Differential geometry 2023

Exercises 5

1. Prove that the mapping $\Phi\colon \mathbb{S}^1 \times \mathbb{E}^1 \to T\mathbb{S}^1$, $\Phi(p,s) = s\,\frac{\partial}{\partial\theta}|_p$ is a smooth diffeomorphism.

Solution. The tangent space $T_p\mathbb{S}^1$ is 1-dimensional for all $p \in \mathbb{S}^1$. Thus, $T_p\mathbb{S}^1$ is generated by the tangent vector $\frac{\partial}{\partial\theta}|_p$. This implies that the mapping Φ is a bijection with inverse $\Phi^{-1}(s\,\frac{\partial}{\partial\theta}|_p) = (p,s)$. Therefore, it suffices to show that Φ is a local diffeomorphism. Let (U,ϕ) be an angle chart on $\mathbb{S}^1$, use the canonical chart of $\mathbb{E}^1$, the product chart on $U \times \mathbb{E}^1$, and the chart associated with (U,ϕ) on $T\mathbb{S}^1$. In these charts Φ corresponds to the identity mapping $(x,s) \mapsto (x,s)$, which is a diffeomorphism.

2. Let M be a smooth n-manifold. Let $X_1, X_2, \ldots, X_n \in \mathfrak{X}(M)$ be smooth vector fields such that the tangent vectors $X_1(p), X_2(p), \ldots, X_n(p) \in T_pM$ are linearly independent for all $p \in M$. Prove that there is a smooth diffeomorphism $F\colon M \times \mathbb{E}^n \to TM$.

Solution. Set $H(p,v) = \sum_{k=1}^n v^k X_k(p)$. To check its smoothness, fix $p \in M$, $v \in \mathbb{E}^n$ and use the chart $(U \times \mathbb{E}^n, \phi \times \mathrm{id}_{\mathbb{E}^n})$ of $M \times \mathbb{E}^n$ around (p,v), as well as the chart (TU, ψ) of TM associated to (U,ϕ) around $H(p,v)$. We compute, for all $(x,u) \in \phi(U) \times \mathbb{E}^n$,

$$\begin{aligned}
\psi \circ H \circ (\phi^{-1} \times \mathrm{id}_{\mathbb{E}^n})(x,u) = \psi(H(\phi^{-1}(x),u)) &= \psi\Big(\sum_{k=1}^n u^k X_k(\phi^{-1}(x))\Big) \\
&= \Big(x, d\phi_{\phi^{-1}(u)}\Big(\sum_{k=1}^n u^k X_k(\phi^{-1}(x))\Big)\Big) \\
&= \Big(x, \sum_{k=1}^n u^k d\phi_{\phi^{-1}(u)}(X_k(\phi^{-1}(x)))\Big),
\end{aligned}$$

which is smooth (since its first coordinate clearly is, and its second one is a combination of smooth coordinates, since $X_1, \ldots, X_n$ are smooth, with coordinates given by the smooth projections $u \mapsto u^k$). Set $A(x,u)$ the $n \times n$ matrix of columns given by $d\phi_{\phi^{-1}(u)}(X_k(\phi^{-1}(x)))$ for $1 \le k \le n$. By the hypothesis on $X_1, \ldots, X_n$, this matrix is always invertible. And since matrix inversion is a smooth map of $\mathrm{GL}_n(\mathbb{R})$, the local inverse of H on $\phi(U) \times \mathbb{E}^n$ is smooth. Finally, since H is globally a bijection, it is indeed a diffeomorphism.

3. Let M be a smooth manifold. Let $p \in M$ and let $v_p \in T_pM$. Prove that there is a smooth vector field $X \in \mathfrak{X}(M)$ such that $X_p = v_p$.

Solution. Let (U,ϕ) be a chart centered at p. Let $v_p = \sum_{k=1}^n c^k \frac{\partial}{\partial x^k}\Big|_p$ and let $Y \in \mathfrak{X}(U)$ be the constant vector field $Y = \sum_{k=1}^n c^k \frac{\partial}{\partial x^k}$. Let $\eta \in \mathfrak{F}(M)$ be a smooth bump at p such that its support is contained in U. Then $Z = \eta|_U Y|_U \in \mathfrak{X}(U)$ and $Z_p = v_p$. Let X be the vector field defined by

$$X(q) = \begin{cases} Z(q) & \text{if } q \in U, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If $q \in M - U$, then there is an open neighbourhood $V \ni q$ such that $X|_V = 0$. Thus X is smooth.

