Differential geometry 2023

Exercises 10

1. Let S be a smooth manifold and let (M, g) be a Riemannian manifold. Let $F\colon S \to M$ be an immersion. Prove that F^*g is a Riemannian metric.

Solution. The pullback F^*g is a covariant 2-tensor field by Proposition 8.11 and Lemma 8.12. It is symmetric because pullback is defined by the slots:

$$(F^*g)_p(v, w) = g_{F(p)}(dF_p v, dF_p w) = g_{F(p)}(dF_p w, dF_p v) = (F^*g)_p(w, v)\,.$$

It remains to check positive-definiteness: Let $v \in T_pM - \{0\}$. Then $dFv \neq 0$ because F is an immersion. Thus,

$$(F^*g)_p(v, v) = g_{F(p)}(dF_p v, dF_p v) > 0\,.$$

The stereographic projection $\mathscr{S}\colon \mathbb{S}^2 - \{e_3\} \to \mathbb{E}^2 = \mathbb{E}^2 \times \{0\}$ from the north pole to the equatorial plane, is the mapping

$$\mathscr{S}(x) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right).$$

The mapping $\mathscr{S}$ is a diffeomorphism that assigns to $x \in \mathbb{S}^2 - \{e^3\}$ the unique point in the plane $\mathbb{E}^2$ (thought of as the hyperplane $\mathbb{E}^2 \times \{0\}$ in $\mathbb{E}^3$) that lies on the line through e_3 and x. The inverse of the stereographic projection is given by

$$\mathscr{S}^{-1}(y) = \frac{1}{1 + \|y\|^2}(2y_1, 2y_2, \|y\|^2 - 1)\,.$$

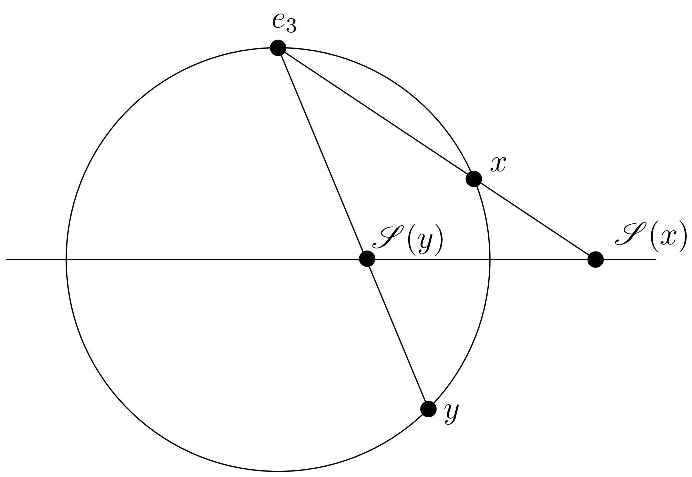

Figure 1: Stereographic projection.

Let $i\colon \mathbb{S}^n \to \mathbb{E}^{n+1}$ be the inclusion mapping. Let $g_{\mathbb{E}} = \sum_{k=1}^{n+1}(dx^k)^2$ be the Euclidean Riemannian metric of $\mathbb{E}^{n+1}$. The Riemannian metric $g_{\mathbb{S}} = i^*g_{\mathbb{E}}$ is the *standard* or *round Riemannian metric* on $\mathbb{S}^2$. The Riemannian manifold $(\mathbb{S}^n, g_{\mathbb{S}})$ is the *standard* or *round n-sphere.*

2. Let $g_{\mathbb{S}}$ be the Riemannian metric of the round 2-sphere $\mathbb{S}^2$. Compute $(\mathscr{S}^{-1})^* g_{\mathbb{S}}$.

