Neg. curved spaces 28.10.2020

$\to$ simplicial graph

length 1

Geom group theory: "Associate a geom. object to a group and use geometry to deduce results on algebra."

Defn G group, $S \subset G$, if $s \in S$ then $s^{-1} \in S$, $e \notin S$.

identity, neutral element of G

If S generates G, then S is a symmetric generating set of G.

G is the smallest subgp of G that contains S.

A group is finitely generated if it has a finite gen. set.

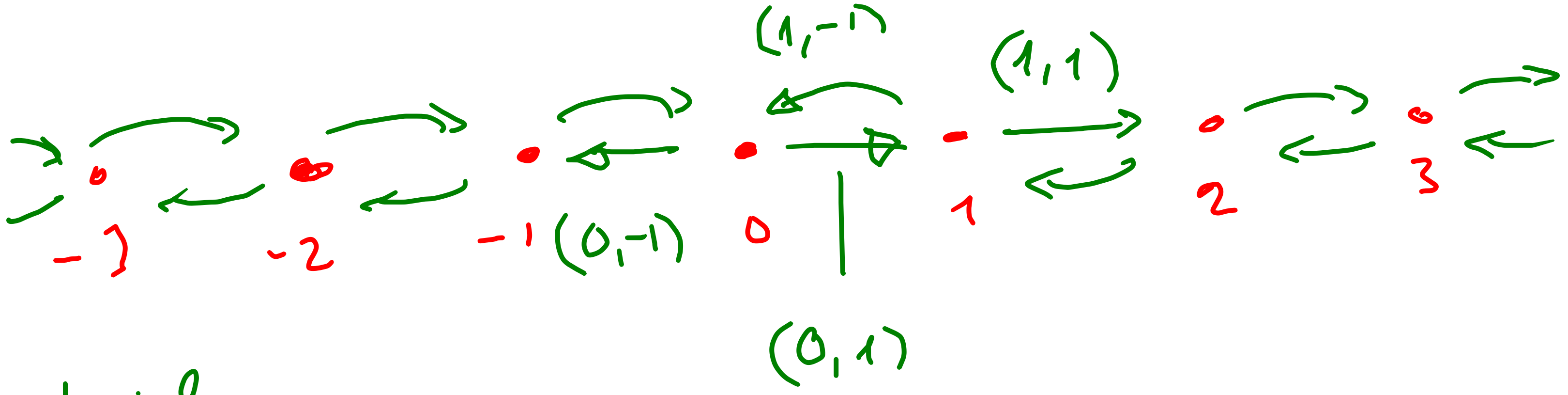
Cayley graph $\mathcal{G}(G,S)$: vertices $= G$

edges $= G \times S$

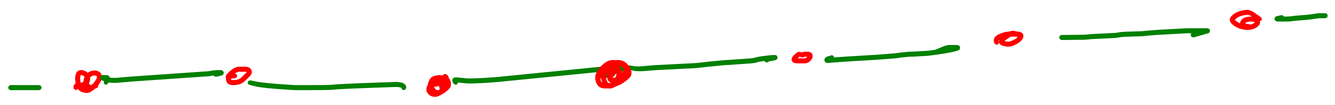
$o(g,s) = g$, $t(g,s) = gs$

(1) $\rightsquigarrow$ simplicial graph

Examples 1) $G=(\mathbb{Z},+)$, $S=\{-1,1\}$ $\qquad$ $\mathcal{G}(\mathbb{Z},\{-1,1\})$

