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Exercise set 2 **Topological Vector Spaces**
tuesday 10.5.2010 probably alraedy 14-16 MaD-355 or somewhere else ??

Unless otherwise stated, E is a tvs.

2.1. A. nswer some of the following questions:
Is the image $f(A)$ of a balanced set A in a continuous linear mapping f always balanced? How about the image of absorbing, convex, closed or compact sets?

2.2. Answer some of the following questions:
In a continuous linear mapping between topological vector spaces, is the pre-image $f^{-1}A$ of a balanced set A always balanced? How about the pre-image of absorbing, convex, closed or compact sets?

2.3. a) Constract an example of a convex set whose balanced hull is not convex.
b) Prove that a convex set $A \subset E$ on is balanced, if $\lambda A \subset A$ for all $\lambda \in \mathbf{K}$, with $|\lambda| = 1$.

2.4. Prove that the balanced hull of a compact set is compact.

2.5. Construct an example of a closed $A \subset \mathbf{R}^2$, whose convex hull is not closed.

2.6. Prove that the supremum $p(x) = \sup_{i \in I} p_i(x)$ of a family seminorms $(p_i)_{i \in I}$ on a vector space E is a seminorm, if $p(x) < \infty$ for all $x \in E$.

2.7. *A basis of continuous seminorms* $\mathcal{N}$ in a locally convex space E is a set of continuous seminorms $\mathcal{N}$ such that for every continuous seminorm p there exists a basis seminorm $q \in \mathcal{N}$ and a number $\lambda > 0$ such that $p \leq \lambda q$.
Prove that in a locally convex space $(E, \mathcal{T})$ every basis of continuous seminorms $\mathcal{N}$ defines the same locally convex topology as $\mathcal{T}$.

2.8. Prove that if there exists a continuous norm in a locally convex space, then there exists a basis of continuous seminorms consisting of norms. Is E a normed space?

2.9. Let E be a vector space and $M \subset E$ a subspace, p a seminorm in M and q a seminorm in the whole space a E such that $p \leq q\big|_M$ meaning $p(x) \leq q(x) \forall x \in M$. Prove that there exists a seminorm $\bar{p}$, defined in the whole space E, such that $p = \bar{p}\big|_M$ and $p \leq q$. (Don't try to use the axiom of choice (yet)!)

2.10. Let $(E, \mathcal{P})$ and $(F, \mathcal{Q})$ be locally convex spaces with bases of continuous seminorms $\mathcal{P}$ and $\mathcal{Q}$. Prove that a linear mapping $T : E \to F$ is continuous if and only if for every $q \in \mathcal{Q}$ there exist $p \in \mathcal{P}$ and $\lambda > 0$ such that $q(Tx) \leq \lambda p(x)$ for all $x \in E$.