Solution. Notice that $(\mathscr{S}^{-1})^* g_{\mathbb{S}} = (\mathscr{S}^{-1})^*(i^* g_{\mathbb{E}}) = (i \circ \mathscr{S}^{-1})^* g_{\mathbb{E}}$. The Jacobian matrix of $i \circ \mathscr{S}^{-1} : \mathbb{E}^2 \to \mathbb{E}^3$ is

$$\frac{1}{(1+\|y\|^2)^2} \begin{pmatrix} 2(1-(y^1)^2+(y^2)^2) & -4y^1y^2 \\ -4y^1y^2 & 2(1+(y^1)^2-(y^2)^2) \\ 4y^1 & 4y^2 \end{pmatrix} .$$

Therefore,

$$\begin{aligned}
&(i \circ \mathscr{S}^{-1})^*((dx^1)^2+(dx^2)^2+(dx^3)^2) \\
&= \frac{1}{(1+\|y\|^2)^4}\Big((2(1-(y^1)^2+(y^2)^2)dy^1 - 4y^1y^2dy^2)^2 \\
&\quad + (-4y^1y^2dy^1 + 2(1+(y^1)^2-(y^2)^2)dy^2)^2 + (4y^1dy^1+4y^2dy^2)^2\Big) \\
&= \frac{1}{(1+\|y\|^2)^4}\Big((4(1-(y^1)^2+(y^2)^2)^2 + 16(y^1y^2)^2 + 16(y^1)^2)(dx^1)^2) \\
&\quad (4(1+(y^1)^2-(y^2)^2)^2 + 16(y^1y^2)^2 + 16(y^2)^2)(dx^2)^2 + 0\, dx^1 \otimes dx^2 + 0\, dx^1 \otimes dx^2\Big) \\
&= \frac{4}{(1+\|y\|^2)^4}\Big(((1+(y^2)^2)^2 - 2(y^1)^2(1+(y^2)^2) + (y^1)^4 + 4(y^1y^2)^2 + 4(y^1)^2)(dx^1)^2) \\
&\quad ((1+(y^1)^2)^2 - 2(y^2)^2(1+(y^1)^2) + (y^2)^4 + 4(y^1y^2)^2 + 4(y^2)^2)(dx^2)^2\Big) \\
&= \frac{1}{(1+\|y\|^2)^4}((1+\|y\|^2)^2(dy^1)^2 + (1+\|y\|^2)^2(dy^2)^2) \\
&= \frac{(dy^1)^2+(dy^2)^2}{(1+\|y\|^2)^2}.
\end{aligned}$$

The Riemannian metric

$$g_{\mathbb{H}} = \frac{1}{(x^n)^2}\sum_{i=1}^{n}(dx^i)^2$$

of the *upper halfspace* $\{x \in \mathbb{R}^n : x^n > 0\}$ defines the *upper halfspace model*

$$\mathbb{H}^n = \big(\{x \in \mathbb{R}^n : x^n > 0\}, g_{\mathbb{H}}\big),$$

of *hyperbolic n-space.*

3. The mapping $F\colon B^n(0,1) \to \mathbb{H}^n$

$$F(y) = -\mathbf{e}_n + 2\frac{y+\mathbf{e}_n}{\|y+\mathbf{e}_n\|^2}$$

is a smooth diffeomorphism.[1] Compute $F^* g_{\mathbb{H}}$. It is sufficient to do the computation just for $n=2$.

Solution. We assume $n=2$. In this case, we can write $F(y) = \frac{(2y_1, 1-y_1^2-y_2^2)}{y_1^2+(y_2+1)^2}$. The Jacobian matrix of the map F is given by

$$\frac{1}{\|y+\mathbf{e}_2\|^4}\begin{pmatrix} 2((y_2+1)^2-(y_1)^2) & -4y_1(y_2+1) \\ -4y_1(y_2+1) & 2(y_1^2-(y_2+1)^2) \end{pmatrix}.$$

[1]This mapping is the restriction of the inversion in the sphere of radius $\sqrt{2}$ centred at $-\mathbf{e}_n$ to the unit ball in $\mathbb{E}^n$.

