Differential geometry 2023

Exercises 3

1. Prove that the quotient mapping $\pi\colon \mathbb{S}^n \to \mathbb{P}^n$, $\pi(x) = [x]$, is a smooth local diffeomorphism.

Solution. Let $p \in \mathbb{S}^n$. We may assume that $p_{n+1} > 0$. Let us use the smooth chart $\mathrm{pr}_{n+1}\colon U^+_{n+1} \to \mathbb{E}^n$, $\mathrm{pr}_{n+1}(x) = (x_1, x_2, \dots, x_n)$ for $\mathbb{S}^n$ and the smooth chart $\phi_{n+1}\colon U_{n+1} \to \mathbb{E}^n$,

$$\phi_{n+1}\big([x_1 : x_2 : \dots : x_{n+1}]\big) = \Big(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\Big)$$

for $\mathbb{P}^n$. We saw in the lectures [1]that

$$\begin{aligned}\phi_{n+1} \circ \pi \circ \mathrm{pr}^{-1}_{n+1}(y) &= \phi_{n+1} \circ \pi(y_1, y_2, \dots, y_n, \sqrt{1 - \|y\|^2}) \\ &= \phi_{n+1}\big([y_1 : y_2 : \dots : y_n : \sqrt{1 - \|y\|^2}]\big) \\ &= \Big(\frac{y_1}{\sqrt{1 - \|y\|^2}}, \frac{y_2}{\sqrt{1 - \|y\|^2}}, \dots, \frac{y_n}{\sqrt{1 - \|y\|^2}}\Big)\end{aligned}$$

defines a smooth mapping, and similarly for other smooth charts $\mathrm{pr}_k\colon U^\pm_k$, so π is smooth.

To show that π is a local diffeomorphism, first notice that it is injective on U^+_{n+1} (and similarly on each $U^\pm_k$). On U_{n+1}, the inverse of this restriction is given by

$$(\pi_{|\,U^+_{n+1}})^{-1} : [x_1 : x_2 : \dots : x_{n+1}] \mapsto \frac{\mathrm{sgn}(x_{n+1})}{\|x\|}(x_1, \dots, x_{n+1}).$$

Then, we compute

$$\begin{aligned}\mathrm{pr}_{n+1} \circ (\pi_{|\,U^+_{n+1}})^{-1} \circ \phi^{-1}_{n+1}(x_1, \dots, x_n) &= \mathrm{pr}_{n+1} \circ (\pi_{|\,U^+_{n+1}})^{-1}([x_1 : \dots : x_n : 1]) \\ &= \mathrm{pr}_{n+1} \frac{1}{\sqrt{1 + \|x\|^2}}(x_1, \dots, x_n, 1) \\ &= \mathrm{pr}_{n+1} \frac{1}{\sqrt{1 + \|x\|^2}}(x_1, \dots, x_n)\end{aligned}$$

which is the formula on a smooth mapping, thus is this local inverse of π and hence π is a local diffeomorphism.

(The mapping π is not a (global) diffeomorphism as it is not injective: $\pi(0, \dots, 0, 1) = \pi(0, \dots, 0, -1)$ by definition of $\mathbb{P}^n$.)

2. Let M and N be smooth manifolds and let $q_0 \in N$. Let $i_{q_0}\colon M \to M \times N$ be the mapping $i_{q_0}(p) = (p, q_0)$. Prove that i_{q_0} is smooth.

Solution. Let $p_0 \in M$. Take a chart (U, ϕ) of M containing p_0 and a chart $(U' \times V, \phi' \times \psi)$ of $M \times N$ containing $i_{q_0}(p_0) = (p_0, q_0)$ (i.e. $p_0 \in U'$ and $q_0 \in V$). Then we compute, for all $x \in \phi(U \cap U')$,

$$(\phi' \times \psi) \circ i_{q_0} \circ \phi^{-1}(x) = (\phi' \times \psi)(\phi^{-1}(x), q_0) = (\phi' \circ \phi^{-1}(x), \psi(q_0)),$$

which defines a smooth map.

[1]Example 2.8(2)

3. Prove that the mapping $F\colon \mathbb{T}^1 \to \mathbb{S}^1$, $F(s+\mathbb{Z}) = (\cos(2\pi s), \sin(2\pi s))$ is smooth.

Solution. Let $\pi : \mathbb{R} \to \mathbb{T}^1$ denote the canonical projection. Take $p \in \mathbb{T}^1$ and $x \in \pi^{-1}(p)$. Around p, we use the chart (U, π^{-1}) given by the local inverse $(\pi_{|\ ]x-\frac{1}{3},x+\frac{1}{3}[\ })^{-1}$ on $U = \pi(\ ]x-\frac{1}{3}, x+\frac{1}{3}[\ )$. Let us assume that $\cos(2\pi x)$ is positive, that is to say $F(p) = (\cos(2\pi x), \sin(2\pi x))$ has a positive first coordinate. Then, we can take the chart (U_1^+, pr_1) of $\mathbb{S}^1$. We compute, for all $s \in\ ]x-\frac{1}{3}, x+\frac{1}{3}[\ \cap \cos^{-1}(\mathbb{R}_+^*)$,

$$\mathrm{pr}_1 \circ F \circ \pi(s) = \sin(2\pi s).$$

4. Let M be a smooth n-manifold and let $F\colon U \to F(U) \subset \mathbb{E}^n$ be a smooth diffeomorphism. Prove that (U, F) is a smooth chart.

Solution. Let $p \in M$ and let (V, ϕ) be a smooth chart at p. As a diffeomorphism, F is a homeomorphism (it is needed in the definition of a chart!). We saw in class that ϕ is a diffeomorphism, so $F \circ (\phi|_{U\cap V})^{-1}$ is a diffeomorphism as the composition of two diffeomorphisms. This shows that F is compatible with the maximal atlas.

5. Let A and B be matrices such that $AB = I_n$ and $BA = I_m$. Prove that $B = A^{-1}$ and $n = m$.

Solution. A priori, we only know that A is of dimensions $n \times m$ and that B is $m \times n$. Suppose $n < m$. Then, A is of rank at most n, and by composition, so is the product BA, which is absurd since the rank of I_m is m. Symmetrically, we cannot have $n > m$, hence $n = m$, these matrices are square and A is then the inverse of B by definition of the inverse of a (square) matrix.

6. Let M be a smooth manifold and let $p \in M$. Prove that $\mathrm{Der}(p)$ is a vector space.

Solution. The set of derivations is nonempty as the 0-operator is clearly a derivation at p. Let $v, w \in \mathrm{Der}(p)$ and let $\lambda, \mu \in \mathbb{R}$. We know from linear algebra that $\lambda v + \mu w$ is a linear function. Let $f \in \mathrm{C}^\infty(M)$. Then by definition of a linear combination of mappings, as v and w are derivations, and by definition of a linear combination of mappings again

$$\begin{aligned}(\lambda v + \mu w)(fg) &= \lambda v(fg) + \mu w(fg) \\ &= \lambda(v(f)\, g(p) + f(p)\, v(g)) + \mu(w(f)g(p) + f(p)w(g)) \\ &= (\lambda v + \mu w)(f)\, g(p) + f(p)(\lambda v + \mu w)(g)\,,\end{aligned}$$

which shows that $(\lambda v + \mu w) \in \mathrm{Der}(p)$.