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Part I: Topological vector spaces

1. General topological vector spaces

1.1. Vector space topologies.

Definition 1.1. $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ with its standard topology. *Vector spaces* have coefficdients in $\mathbb{K}$.

A topological vector space is a vector space E with a topology $\mathcal{T}$, such that

$$+ : E \times E \to E \quad \text{and} \tag{1.1}$$

$$\cdot : \mathbb{K} \times E \to E \tag{1.2}$$

are continuous. This kind of topology is E is called a *vector space topology.*

Example 1.2. Examples:
- normed spaces,
- Lebesgue spaces (0<p<1 included!) [1]
- weak topology

Remark 1.3. In a topological vector space every *translation map* $E \to E : x \mapsto x + a$ is a homeomorphism, so $\mathcal{T}$ is *translation invariant* for $a \in E$, we have

$$A \in \mathcal{T} \text{ if and only if } A + a \in \mathcal{T}.$$

Similarly, $\mathcal{T}$ is also *homothety invariant*: for $\lambda \in \mathbb{K} \setminus \{0\}$, we have

$$A \in \mathcal{T} \text{ if and only if } \lambda A \in \mathcal{T}.$$

Corollary of transla tion invariance:

Corollary 1.4. *Between topological vector spaces $(E, \mathcal{T}_E)$ and $(F, \mathcal{T}_F)$ a linear map $L : E \to F$ is continuous at any $a \in E$ if and only if it is continuous at the origin. Furthermore L i such that en* uniformly continuous *in the following sense: For each open neighbourhood A of the origin of F there exists an open neighbourhood B of the origin of E such that*

$$(x - y) \in B \implies (Lx - Ly) \in A.$$

Proof. If L is continuous at 0, then

$$x \mapsto x - a \mapsto L(x - a) \mapsto L(x - a) + La = Lx$$

— same as L itself — is continuous at a. Similarly prove the converse; if L is continuous at a then it is also continuous at 0. "Uniformly continuous" is exercise. □

1.2. Neighbourhoods and filters.

Definition 1.5. Let $(X, \mathcal{T})$ be a topological space and $x \in X$.

(1) The set $U \subset X$ is a neighbourhood of the point x if x is an *interior point of* U which means, there exists an open set $A \in \mathcal{T}$, such, that $x \in A \subset U$. In particular every open set containing x is a neighbourhood of x.

[1] Draw unit ball in 2-dimensional $\ell_2^{\frac{1}{2}}$!

(2) The set of all neighbourhoods of a point x is the *neighbourhood filter* of x:

$$\mathcal{U}_x := \{U \subset E \mid U \text{ is a neighbourhood of } x\}.$$

In particular, in a topological vector space the neighbourhood filter of the origin is called $\mathcal{U}_0$.[2]

(3) *A basis* $\mathcal{K}$ *of a topology* $\mathcal{T}$ is a subset $\mathcal{K} \subset \mathcal{T}$ such that every open set $A \in \mathcal{T}$ is a unuion of some basis sets: $A = \bigcup \mathcal{A}$ for some $\mathcal{A} \subset \mathcal{K}$.

Remark 1.6. The classical example of a basis for a topology are the open balls in a metric space. Another example is the product of two topological spaces, where the basic open sets are the finite intersections of Cartesian products of open sets.

$A \subset X$ is a neighbourhood of x if and only if there exists a basis open set $K \subset X$, such that $x \in K \subset A$.

$A \subset X$ is open if and only if it is a neighbourhood of each of its points.

In a topological vector space, by translation invariance: $\forall x \in E$:

$$\mathcal{U}_x = x + \mathcal{U}_0 = \{x + A \mid A \in \mathcal{U}_0\}.$$

Definition 1.7. A *neighbourhood basis* of a point $x \in E$ is a set $\mathcal{K}_x \subset \mathcal{U}_x$ of neighbourhoods such that . every neighbourhood contains one of the basic ones:

$$\forall A \in \mathcal{U}_x \, \exists K \in \mathcal{K}_x : \; K \subset A,$$

in other words: The neighbourhood filter consists of all basis neighbourhoods and their **supsets**.

Example 1.8. In a normed spaxe $(E, \|\cdot\|)$ a neighbourhood basis of x is given by all x-centered balls. The same holds in any metric space.

