JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Exercise help set 1 **Topological Vector Spaces**

1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be filter bases in a set E.*
a) Is $\{A \cup B \mid A \in \mathcal{A},\, B \in \mathcal{B}\}$ a filter basis in the set E?
No. It can contain $\emptyset$ as an element.
The filter basis axioms are ovat

$$\emptyset \notin \mathcal{A} \quad \text{ja } \mathcal{A} \neq \emptyset \tag{1}$$

$$A, A' \in \mathcal{A} \quad \Longrightarrow \quad \exists A'' \in \mathcal{A} : A'' \subset A \cap A' \tag{2}$$

the first is easy, the second is not difficult:

$$(A \cup B) \cap (A' \cup B') \supset (A \cap A') \cup (B \cap B') \supset A'' \cup B''.$$

b) Is $\{A \cap B \mid A \in \mathcal{A},\, B \in \mathcal{B}\}$ a filter basis in the set E?
No. It can contain $\emptyset$ as an element.

Notation. .
Unless otherwise stated, E is a tvs, $\mathcal{F}(0)$ the neighbourhood filter of its origin.
$\mathbf{N} = \{0, 1, 2, \dots\}$. $\mathbf{N}^* = \{1, 2, \dots\}$

1.2. *Prove: E is connected. (PS: How about general topological groups?)*
Even pathwise connected!. Let $x, y \in E$. The mapping $\gamma : [0.1] \to E : t \mapsto ty + (1-t)x$ is a continuoius path from $\gamma(0) = x$ to $\gamma(1) = y$. *(Added question : Is a topological group always connected.* No, Counterexample: Discrete finite grpous.

1.3. *Prove that a) $\bigcap \mathcal{F}(0) = \overline{\{0\}}$ and b) that this is a vector subspace.*

$$\begin{aligned}
x \in \overline{\{0\}} &\iff 0 \in U \quad \forall U \in U_x \\
&\iff 0 \in x + V \quad \forall V \in U_o \\
&\iff x \in -V \quad \forall V \in U_o \\
&\iff x \in V \quad \forall V \in U_o \text{ (homotetiainvarianssi)} \\
&\iff x \in \bigcap \mathcal{F}(0)
\end{aligned}$$

b) follows drom the next exercise. there is also a direct proof:

1°) $0 \in \overline{\{0\}}$.

2°) $x \in \overline{\{0\}} = \bigcap \mathcal{F}(0) \iff x \in V \quad \forall V \in U_o$

$$\begin{aligned}
&\iff \lambda x \in \lambda V \quad \forall V \in U_o, \lambda \neq 0 \\
&\iff \lambda x \in U \quad \forall U \in U_o, \lambda \neq 0 \text{ (homotety invariance)} \\
&\iff \lambda x \in \overline{\{0\}}.
\end{aligned}$$

3°) Let $x, y \in \overline{\{0\}}$, t.s. $x, y \in V \quad \forall V \in U_o$. We will prove that $x, y \in \overline{\{0\}}$. Take $U \in U_o$. Choose $V \in U_o$ s.th. $V + V \subset U$.Now $x + y \in V + V \subset U$. □

1.4. *Prove that the closure $\overline{F}$ of a vector subspace $F \subset E$ is a vector subspace.*

At least $0 \subset F \subset \bar{F}$. Remeber from topology that a mapping f is continuous iff $f(\bar{A}) \subset \overline{f(A)}$ fo all A, and that in the product topology: $\overline{A \times B} = \bar{A} \times \bar{B}$. By continuity of multiplication , every $\lambda\cdot : E \to E : x \mapsto \lambda x$ is continuous, and — because $\lambda \cdot F \subset F$, — also $\lambda \cdot \bar{F} \subset \overline{\lambda \cdot F} \subset \overline{F}$. because addition $+ : E \times E \to E$ is continuous, we have $\bar{F} + \bar{F} = +(\bar{F} \times \bar{F}) = +(\overline{F \times F}) \subset \overline{+(F \times F)} \subset \overline{F}$. $\square$

1.5. *Is the balanced hull* $\operatorname{bal} A = \{\lambda x \mid \lambda \in \mathbf{K}\}$ *of any open set $A \subset E$ open? (Hint. no, but if . . .)*

No. Counterexample . In the normaed space $\mathbf{R}^2$ the bal hull of $]0, 1[\times]0, 1[$ contains 0.

But if 0 already is contained in the open set A, then $\operatorname{bal} A = \{\lambda x \mid \lambda \in \mathbf{K}\}$ is open, since in that case $\operatorname{bal} A = \bigcup_{|\lambda|\le 1} \lambda A = \bigcup_{0\ne|\lambda|\le 1} \lambda A$ and for $\lambda \ne 0$ every λA is open .

