Funktionaalianalyysi
Exercises 10, 19.3.2018

1. Let X be a dense subspace of a normed space V. Let W be a Banach space and let $T \in \mathrm{Lin}_b(X, W)$ be an isometric embedding. Prove that the extension of T is an isometric embedding.

2. Let H be a Hilbert space and let $U \subset H$ vector subspace. Prove that every bounded functional $u' \in U'$ has a bounded extension $\widehat{u}' \in H'$.

3. Let E be an orthonormal set in a Hilbert space H. Prove that H has an orthonormal basis that contains E.[1]

4. Let E be an orthonormal set in a Hilbert space H such that

$$\sum_{e \in E} |(h \mid e)|^2 = \|h\|^2 .$$

for all $h \in H$. Prove that

$$\sum_{e \in E} (x \mid e)(e \mid y) = (x \mid y) .$$

for all $x, y \in H$.

5. Let E be an orthonormal set in a Hilbert space H such that

$$\sum_{e \in E} (x \mid e)(e \mid y) = (x \mid y)$$

for all $x, y \in H$. Prove that E is an orthonormal Hilbert basis.

6. Let $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$. Give an example of an isometric isomorphism $g \colon L^2([a, b]) \to L^2([c, d])$.

7. Determine the Fourier series of $f \in L^2([-\pi, \pi])$, $f(t) = |t|$.

8. Let $A \subset [-\pi, \pi]$be a measurable set. Prove that

$$\lim_{k \to \infty} \int_A \cos(kt)\, dt = 0 .$$

[1]Use Zorn's lemma as in the proof of the correspondin result for Hamel bases.