Definition 1.9. $A \subset E$ is *balanced*, if and only if

$$\alpha A \subset A \qquad \forall |\alpha| \leq 1.$$

The *balanced hull* of $A \subset E$ is the smallest balanced set containing A. It exists, since the intersection of arbitrarily (even ∞) many balanced sets is balanced.

$$\operatorname{bal}(A) = \bigcap \{B \mid B \text{ is balanced and } B \supset A\}.$$

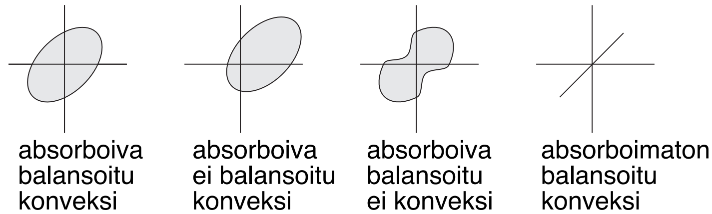
Homework problem: Is $\operatorname{bal}(A) = \bigcup_{|\alpha| \leq 1} \alpha A \quad ?$

Definition 1.10. $A \subset E$ is *absorbing* if for all $x \in E$ there exists a number $\lambda > 0$, such that

$$\lambda < |\alpha| \implies x \in \alpha A.$$

This is equivalent to the claim that every line through the origin $\{\lambda x \mid \lambda \in \mathbb{K}\}$ $(x \in E \setminus \{0\})$ contains an origin-centered segment in A, ie. $\{\alpha x \mid |\alpha| < \epsilon\}$, where $\epsilon > 0$. The definition is equivalent to that every origon kautta kulkeva suora $\{\lambda x \mid \lambda \in \mathbb{K}\}$ $(x \in E \setminus \{0\})$ sisältää jonkin 0-keskisen välin $\{\alpha x \mid |\alpha| < \epsilon\}$, where $\epsilon > 0$. (If $\mathbb{K} = \mathbb{C}$, the "segment" looks more like a "disk"'.)

[2] In some texts $\mathcal{F}(0)$.



Kuva 1. Geometry in a vector space

A balanced set $B \subset E$ is absorbing if and only if

$$E = \bigcup_{\alpha > 0} \alpha B$$

Definition 1.11. A subset $A \subset E$ is *convex*, if and only if it contains all segments between its points:

$$x \in A, y \in A, 0 < \alpha < 1 \implies \alpha x + (1-\alpha) y \in A.$$

The *convex hull* of a set $A \subset E$ is the smallest convex set containing A.

It exists, since the intersection of arbitrarily (even ∞) many convex sets is convex. So

$$\mathrm{co}(A) = \bigcap \{B \mid B \text{ is convex and } B \supset A\}.$$

Theorem 1.12. *The neighbourhood filter of the origin has the following properties:*

$$A \in U_0 \implies A \textit{ is absorbing.} \tag{1.3}$$

$$\forall A \in \mathcal{U}_0 \quad \exists B \in \mathcal{U}_0 : B + B \subset A. \tag{1.4}$$

$$\forall A \in \mathcal{U}_0 \quad \exists B \in \mathcal{U}_0 : B \textit{ is balanced and closed and } B \subset A. \tag{1.5}$$

Proof. (1.3): Let $x \in E$. By continuity if the product and $0x = 0$, there exists an interval

$$B = B_{\mathbb{K}}(0, \varepsilon) = \{\alpha \in \mathbb{K} \mid |\alpha| < \varepsilon\}$$

around the origin $0 \in \mathbb{K}$ such, that

$$Bx \subset A.$$

(1.4): Addition $+ : E \times E \to E$ is continuous and $+(0,0) = 0 + 0 = 0$. Therefore there exists a standard basis neighbourhood of $(0,0)$, like $W = C \times D$ such, that

$$C + D = +(C \times D) \subset A$$

Choose $B = C \cap D$.

(1.5): It is sufficient to prove that

(i) $\forall A \in \mathcal{U}_0 \quad \exists$ balanced $B \in \mathcal{U}_0 : B \subset A$,
(ii) $\forall A \in \mathcal{U}_0 \quad \exists$ closed $S \in \mathcal{U}_0 : S \subset A$,
(iii) A balanced $\implies \overline{A}$ balanced.

All these are true:

(i) The product map $\cdot : \mathbb{K} \times E \to E$ is continuous and $\cdot(0_{\mathbb{K}}, 0_E) = 0$. Therefore in the product topology there exists a neighbourhood of the origin $(0_{\mathbb{K}}, 0_E)$ like $W = C \times D$ such, that

$$CD = \cdot(C \times D) \subset A$$

here C can be chosen to be

$$C = B_{\mathbb{K}}(0, \varepsilon).$$

Now we can choose $B = \bigcup_{|\alpha| \le \varepsilon} \varepsilon D$. It works!.

