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Exercise set7 **Topological vector spaces**
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7.1. *Let E locally convex Hausdorff-space and $A \subset E$. Find a necessary and sufficient condition for that the polar of A:n in E^* (or E') is $\{0\}$. (Some help:: bipolaari?)*

Solution: Let $A^\circ = A$be the polar in E^*.
Claim: $A^\circ = \{0\} \iff A^{\circ\circ} = E$.

PROOF $A^\circ = \{0\} \implies A^{\circ\circ} = \big\{x \in E \;\big|\; |\langle x, y\rangle| \leq 1 \forall x \in \{0\}\big\} = E$.
$A^{\circ\circ} = E \implies A^{\circ\circ\circ} = E^\circ = \{0\}$.

Notice: (E, E^*) separates E . (So doesn E'). The result can be generalized!

7.2. *Let E be a locally convex Hausdorff-space and $B \subset E$ balansoitu, konveksi and bounded – and complete. Define Therefore is a seminormi, since B is balansoitu, konveksi and in E_B s absorboiva. In fact the gauge p is a norm, since B is bounded in tehe original Hausdorff-topology τ of E , so we write $p = \|\cdot\|$. Is the unit ball of E_B same as the closure $\bar{B}$?*

Solution: Of course $p_B = \inf_{x \in \lambda B} \lambda$ is seminormi. If $x \in E_B$ and $p_B(x) = 0$, then $x \in \lambda B$, for all $\lambda > 0$. But B is bounded, so for all neighbourhoods of the origin U exists $\lambda > 0$ such that $\lambda B \subset U$, so $x \in U$. Now x sits inside every neighbourhood of the origin and is therefore 0, since the space is Hausdorff. So p_M is a norm in E_B , so call it $\|\cdot\|$. B iscomlete, so closed in E. Canoninen injektio $E_B \to E$ is continuous, since for all neighbourhoods of the origin U exists $\lambda > 0$ such that $\lambda B \subset U$. So B is preimage of closed, so closed in norm topology. Finlly generally every balansoitu konveksi is inside the closed ballof its gauge, and contains the open ball. This implies the claim. □

7.3. *Let E be a locally convex Hausdorff-space. Then (E, E^*) is separoiva. Therefore the completion of $E_{\sigma(E,E^*)}$:is the algebrallinen duaali $(E^*)'$. Can E_σ be complete?*

Solution:
Take $E = F'_{\sigma(F',F)}$.

- Take any vector space F.
- Let E be its algebrallinen duaali F'.
- Topologize E with locally convex topology $\sigma(E, F) = \sigma(F', F)$, which is locally convex , Hausdorff and compatible ,
- $E^* = (E_{\sigma(E,F)})^* = F$.
- $(E^*)' = F' = E$.

Is this OK?

7.4. *Let (E, F) separoituva dualiteetti and $M \subset E$ vector subpace. Prove that $M^{\perp\perp} = M$ if and only if M is closed in some compatible topology wrt duality (E, F).*

Solution: Let i $M^{\perp\perp} = M$. Now M is closed in topology $\sigma(E, F)$, since the orthogonal complement of any set $B \subset F$

$$B^{\perp} = \{x \in E \mid \langle x, x^* \rangle = 0 \, \forall x^* \in B\} = \bigcap_{x^* \in B} \operatorname{Ker}\langle \cdot, x^* \rangle$$

is $\sigma(E, F)$ is an intersection of closed sets in $\sigma(E, F)$.

Let next M be closed in some compatible topology. M is weakly closed, since a subspace is convex and convex closed sets are the same in any compatible topology. For a subspace, $M^{\perp} = M^{\circ}$ (!)is a subspace , so $M^{\perp\perp} = M^{\circ\circ} = \overline{M} = M$, where the closure is in the weak topology,

7.5. *Prove that*

a) $\mathfrak{S}-$topology is locally convex and is given by the gaugesof the polars of the $A \in \mathfrak{S}$, $p_A(y) = \sup_{x \in A} |\langle x, y \rangle|$.

