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Exercise help set 2 **Topological Vector Spaces**

1.1. *Is the image $f(A)$ of a balanced set A in a continuous linear mapping f always balanced? How about the image of absorbing, convex, closed or compact sets?*

Easy. The image $f(A)$ of a balanced set A in any linear mapping f is always balanced. The image of an absorbing set is not absorbing, unless f is surjective!. Convex goes to convex by linearity, and compact to compact by continuity of f. Closed sets need not go to closed – think of the identical mapping between the same space with 2 norms giving different topologies.

1.2. *In a continuous linear mapping between topological vector spaces, is the pre-image $f^{-1}A$ of a balanced set A always balanced? How about the pre-image of absorbing, convex, closed or compact sets?*

Bal yes, absorbing yes, convex yes, closed yes, compact no. All easy.

1.3. *a) Constract an example of a convex set whose balanced hull is not convex.*

b) Prove that a convex set $A \subset E$ on is balanced, if $\lambda A \subset A$ for all $\lambda \in \mathbf{K}$, with $|\lambda| = 1$.

a) An interval in $\mathbf{R}^2$ not containing the origin.

b) Yes. Let $|\mu| \le 1$ and$x \in A$. We prove that $\mu x \in A$. By assumption $-x \in A$, so by convexity $0 = \frac{1}{2}x + \frac{1}{2}(-x) \in A$. So the claim is true for $\mu = 0$. By convexity $\mu x = (1 - |\mu|) \cdot 0 + |\mu| x \in A$. By assumption $\mu x = \frac{\mu}{||\mu|}|\mu| x \in A$, kun $\mu \neq 0$.

1.4. *Prove that the balanced hull of a compact set is compact.*

$\operatorname{bal} A = \bigcup_{\lambda \le 1} \lambda A = f(\{\lambda \in \mathbf{K} \mid |\lambda| \le 1\} \times A)$, in the continuuous image of th eproduct of 2 compact sets, hence compact.

1.5. *Construct an example of a closed $A \subset \mathbf{R}^2$, whose convex hull is not closed. The graph of $\frac{1}{1+x^2}$ in $\mathbf{R}^2$.*

1.6. *Prove that the supremum $p(x) = \sup_{i \in I} p_i(x)$ of a family seminorms $(p_i)_{i \in I}$ on a vector space E is a seminorm, if $p(x) < \infty$ for all $x \in E$.*

$$p(x+y) = \sup_{i \in I} p_i(x+y) \le \sup_{i \in I}(p_i(x) + p_i(y)) \le \sup_{i \in I} p_i(x) + \sup_{i \in I} p_i(y) = p(x) + p(y)$$

$$p(\lambda x) = \sup_{i \in I} p_i(\lambda x) = \sup_{i \in I} |\lambda|(p_i(x) = |\lambda| \sup_{i \in I}(p_i(x) = |\lambda| p(x)$$

1.7. A basis of continuous seminorms $\mathcal{N}$ *in a locally convex space E is a set of continuous seminorms $\mathcal{N}$ such that for every continuous seminorm p there exists a basis seminorm $q \in \mathcal{N}$ and a number $\lambda > 0$ such that $p \le \lambda q$.*

Prove that in a locally convex space $(E, \mathcal{T})$ every basis of continuous seminorms $\mathcal{N}$ defines the same locally convex topology as $\mathcal{T}$.

Let $\mathcal{J}$ be a basis of continuous seminorms in $(E, \mathcal{N})$. Denote the resp nbhd filters of the origin by $\mathcal{J}$ and $\mathcal{N}$. We will prove $\mathcal{J} = \mathcal{N}$.

Assume first that i $U \in \mathcal{U}_{\mathcal{J}}$. Without loss of generality U is a $\mathcal{J}-$ basis set, so there exists a number $\varepsilon > 0$ and continuous seminormst $p_1, \dots, p_n \in \mathcal{J}$ s.th.

$$U = \varepsilon \bigcap_{i=1}^{n} B_{p_i}$$

Because the seminorms p_i are continuous in $(E, \mathcal{N})$ by assumption there are for each i seminorms $q_{i,j} \in \mathcal{N}$ and numbers $\lambda_i > 0$ $(j = 1, \dots, n_i)$ s.th.

$$p_i \leq \lambda_i (q_{i,1} + \dots + q_{i,n_i}),$$

so for suitable $\varepsilon' > 0$ we have

$$U \supset \varepsilon' \bigcap_{i=1}^{n} \bigcap_{j=1}^{n_j} B_{q_{i,j}},$$

which is a $\mathcal{N}$-basis set. Therefore $U \in \mathcal{U}_{\mathcal{N}}$, so we have proved $\mathcal{U}_{\mathcal{J}} \subset \mathcal{U}_{\mathcal{N}}$.