(ii) We have proved that we can assume A is balanced. Also, we already know there exists a balanced neighbourhood of the origin B such, that $B + B \subset A$. Now $S := \overline{B} \subset A$, since

$$\begin{aligned} x \in S &\implies \emptyset \neq B \cap (x + B) \\ &\implies \exists x, y \in B : z = x + y \\ &\implies \exists x, y \in B : x = z - y \in B - B \subset B + B \subset A. \end{aligned}$$

(iii) Let $x \in \overline{A}$ and $0 < |\alpha| < 1$, and $U \in \mathcal{U}_{\alpha x}$. By omotety invariance $\frac{1}{\alpha} U$ is a neighbourhood of x, so

$$\frac{1}{\alpha} U \cap A \neq \emptyset.$$

Let $y \in \frac{1}{\alpha} U \cap A$. Then

$$\alpha y \in U \cap \alpha A \subset U \cap A,$$

because A is balanced. □

Definition 1.13. Let X be a set.
a) A collection of $\mathcal{F} \subset \mathcal{P}(X)$ is a *filter*, if it satisfies the *filter axioms*:

$$\emptyset \notin \mathcal{F} \quad \text{and } \mathcal{F} \neq \emptyset \tag{1.6}$$

$$A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F} \tag{1.7}$$

$$A \supset B \in \mathcal{F} \implies A \in \mathcal{F} \tag{1.8}$$

b) A subset of a filter $\mathcal{K} \subset \mathcal{F}$ is its *filter basis* , if every set in the filter contains a basis set. The same is called: the filter is *spanned by* the basis.
c) A collection of subsets $\mathcal{K} \subset \mathcal{P}(X)$ is an abstract *filter basis* , if it satisfies the *filter basis -axioms*:

$$\emptyset \notin \mathcal{K} \quad \text{and } \mathcal{K} \neq \emptyset \tag{1.9}$$

$$A, B \in \mathcal{K} \implies \exists C \in \mathcal{K} : C \subset A \cap B \tag{1.10}$$

Evidently, each basis of a filter satisfie such that ese axioms and each abstract filter basis spans a filter consisting of all its supsets.
d) The open sets of a topological space are defined by all neighbourhoods of all points. One can define a topology by giving the neighbourhood filter $\mathcal{U}_x$ for all points $x \in X$. To get a topology, onee needs to have:
Y1) $U \in \mathcal{U}_x \implies x \in U$
Y2) $U \in \mathcal{U}_x \implies \exists\, V \in \mathcal{U}_x$ such, that for all $y \in V$ is $V \in \mathcal{U}_y$.

Theorem 1.14. *If E is a vector space and $\mathcal{F}$ is filter, whose all elements are i) absorbing sets and*
ii) each of them contains a balanced set in $\mathcal{F}$ and
iii) for all $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such, that $B + B \subset A$, and
iv) for all $\alpha \in \mathbb{K} \setminus \{0\}$ and $A \in \mathcal{F}$holds: $\alpha A \in \mathcal{F}$,

then there exists exactly one topology in E such that $(E, \mathcal{T})$ is a topological vector space and

$$\mathcal{F} = \mathcal{U}_0.$$

Proof. If it exists, then every neighbourhood filter must be $\mathcal{U}_x = x + \mathcal{U}_0 x + \mathcal{F}$, so this is the only possible topology. Let us prove that it makes E a topological vector space .

First prove that the $\mathcal{U}_x = x + \mathcal{F}$ form a topology at all. We must verify Y1) and Y2) in 1.13 d) . Since $\mathcal{F}$ consists of absorbing sets, they all contain the origin, so at least $x \in U$ for all $U \in x + \mathcal{F}$, jso Y1) is OK. If $U = x + A \in x + \mathcal{F}$, then by iii) there exists $B \in \mathcal{F}$ such, that $B + B \subset A$. Let us prove that for all $y \in V = x + B$ we have $U \in U_y$. Easy:

$$U = x + A \supset x + B + B = V + B \supset y + B \in \mathcal{U}_y.$$

to select $B \in \mathcal{F}$ such, that $B + B \subset A$, and notice

$$+((x, y) + B \times B) = x + y + B + B \subset (x + y) + A.$$

Continuity of multiplication can be proved similarly by ii), iii) and iv). this You can do as an exercise, since there is a more challenging way to do it, namely without using iv) at all. As a corollary we understand that iv) follows from the other three.