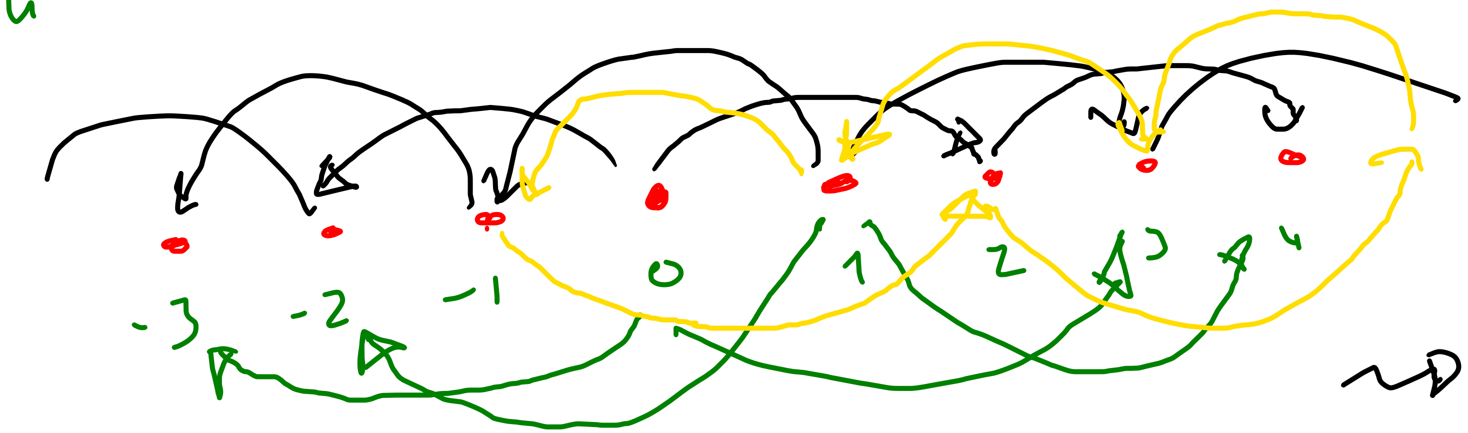


simplicial $\rightsquigarrow$ graph



isometric with $\mathbb{E}^1$. 0-hyp.

2)



$\tilde{S}=\{-3,-2,2,3\}$

$\rightsquigarrow$ simplicial graph $\mathcal{G}(\mathbb{Z},\tilde{S})$, not isometric with $\mathcal{G}(\mathbb{Z},\{\pm 1\})$.

Prop. 7.17: $\mathcal{G}(\mathbb{Z},S)$ and $\mathcal{G}(\mathbb{Z},\tilde{S})$ are quasi-isometric.

Thm. 7.8: $\mathcal{G}(\mathbb{Z},S)$ is Gromov-hyperbolic.

3) Free group on two generators.

Alphabet $\mathcal{A} = \{\underbrace{a, b, a^{-1}, b^{-1}}_{\text{symbols/letters}}\}$ $\rightsquigarrow$ a word in $\mathcal{A}$ is a finite sequence $s_1 s_2 \cdots s_n$, $s_i \in \mathcal{A}$.

+ $\boxed{\text{the empty word } e}$.

A word is reduced if it has no subwords $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$.

$\underbrace{abba}_{\text{reduced}}$ $\qquad$ $\underbrace{a\cancel{bb^{-1}}a}_{\text{not reduced}}$.

$\mathcal{R}(\mathcal{A})$ = set of reduced words on $\mathcal{A}$.

binary operation $*$ on $\mathcal{R}(\mathcal{A})$:

If $u = s_1 s_2 \cdots s_m$, $w = t_1 \cdots t_n \in \mathcal{R}(\mathcal{A})$

$u * w$ = reduced word obtained from $s_1 \cdots s_m t_1 \cdots t_n$ by deleting successively the forbidden subwords

identity

$(ab) * (b^{-1}a) = a\cancel{bb^{-1}}a = aa = a^2$. / $(ab)*(b^{-1}a^{-1}) = \cancel{a}\cancel{bb^{-1}}\cancel{a^{-1}} = e$.