1.6. *Consider $E = \mathcal{C}(\mathbf{R}, \mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} \mid f$ is continuous$\}$. Denote $V_m = \{f \in E \mid f(t)| \le m(t)\, \forall t \in \mathbf{R}\}$, where $m \in E$ and $m(t) > 0\, \forall t \in \mathbf{R}$.*

Prove the existence of a topology $\mathcal{T}$ in E such that addition is continuous (so E is a topological abelian group) and $\mathcal{F} = \{V_m \mid m \in E$ and $m(t) > 0\, \forall t \in \mathbf{R}\}$ is a neighbourhood basis of the origin. Is $(E, \mathcal{T})$ a tvs? Is the subspace

$$D = \{f \in E \mid \operatorname{supp} f \text{ is compact } \} \subset E$$

a tvs? (Does it have a countable neighbourhood basis of the origin? Why do I ask?)

There is only one choice for topology — the one given by neighbouhood fileters $U_f = \{f + \mathcal{V}_m \mid m(t) > 0\, \forall t \in \mathbf{R}\}$.

At least $\mathcal{F}$ satisfies the filter basis axioms. (For the intersection,property, choose $m'' = \min(m, m')$.) and every V_m contains the origin. so a translation invariant topology eists.

Cont of sum:

$$(x + \tfrac{1}{2}V_m) + (y + \tfrac{1}{2}V_m) \subset (x + y) + V_m.$$

Counterexample proving discontinuity of product: $f(t) = e^t$. At $f \in E$ the partial mapping $\lambda \mapsto \lambda f$ is not continuous $\mathbf{R} \to E$, sillä , since for $m(t) = 1\forall t$ we have $|\lambda f(t) - f(t) = (|\lambda - 1|)e^t$ which is outside V_m unless λ ole 1. so neighbourhoods are not absorbing.

b) $D = \{f \in E \mid \operatorname{supp} f \text{ is c}\} \subset E$ with the subspace topology is a tvs, since $f + V_m \in \mathcal{F}_f,\ g \in D \cap (f + V_n)$ for some $n(t) \ge 0 \forall t \in \mathbf{R}$ ja $|\lambda - 1| \le \epsilon$ and $t \in \mathbf{R}$.

1) $\quad t \notin \operatorname{supp} f \implies |\lambda g(t) - f(t)| = |\lambda g(t) - 0| = |\lambda||g(t)| \le |1 + \epsilon| n(t) < m(t)$, for the choice $n(t) = \frac{1}{1+\epsilon} m(t)$

$$\begin{aligned} 2) \quad t \in \operatorname{supp} f \implies |\lambda g(t) - f(t)| &= |\lambda g(t) - \lambda f(t) + \lambda f(t) - f(t)| \le \\ &\le |\lambda| g(t) - f(t)| + (\lambda - 1)|f(t)| \\ &\le (1 + \epsilon) m(t) + \epsilon \|f\|_\infty < m(t), \end{aligned}$$

for the choice $n(t) = \frac{1}{3} m(t)$ and $\epsilon = \min\{1, \frac{1}{3\|f\|_\infty} \inf_{t\in \operatorname{supp} f} n(t)\}$.

The space is not metrizable — in fact nbot even = has a denumerable neighbourhood basis: If it would hjave a neighbourhood basis like $\{V_{m_k} \mid k \in \mathbf{N}\}$,then we woud choose $m \in \mathcal{C}(\mathbf{R}, \mathbf{R})$ s.th.

$$m(t) > 0 \forall t$$

and

$$m(k) < m_k(k) \forall k \in \mathbf{N}.$$

Now there woud not exist any V_{m_k}, s. that $V_{m_k} \subset V_m$.

1.7. *Let $U \subset E$ be convex, balanced and absorbing. Prove that $\{\frac{1}{n}U \mid n \in \mathbf{N}^*\}$ is a neighbourhood basis of the origin in some tvs-topology. (Do we need all 3 assumptions?)*

Jus ta solution sketch:

$\mathcal{K} = \{\frac{1}{n}U \mid n \in \mathbf{N}*\}$ *is a nbhd basis in some tr inv topology, since it satisfies the properties in theorem.......... (Check this)*

a) $V \in \mathcal{K} \& \lambda \neq 0 \implies \exists U \in \mathcal{K} : \lambda U \subset V$ *(?)*

b) $V \in \mathcal{K} \implies$ V *absorb.*

(Really: U *abs* $\implies \frac{1}{n}U$ *abs* $\implies V$ *abs, for* $\frac{1}{n}U \subset V$*.)*

c) $U' \in \mathcal{K} \implies \exists n, U : \frac{1}{n}U \subset U' \implies \frac{1}{2n}U + \frac{1}{2n}U \subset \operatorname{co}(\frac{1}{n}U) = \frac{1}{n}\operatorname{co} U = \frac{1}{n}U \subset U'$.

d) $U \in \mathcal{K} \implies \exists$ *bal* $V \in \mathcal{K}$ *s. th.* $V \subset U$*. OK.*

1.8. A linear mapping $L : E \to F$ between 2 topological vector spaces $(E, \mathcal{T}_E)$ ja $(F, \mathcal{T}_F)$ is continuous at any point $a \in E$ iff it is continuous at the origin. Prove that in this case L is uniformly continuous in the following sense

$$\forall A \in \mathcal{U}_{0,F} \quad \exists B \in \mathcal{U}_{0,E} : \quad (x - y) \in B \implies (Tx - Ty) \in A.$$

TODISTUS. : easy!