4. Let M be a smooth n-manifold. Prove that $\mathfrak{X}(M)$ is a real vector space and an $\mathfrak{F}(M)$-module.

Solution. Let $p \in M$. Take (U, ϕ) a chart containing p and (TU, ψ) the associated map of TM. We recall the definition

$$\psi : (q, v) \mapsto (\phi(q), d\phi_q(v)) \in \phi(U) \times \mathbb{E}^n \subset \mathbb{E}^{2n},$$

where the derivation $d\phi_q(v)$ is identified with a vector in $\mathbb{E}^n$ by the bijection $v' \mapsto \partial_{v'|\phi(q)}$. Let $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \mathbb{R}$. Then, we compute, for all $u \in \phi(U)$,

$$\begin{aligned}\psi \circ (\alpha X) \circ \phi^{-1}(u) &= \psi(\alpha X_{\phi^{-1}(u)}) = (u, d\phi_{\phi^{-1}(u)}(\alpha X_{\phi^{-1}(u)})) = (u, \alpha d\phi_{\phi^{-1}(u)}(\alpha X_{\phi^{-1}(u)}),\\ \text{and } \psi \circ (X + Y) \circ \phi^{-1}(u) &= (u, d\phi_{\phi^{-1}(u)}(X_{\phi^{-1}(u)} + d\phi_{\phi^{-1}(u)}(Y_{\phi^{-1}(u)}),\end{aligned}$$

because of the linearity of $d\phi_{\phi^{-1}(u)}$. These two formulae give smooth maps since they come from linear combinations of the coordinates of

$$\psi \circ X \circ \phi^{-1}(u) = (u, d\phi_{\phi^{-1}(u)}(X_{\phi^{-1}(u)})$$

and

$$\psi \circ Y \circ \phi^{-1}(u) = (u, d\phi_{\phi^{-1}(u)}(Y_{\phi^{-1}(u)})$$

(which are smooth by definition of vector fields). This proves that $\mathfrak{X}M$ is a real vector space.

For all $f \in \mathfrak{F}(M)$ $(= C^\infty(M))$, we define $fX : M \to TM$ by the formula, for all $p \in M$, $(fX)(p) = f(p)X_p \in T_pM$. With this definition, all the axioms of an $\mathfrak{F}(M)$-module are verified for the space $\mathfrak{X}(M)$. It only remains to show that $(f, X) \mapsto fX$ is well defined from $\mathfrak{X}(M)$ to $\mathfrak{X}(M)$. Fix $f \in \mathfrak{F}(M)$. In other words, we want to show that fX is smooth. Take the charts (U, ϕ) and (TU, ψ) defined above. Then we compute

$$\begin{aligned}\psi \circ (fX) \circ \phi^{-1}(u) = \psi(f(\phi^{-1}(u))X_{\phi^{-1}(u)}) &= (u, d\phi_{\phi^{-1}(u)}(f(\phi^{-1}(u))X_{\phi^{-1}(u)}))\\ &= (u,\ f(\phi^{-1}(u))\, d\phi_{\phi^{-1}(u)}(X_{\phi^{-1}(u)})),\end{aligned}$$

since $d\phi_{\phi^{-1}(u)}$ is linear. This is the formula of a smooth map since X and f are smooth.

5. (1) Let v be a derivation of an algebra A. Prove that $v(1) = 0$.
(2) Let v_1 and v_2 be derivations of an algebra A. Prove that $v_1v_2 - v_2v_1$ is a derivation.

Solution. (1) $v(1) = v(1 \cdot 1) = 1v(1) + v(1)1 = 2v(1)$ implies $v(1) = 0$.
(2) Let $f \in A$. Then

$$\begin{aligned}(v_1v_2 - v_2v_1)(ab) = v_1v_2(ab) - v_2v_1(ab) &= v_1(v_2(a)b + av_2(b)) - v_2(v_1(a)b + av_1(b)\\ &= (v_1v_2(a)b + v_2(a)v_1(b) + v_1(a)v_2(b) + av_1v_2(b))\\ &\quad - (v_2v_1(a)b + v_1(a)v_2(b) + v_2(a)v_1(b) + av_2v_1(b)\\ &= (v_1v_2(a)b + av_1v_2(b)) - (v_2v_1(a)b + av_2v_1(b))\\ &= (v_1v_2 - v_2v_1)(a)b + a(v_1v_2 - v_2v_1)(b)\,.\end{aligned}$$