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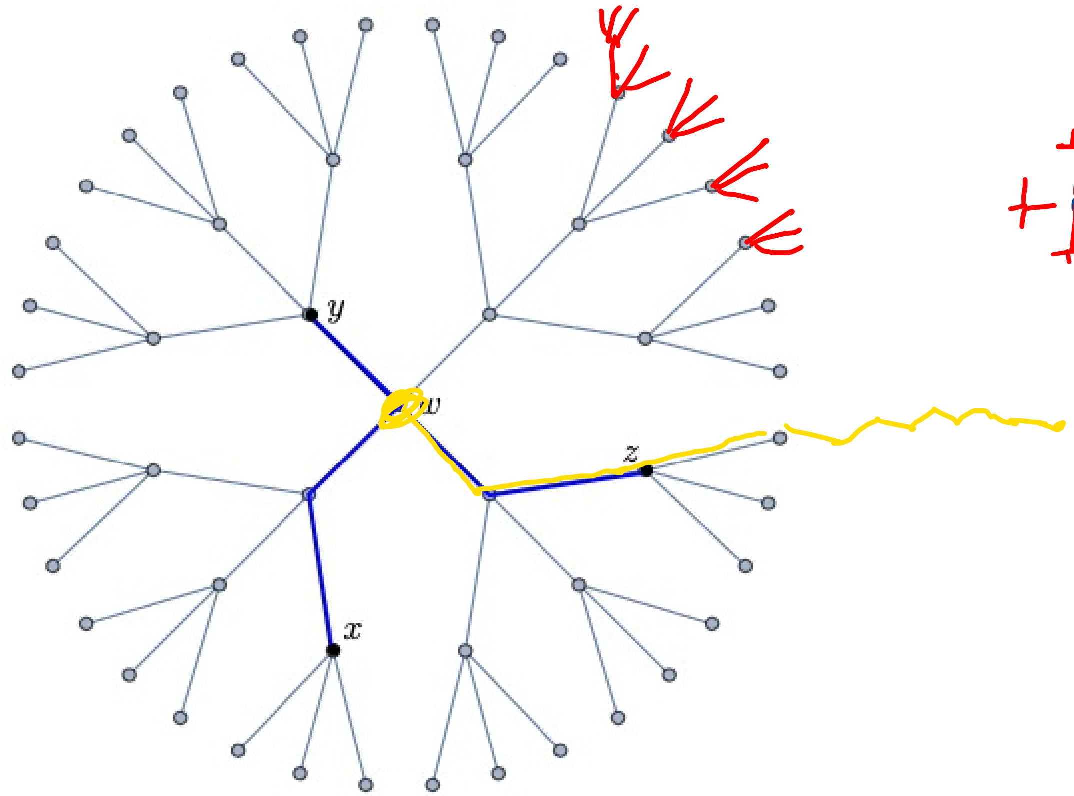
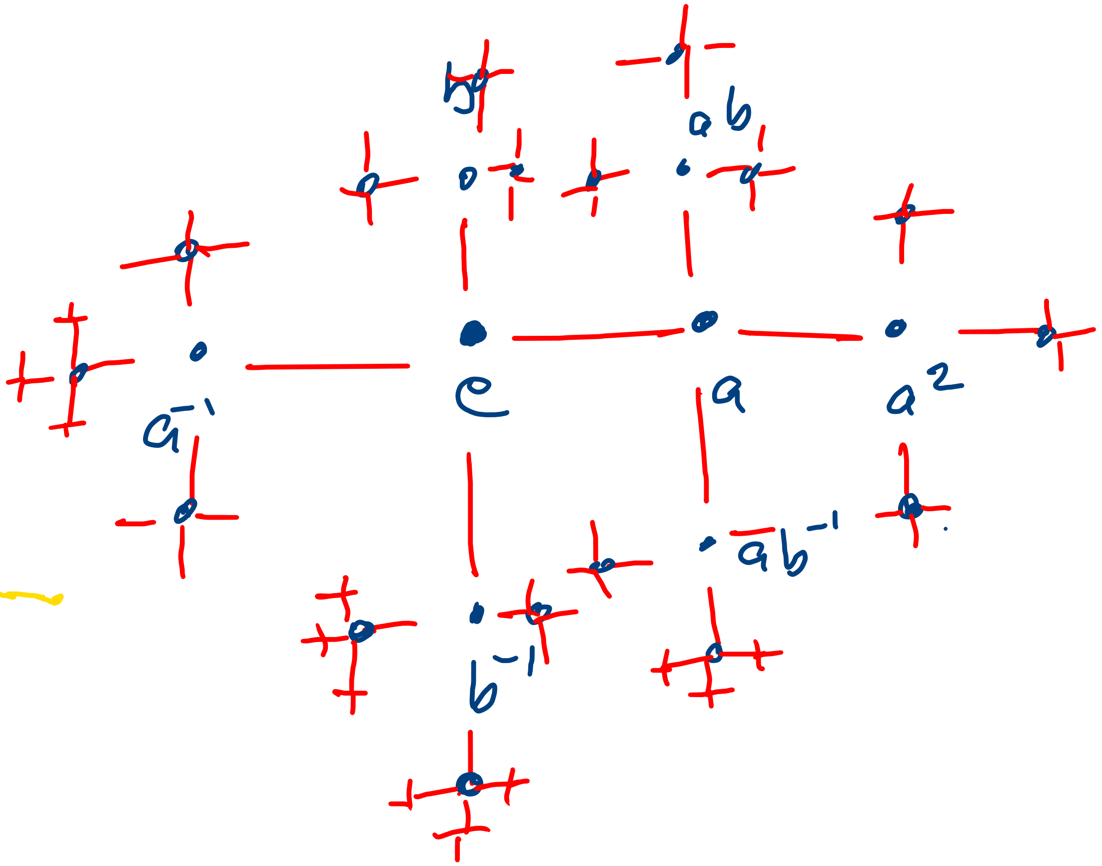
$F_2 = (\mathcal{R}(A), *)$ is the free group on two generators (a and b)