Polara A° are always balansoituja and konvekseja, and since $\mathfrak{S}-$j-sets are weakly biunded, the polars also are absorbing. So their gauges are seminorms and define the topology.

b) if $\mathfrak{S}$ satisfies the conditions

(1) *$A, B \in \mathfrak{S} \implies \exists C \in \mathfrak{S}$ such that $A \cup B \subset C$ and*
(2) *$A \in \mathfrak{S}$, $\lambda \in \mathbb{K} \implies \exists B \in \mathfrak{S}$ such that $\lambda A \subset B$,*

then $\{A^{\circ} \mid A \in \mathfrak{S}\}$ is a basis of neighbourhoods of the origin in the $\mathfrak{S}-$topologiy.

Solution: Let $\epsilon \bigcap_I A_i^{\circ}$ belong to a basis of neighbourhoods of the origin in the $\mathfrak{S}-$topologiy

$$\mathcal{U}_{\mathfrak{S}} = \Big\{\epsilon \bigcap_I A_i^{\circ} \mid A_i \in \mathfrak{S},\ \epsilon > 0 \ I \text{ finite}\Big\}.$$

If for all $A, B \in \mathfrak{S}$ there exists a $C \in \mathfrak{S}$ such that $A \cup B \subset C$, then (induktio!) for a finite family $\{A_i \mid i \in I\} \subset \mathfrak{S}$ exists $C \in \mathfrak{S}$ such that $\bigcup_{i \in I} A_i \subset C$. If also for all $C \in \mathfrak{S}$ and $\lambda \in \mathbb{K}$ exists $B \in \mathfrak{S}$ such that $\lambda C \subset B$, then choose $B \in \mathfrak{S}$ such that $\frac{1}{\epsilon} C \subset B$. Now

$$B^{\circ} \subset \epsilon C^{\circ} \subset \epsilon \bigcap_{i \in I} A_i^{\circ}.$$

c) If $\mathfrak{S}$ satisfies

$$\bigcup_{\mathfrak{S}} A = E,$$

then $\mathfrak{S}$-topologia is finer than weak topology $\sigma(F, E)$.

Solution: Prove that for every point $x \in E$ the polar polaari $\{x\}^{\circ} = \{y \in F \mid |\langle x, y \rangle| \leq 1\}$ is $\mathfrak{S}$-open , since in the weak topology a basis of the neighbourhoods of the origin are finite intersections of these polars. . Let $x \in E$. By assumption there exists $A \in \mathfrak{S}$, s th. $x \in A$. Now $A^{\circ} \subset \{x\}^{\circ}$. □

7.6. *Prove directly (No Alaoglun and Bourbakin), that anequicontinuous set $A \subset E^*$ is weakly bounded .*

Solution: Let $A \subset E^*$ be equicontinuous. $A \subset E^*$ is equicontinuous exactly when $A \subset U^\circ$ for some $U \in \mathcal{U}_{\sigma(E,E^*)}$.

Let V be aweak environment.There exists a finite set $S \subset E$ such that $S^\circ \subset V$. since finite sets are bounded, exists $\lambda > 0$ such that $S \subset \lambda U$, and therefore

$$A \subset U^\circ \subset \lambda S^\circ \subset \lambda V.$$

qed

7.7. *Let E non-complete locally convex Hausdorff-space and $\hat{E}$ its completion. Prove that the topology $\sigma(E', \hat{E})$ is strictly finer than $\sigma(E', E)$ andsimilarly for E^*.*

Solution: The weak topologia is compatible , so the dual of E' : is E in one topology and $\hat{E}$ in the other. The topologies must be different!

7.8. *Let E be a Banach space. Prove that $b(E, E') = \tau(E, E')$.*

Mackeyn and Arensin corollary "Lause 7.46" space is barreled if and only if $b(E, E') = \tau(E, E')$. Every Banach-space is barreled. $\square$.

Solution:

7.9. **Is Schwartzin testifunktiospace $D(\mathbb{R})$ normable? How about the spaces $D(K)$?*

Solution: Done before. $D(K)$ was called $\mathcal{C}^\infty(K)$. No normed space! ($\mathcal{C}n(K)$ is normispace, for $n < \infty$). $D(\mathbb{R})$ is not even metr.