Assume next that U is a neighbourhood of the origin in the topology of $\mathcal{N}$, so $U \in \mathcal{U}_{\mathcal{N}}$. Without loss of generality U on kantajoukko so there exists a number $\varepsilon > 0$ and seminorms $q_1, \dots, q_n \in \mathcal{J}$ s.th.

$$U = \varepsilon \bigcap_{i=1}^{n} B_{q_i}$$

Since, by assumption the seminorms p_i form a basis of cont seminorms in $(E, \mathcal{N})$ there exist for each i a seminorm $p_i \in \mathcal{J}$ and la number $\lambda_i > 0$ s.th $q_i \leq \lambda_i p_i$ so $B_{q_i} \supset B_{\lambda_i p_i} = \frac{1}{\lambda_i} B_{p_i}$ and therefore for a suitable $\varepsilon' > 0$ we have

$$U \supset \varepsilon' \bigcap_{i=1}^{n} B_{p_i},$$

which is a $\mathcal{J}$-basis set. So $U \in \mathcal{U}_{\mathcal{J}}$, and we have proven $\mathcal{U}_{\mathcal{N}} \subset \mathcal{U}_{\mathcal{J}}$.

REMARK. By similar arguments: The following are equivalent:

(1) $\mathcal{J}$ is a basis of continuous seminorms
(2) $\{\varepsilon U_p \mid p \in \mathcal{J}, \varepsilon > 0\}$ is a neighbourhood basis of the origin.

1.8. *Prove that if there exists a continuous norm in a locally convex space, then there exists a basis of continuous seminorms consisting of norms. Is E a normed space?*

1. method: Let n be a continuous norm. in lc space $(E, \mathcal{P})$. its closed 1-pallo $B = B_n$ is the preimage of a closed interval, hence closed. It also is a neighbourhood of the origin. In fact, by exercise 2.1. it is a barrel (closed, absorbing, convex, balanced) Itse asiassa se on tehtävän 1 nojalla tynnyri. There is a neighbourhood basis $\mathcal{U}$ of the origin consisiting of barrels in E. Intersect all members of $\mathcal{U}$ with the ball B and you get a new neighbourhood basis the origin consisiting of smaller barrels. Their gauges are seminorms, but larger than the given norm — therefore they are norms. They obviously form a basis of cont seminorms.

2. method: take abasis of cont seminorms $\mathcal{J}$. and define $\{\max\{n, p\} \mid p \in \mathcal{J}\}$. This works (much like in method 1)

1.9. *Let E be a vector space and $M \subset E$ a subspace, p a seminorm in M and q a seminorm in the whole space a E such that $p \leq q\big|_M$ meaning $p(x) \leq q(x) \forall x \in M$. Prove that there exists a seminorm $\bar{p}$, defined in the whole space E, such that $p = \bar{p}\big|_M$ and $p \leq q$. (Don't try to use the axiom of choice (yet)!)*

Let $U = \operatorname{co}(U_p \cup U_q)$. DRAW A PICTURE! Easily, U is abs, bal, convex, so its gauge $\bar{p}$ is a seminorm. Obviously $U \supset U_q \implies \bar{p} \leq q$ and finally $\bar{p}(x) = p(x)$ for all $x \in M$, since **in the subspace** M we have $x \in U \iff x \in U_p$.

1.10. *Let $(E, \mathcal{P})$ and $(F, \mathcal{Q})$ be locally convex spaces with bases of continuous seminorms $\mathcal{P}$ and $\mathcal{Q}$. Prove that a linear mapping $T : E \to F$ is continuous if and only if for every $q \in \mathcal{Q}$ there exist $p \in \mathcal{P}$ and $\lambda > 0$ such that $q(Tx) \leq \lambda p(x)$ for all $x \in E$.*

Let the condition hold. To prove that T is continuous it is sufficient to prove that every $q \circ T$ on is continuous, where $q \in \mathcal{Q}$ on kantaseminormi. This means that for all $q \in \mathcal{Q}$, $\varepsilon > 0$ there exists $U \in \mathcal{U}_E$ such that $|q \circ T| \leq \varepsilon$ in U.

Consider a seminorm p like in the assumption, satisfying $q(Tx) \leq \lambda p(x)$ for all $x \in E$. Now a suitable μU_p can be chosen for Ui. To verify this, you must calculate a little.

The inverse implication is similar.