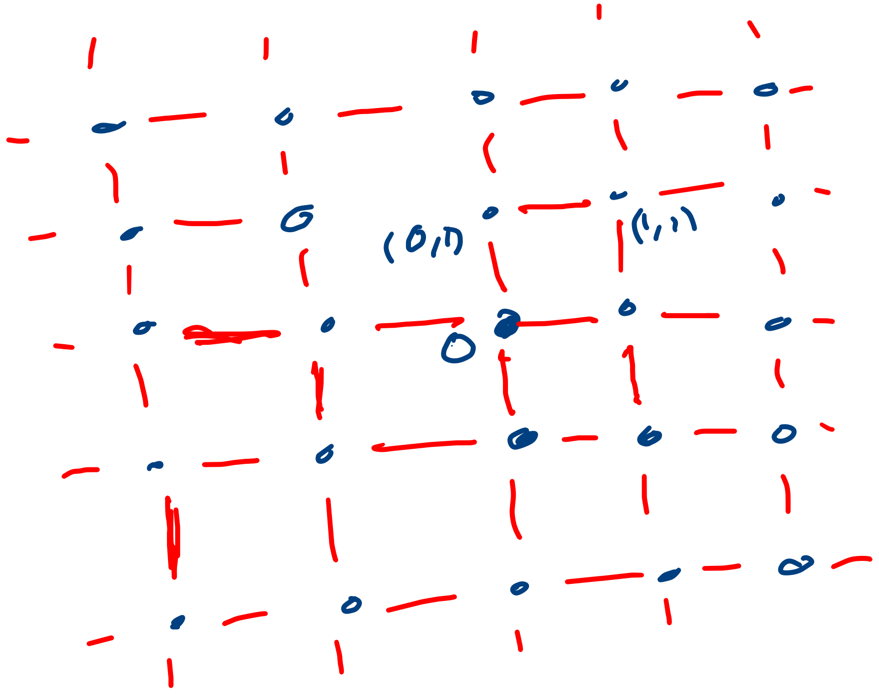
$\mathcal{G}(F_2, \{a, b, a^{-1}, b^{-1}\})$ is an infinite regular tree where the degree of each vertex is 4.

<u>0-hyperbolic</u>





4) $\mathbb{Z}^2$ $\quad S = \{(\pm 1, 0), (0, \pm 1)\}$



Word metric in G: S symm. gen. set

Exercise

$$d_S(g,h) = \min\{ n \in \mathbb{N} : g^{-1}h = s_1 s_2 \cdots s_n, \; s_i \in S \}$$