Then

$$\begin{aligned}
F^*g_{\mathbb{H}} =& F^*\Big(\frac{1}{(y^2)^2}((dy^1)^2+(dy^2)^2)\Big)\\
=&\frac{1}{F_2(y^1,y^2)^2}\Big(\big(\frac{\partial F_1}{\partial y^1}dy^1+\frac{\partial F_1}{\partial y^2}dy^2\big)^2+\big(\frac{\partial F_2}{\partial y^1}dy^1+\frac{\partial F_2}{\partial y^2}dy^2\big)^2\Big)\\
=&\frac{\|y+\mathbf{e}_2\|^4}{(1-\|y\|^2)^2}\frac{1}{\|y+\mathbf{e}_2\|^8}((2((y^2+1)^2-(y^1)^2)dy^1+-4y^1(y^2+1)dy^2)^2\\
&+(-4y^1(y^2+1)dy_1+2((y^1)^2-(y^2+1)^2)dy^2)^2)\\
=&\frac{4((dy^1)^2+(dy^2)^2)}{(1-\|y\|^2)^2}.
\end{aligned}$$

For a general dimension n, a similar computation gives

$$F^*g_{\mathbb{H}} = \frac{4}{(1-\|y\|^2)^2}\sum_{i=1}^{n}(dx^i)^2.$$

The unit ball (or disk if $n=2$) $B(0,1)$ endowed with this metric is known as the Poincaré ball (or disk).

4. Give an example of a 2-covector $\omega \in A^2(\mathbb{R}^4)$ such that $\omega\wedge\omega \neq 0$.

Solution. Let $(\epsilon^1,\epsilon^2,\epsilon^3,\epsilon^4)$ be a basis of covectors dual to the standard basis of $\mathbb{R}^4$. Then

$$\begin{aligned}
&(\epsilon^1\wedge\epsilon^2+\epsilon^3\wedge\epsilon^4)\wedge(\epsilon^1\wedge\epsilon^2+\epsilon^3\wedge\epsilon^4)\\
&=\epsilon^1\wedge\epsilon^2\wedge\epsilon^1\wedge\epsilon^2+\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4+\epsilon^3\wedge\epsilon^4\wedge\epsilon^1\wedge\epsilon^2+\epsilon^3\wedge\epsilon^4\wedge\epsilon^3\wedge\epsilon^4\\
&=2\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4
\end{aligned}$$

because the first and fourth summand have repeated covectors in the wedge products and because (13)(24) is an even permutation. As $\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4(e_1,e_2,e_3,e_4)=1$, the 4-covector $2\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4$ is nonzero.

5. Let $\omega^1,\dots\omega^k \in A^1(V)$. Assume that $\omega^i=\omega^j$ for some indices $1\le i,j\le k$, $i\neq j$. Prove that $\omega^1\wedge\cdots\wedge\omega^k=0$.

Solution. Using Proposition 7.17(2) several times and associtivity of the wedge product, we can assume $\omega^1=\omega^2$ Then

$$(\omega^1\wedge\omega^1)\wedge(\omega^3\wedge\cdots\wedge\omega^k)=0\wedge(\omega^3\wedge\cdots\wedge\omega^k)=0\,,$$

because $0\otimes A=0$ for any tensor A.

6. Assume that $\omega^1,\dots,\omega^k\in A^1(V)$ are linearly dependent. Prove that $\omega^1\wedge\cdots\wedge\omega^k=0$.

Solution. Up to using the anticommutativity formula $\omega\wedge\omega'=(-1)^{1^2}\omega'\wedge\omega=-\omega'\wedge\omega$ (for all $\omega,\omega'\in A^1(V)$) several times, we can assume that ω^1 is equal to the linear combination $\sum_{i=2}^k\lambda_i\omega^i$ (where $\lambda_1,\dots,\lambda_k\in\mathbb{R}$). Then, by linearity of the wedge product and the result of Exercise 5, we obtain

$$\omega^1\wedge\cdots\wedge\omega^k=\sum_{i=2}^{k}\lambda_i(\omega^i\wedge\cdots\wedge\omega^k)=\sum_{i=2}^{k}\lambda_i\,0=0.$$