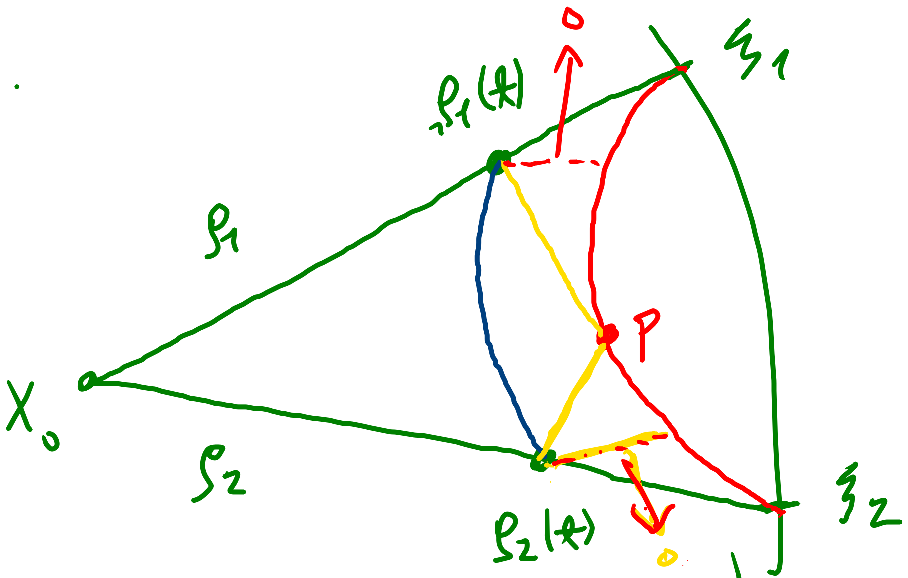
**Prop. 11.11.** Horoballs are convex.

**Proof.** Exercise.

Gromov product $\quad x_0, x, y \quad (x|y)_{x_0} = \frac{1}{2}\big(d(x_0,x) + d(x_0,y) - d(x,y)\big)$.




Lemma II.13.




X proper CAT(-1).

$p \in \,]\xi_1, \xi_2[\,, \quad \rho_1(\infty) = \xi_1$

$\rho_2(\infty) = \xi_2$.

$$\frac{1}{2}\big(d(x_0,\rho_1(t)) + d(x_0,\rho_2(t)) - d(\rho_1(t),\rho_2(t))\big) \Big\} \partial_\infty X$$

$$\|$$

$$(\xi_1|\xi_2)_{x_0} = \lim_{t\to\infty} (\rho_1(t)|\rho_2(t))_{x_0} = \frac{1}{2}\big(\beta_{\xi_1}(x_0,p) + \beta_{\xi_2}(x_0,p)\big)$$

**Proof:** $\frac{1}{2}\big(\beta_{\xi_1}(x_0,p) + \beta_{\xi_2}(x_0,p)\big) = \frac{1}{2}\Big[\lim_{t\to\infty}\big(d(\rho_1(t),x_0) - d(\rho_1(t),p)\big) + \lim_{t\to\infty}\big(d(\rho_2(t),x_0) - d(\rho_2(t),p)\big)\Big]$

$= \lim_{t\to\infty} (\rho_1(t)|\rho_2(t))_{x_0}$ by Prop. II.6. $\square$

We defined a metric in the boundary at $\infty$ of a tree using the Gromov product at $\infty$:

$$d_{x_0}(\xi_1,\xi_2) = \begin{cases} 0 & \text{if } \xi_1 = \xi_2 \\ e^{-(\xi_1|\xi_2)_{x_0}} & \text{otherwise.} \end{cases}$$

If $X$ is a proper CAT(-1)-space, this expression defines the / Gromov-Bourdon/visual metric in $\partial_\infty X$.

Thm 11.16. $d_{x_0}$ is a metric.

Proof. $\alpha_{x_0}(x_1,x_2) = \sin \dfrac{\measuredangle_{\bar{x}_0}(\bar{x}_1,\bar{x}_2)}{2}$ where $\bar{x}_0, \bar{x}_1, \bar{x}_2$ are the vertices of a comp. triangle of $x_0, x_1, x_2$. in $\mathbb{H}^2$.

$\alpha_{x_0}$ defines a metric in $\partial B(x_0,t)$ for any $t>0$:

Compute assuming $d(x_0,x_1) = d(x_0,x_2) = t$:

$$\alpha_{x_0}(x_1,x_2) = \frac{\sqrt{\cosh d(x_1,x_2) - 1}}{\sqrt{2}\,\sinh t}$$

$t \longmapsto \sqrt{\cosh t - 1}$ is convex, increasing and $0 \longmapsto 0$

$\Rightarrow \alpha_{x_0}$ is a metric.

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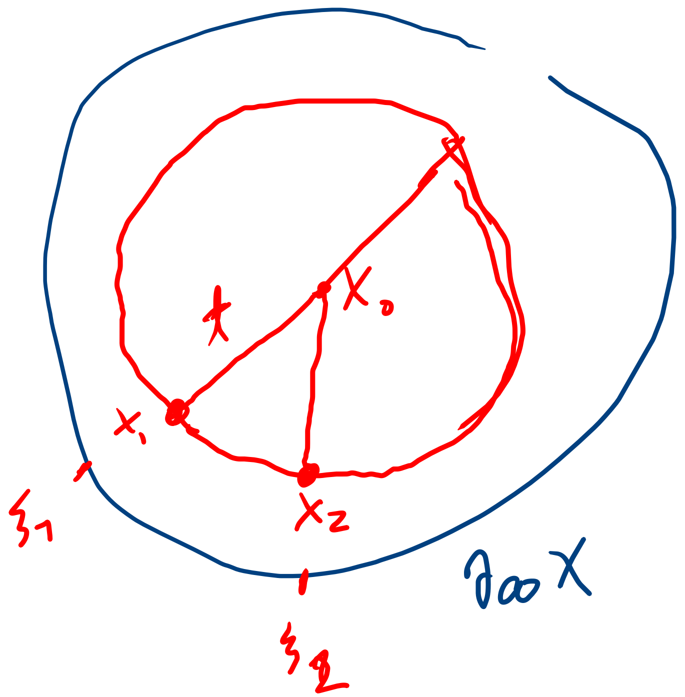


$$\alpha_{x_0}(\rho_1(t), \rho_2(t)) = \sqrt{\frac{\cosh d(\rho_1(t), \rho_2(t)) - 1}{\sqrt{2}\sinh t}} \qquad \sinh t \sim \frac{e^t}{2} \quad t \text{ big.}$$

$$\rho_1(0) = \rho_2(0) = x_0$$
$$\rho_1(\infty) = \xi_1, \ \rho_2(\infty) = \xi_2$$

$$\cosh d(\rho_1(t),\rho_2(t)) - 1 \sim \frac{e^{d(\rho_1(t), \rho_2(t))}}{2}$$

$$= \sqrt{e^{-2t + d(\rho_1(t), \rho_2(t))}}$$

$$= e^{-\frac{1}{2}(t - d(\rho_1(t), \rho_2(t)))} \xrightarrow[t \to \infty]{} \underline{e^{-(\xi_1|\xi_2)_{x_0}}}$$

$\square$

<u>EX</u>. $\mathbb{H}^2$, disk model

$$\alpha_{x_0} = \sin\frac{\measuredangle(\xi_1, \xi_2)}{2} = \underline{\frac{1}{2}\|\xi_1 - \xi_2\|}.$$