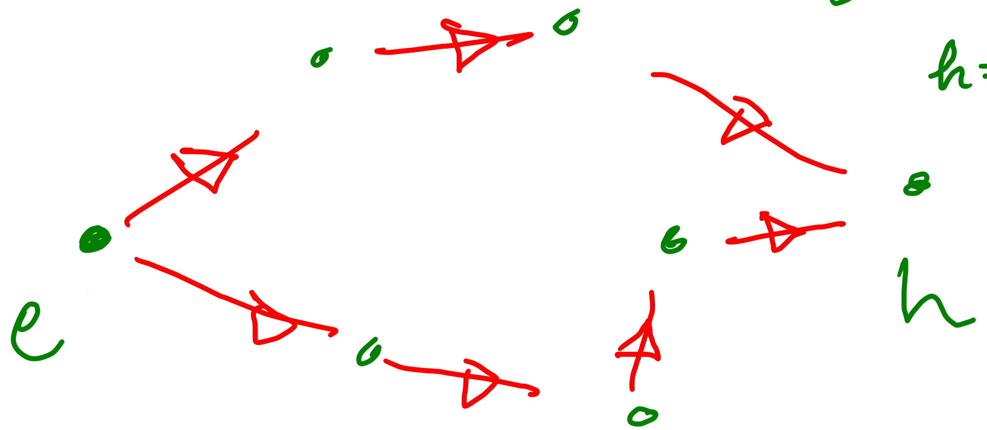
Note

$$\boxed{d_S(g,h) = d_S(\underbrace{e}_{= g^{-1}g}, g^{-1}h)}$$

Lemma The metric of $\mathcal{G}(G,S)$ induces the word metric on $G = V\mathcal{G}(G,S)$.

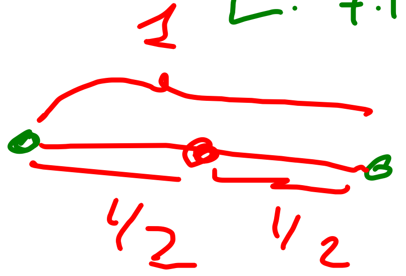
~~Exercise~~

$$d_S(e,h) = \min\{ n : h = s_1 \cdots s_n, \; s_i \in S \}.$$



Lemma 7.15 G gp, S symm. set of generators of G. The spaces $\mathcal{G}(G,S)$ and (G, d_S) are quasi-isometric.

Proof. $G \xrightarrow{i} V\mathcal{G}(G,S) \subset \mathcal{G}(G,S)$ isom. embedding by L. 7.14



For any $x \in \mathcal{G}(G,S)$ $d(x, V\mathcal{G}(G,S)) \leq \frac{1}{2}$

Prop. 7.6 $\Rightarrow$ i is a quasi-isometry.

Lemma 7.16 If S, T are finite symmetric gen. sets of G, then (G, d_S) and (G, d_T) are quasi-isometric.

Proof. Exercise.

Prop. 7.18 S, T finite symm. gen. sets of G. Then $\mathcal{G}(G,S)$ and $\mathcal{G}(G,T)$ are quasi-isometric.

Proof:

$$\mathcal{G}(G,S) \overset{L.7.15}{\text{q.i.}} (G, d_S) \overset{L.7.16}{\text{q.i.}} (G, d_T) \overset{L.7.15}{\text{q.i.}} \mathcal{G}(G,T)$$

L. 7.4

Defn A finitely generated group G is a **hyperbolic group** if $\mathcal{G}(G,S)$ is a Gromov-hyperbolic space for some finite symm. generating set S.

Ex. 1) $(\mathbb{Z}, +)$ Gromov-hyp.

2) F_2 — " — $\rightsquigarrow$ F_n free gp on n generators is also Gromov-hyp. (deg of a vertex is $2n$.)

3) The word metric on $\mathbb{Z}^2$ of $S = \{(\pm 1, 0), (0, \pm 1)\}$ is the same as the metric induced by the norm $\|\cdot\|_1$. $\rightsquigarrow (\mathbb{Z}^2, d_S)$ $\underline{\underline{q.i.}}$ $(\mathbb{R}^2, \|\cdot\|_1)$

equivalent with $\|\cdot\|_2$

$\underline{\underline{q.i}}$ $\mathbb{E}^2$

$\rightsquigarrow (\mathbb{Z}^2, d_S)$ q.i. $\mathbb{E}^2$

not Gromov-hyperbolic. $\Leftarrow$ not Gromov-hyp

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