Funktionaalianalyysi
Exercises 11, 9.4.2018

1. Let V be a complex vector space.
(a) Let $f\colon V \to \mathbb{R}$ be a real functional. Prove that $f_{\mathbb{C}}\colon V \to \mathbb{C}$,

$$f_{\mathbb{C}}(z) = f(z) - if(iz)\,,$$

is a complex functional.
(b) Let $F\colon V \to \mathbb{C}$ compex functional. Prove that $\operatorname{Re} F$ is a real functional and that $(\operatorname{Re} F)_{\mathbb{C}} = F$.

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2. Let V be a normed space and let $x, y \in V$ such that $v'(x) = v'(y)$ for all $v' \in V'$. Prove that $x = y$.

3. What is the relation of problem 2 of exercises 7 and problem 2 above?

4. Let V be a normed space, let $x_1, x_2, \ldots, x_n \in V$ be linearly independent and let $a_1, a_2, \ldots, a_n \in \mathbb{K}$. Prove that a functional $f \in V'$ exists such that $f(x_i) = a_i$ for all $1 \le i \le n$.

5. Let U be a vector subspace of a normed space V. Prove that

$$\overline{U} = \bigcap_{\substack{v' \in V' \\ U \subset \ker v'}} \ker v'\,.$$

6. Let X be a normed space. The annihilator of a normed vector subspace $U \subset X$ in the dual X' is

$$U^{\perp} = \{x' \in X' : x'|_U = 0\}\,.$$

Prove that U is dense if and only if $U^{\perp} = \{0\}$.

7. Give an example of a functional $f \in \ell^1(\mathbb{K})'' - \iota_{\ell^1(\mathbb{K})}(\ell^1(\mathbb{K}))$. [1]

8. Prove that the weak limit of a weakly convergent sequence is unique.[2]

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[1] Recall that $\ell^1(\mathbb{K})'$ is isometrically isomorphic with $\ell^\infty(\mathbb{K})$. Example **??** and the theorem of Hahn Banach may be useful.

[2] A consequence of the theorem of Hahn and Banach.