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Exercise set 3 **Topological Vector Spaces**
Tuesday Oct.12.2010 2.30-4.00 PM MaD-355

Unless otherwise stated, E is a tvs.

3.1. Every compact set $K \subset \mathbf{R}^n$ defines a seminorm $p_K(f) = \sup f(K)$ $(= \max f(K))$ in the space $E = \mathcal{C}(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{R} \mid f \text{ is continuous}\}$. These seminorms give rise to a locally convex topology $\mathcal{T}$.

a) Is $\mathcal{T}$ a Hausdorff-topology?

b) Is the sequence $f_n(x) = \frac{1}{n}e^x$ convergent in the topology $\mathcal{T}$?

c) Does there exist in E a normi, giving the topology $\mathcal{T}$, eli onko E normeerautuva? (Is $\mathcal{T}$ normable?) (answer : no. Why?)

3.2. The seminorms $p_n(f) = \sup_{0 \le t \le 1} |f^{(n)}(t)|$ $(n = 0, 1, 2, \dots)$ define a cocally convex topology $\mathcal{T}$ in the space $E = \mathcal{C}^\infty([0,1]) = \{f : [0,1] \to \mathbf{R} \mid f \text{ is infintely many times derivable}\}$. For $f \in E$, denote

$$Tf(x) = \int_0^x f(t)\,dt.$$

T so is a linear mapping (also called an operaator or eli transformation) $E \to E$.

a) Is T continuous?

b) Is the topology $\mathcal{T}$ normable?

3.3. Let E be a real locally convex space and $A \subset E$ concex. Prove that A is closed if and only if A is the intersection of some closed half spaces in E.

3.4. Let E be a normed space. Prove that the norm $x \mapsto \|x\|$ is discontinuous E in the weak topology of E (weakly continuous). (The weak topology is defined by the seminorms $x \mapsto |\langle x, x*\rangle|$, where $x* \in E^* =$\{continuous lin forms\}.)

Is it lower semicontinuous? For this it is sufficient that it is the pointwise supremum of a family of continuopuis mappings.

3.5. Let (E, P) be a locally convex space. Prove that teh sequence $(x_n)_{\mathbf{N}}$ in E is a Cauchy-sequence if and only if

$$\forall p \in P \text{ and } \forall \epsilon > 0 \; \exists n_0 \in \mathbf{N} \text{ s.e. } q, r \ge n_0 \implies p(x_q - x_r) \le \epsilon.$$

3.6. Let $E = \prod_{i \in I} E_i$ be the product of topological vector spaces (product topology!) and $\pi_i : E \to E_i$ a standard projection $(i \in I)$. Prove that $\mathcal{F}$ is a Cauchy filter in E if and only if every $\pi(\mathcal{F}) \subset E_i$ is a Cauchy filter in E_i. (Take $I = \{1, 2\}$ if you want an easy case)

3.7. (continue) Prove that $E = \prod_{i \in I} E_i$ is complete if and only if each E_i is complete.

3.8. Let

$$E = \{f \in \mathcal{C}[0,1] \bigm| \exists \epsilon_f > 0 \text{ such that } f(t) = 0 \; \forall\, 0 \le t \le \epsilon(f)\}$$

withthe norm $\|f\| = \sup |f|$. Is

$$T = f \in E \bigm| |f(\tfrac{1}{n}) \le \tfrac{1}{n} \,\forall\, n \in \mathbf{N}^*\}$$

a barrel? Is it a neighbourhood of the origin? (Why do I ask?)

3.9. An example of a subset of a locally convex space which is sequentially complete but not complete: $E = \mathcal{F}([0,1], \mathbf{R}) = \mathbf{R}^{[0,1]} = \{$allo functions $[0,1] \to \mathbf{R}\}$. Topology of pointwise convergence ie seminorms $p_x = |f(x)|$. $M = \{f \in E \bigm| f(x) \neq 0$ for at most countably many $x \in [0,1]\}$.

3.10. If You like to do more. Let $K \subset \mathbf{R}^n$ be compact. In the space

$$E = \mathcal{C}_c^\infty(K) = \{f : \mathbf{R}^n \to \mathbf{R} \bigm| f \in \mathcal{C}^\infty, \; \operatorname{supp} f \subset K\}$$

use the semonorms

$$q_\alpha(f) = \sup_{x \in K} \left| \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right|,$$

where $\left(\frac{\partial}{\partial x}\right)^\alpha f(x)$ is the (higher) partial derivative corresponding to the multi-index $\alpha \in \mathbf{N}^n$ (You can take $\mathbf{R}^1$ and usual lhigher derivatives - it makes no difference) . Write $\mathcal{Q} = \{q_\alpha \bigm| \alpha \in \mathbf{N}^n\}$. Prove that a $(E, \mathcal{Q})$ is Fréchet space. (loc-con, metr, compl)