Let $x_0 \in E$, $\lambda_0 \in \mathbb{K}$ and $A \in \mathcal{F}$. Try to find $B \in \mathcal{F}$ and $\epsilon > 0$ such, that $([\lambda - \epsilon, \lambda + \epsilon] \times (x_0 + B)$ is mapped inside $\lambda_0 x_0 + A$ by multiplication.

Choose $n \in \mathbb{N}^*$ such, that $|\lambda_0| < n$. By induction: there exists a balanced $B \in \mathcal{F}$ such, that $B + B + \cdots + B$ $(n + 2$ kpl$)$ $\subset A$. Since B is absorbing, there exists a number $\epsilon \in]0, 1]$ such, that $|\lambda| \leq \epsilon \implies \lambda x_0 \in B$. Since B is balanced and $|\frac{\lambda_0}{n}| \leq 1$, we have

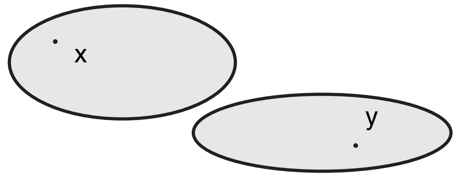
$$x \in B \implies \lambda_0 x = n\frac{\lambda_0}{n} x \in nB \subset B + B + \cdots + B \text{ } (n \text{ kpl}).$$

So $|\lambda| \leq \epsilon$ and $x \in B \implies$

$$\begin{aligned}(\lambda_0 + \lambda)(x_0 + x) &= \lambda_0 x_0 + \lambda_0 x + \lambda x_0 + \lambda x \\ &\in \lambda_0 x_0 + (B + \cdots + B) \text{ (n kpl) } + B + B \subset \lambda_0 x_0 + A. \quad \square\end{aligned}$$

Remark 1.15. Background information

- Filters:
  - $\{B \subset X \mid A \subset b\}$
  - *Fréchet filter* $\{B \subset \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$
  - *Image of filter*: If $\mathcal{F} \subset \mathcal{P}(X)$ is a filter and $\phi : X \to Y$ is a mapping, then $\{\phi(A) \mid A \in \mathcal{F}\}$ is a filter basis in F. It spans what is called the *image* $\phi(\mathcal{F})$ *of* $\mathcal{F}$.
  - *A sequence filter* is the image of a Fréchet filter in a mapping $\mathbb{N} \to X$ (which is a sequence).



KUVA 2. Distinct meighbouthoods

– *An ultrafilter* also called a *maximal filter* is a filter, where you can add no more set without it becoming no filter anymore.

- *Filter convergence*: In a topological space, a filter $\mathcal{F} \subset \mathcal{P}(X)$ *converges to* $x \in X$, if $U_x \subset \mathcal{F}$.
- *Filter basis convergence*: In a topological space, a filter basis $\mathcal{K} \subset \mathcal{P}(X)$ *converges to* $x \in X$, if for all $U \in U_x$ there exists $B \in \mathcal{K}$ such, that $B \subset A$. this is equivalent to the filter spanned by $\mathcal{K}$ converging to x.
- Same concepts! in a topological space, a sequence filter converges if and only if the sequence converges (to the same point x).
- Importance: in a topological space, filters replace sequences for characterizing various objects like we use sequences in metric spaces. Example: apoint x belongs to the closure $\bar{A}$ if and only if there exists a filter basis of sets in A converging to x.

1.3. **Finite dimensional topological vector spaces.**

Definition 1.16. A topological space $(X, \mathcal{T})$ is *Hausdorff*[3] also called T_2, if two distinct points always have disjoint neighbourhoods.

Example: A metric space is always Hausdorff. In any set, the discrete topology is always Hausdorff.

A topological space is Hausdorff if and only if no filter has more than one limit.

A topological vector space is Hausdorff if and only if all 1-point sets are closed.[4]

Theorem 1.17. (Tihonov 1935)[5] *Every n-dimensional topological Hausdorff–vector space $(E, \mathcal{T})$ is llinearly homeomorphic to Euclidean space $\mathbb{K}^n$ kanssa.*

PROOF. Let $(e_1, \ldots, e_n)$ be a basis of the vector space E Every mapping .

$$\mathbb{K} \to E : \lambda \mapsto \lambda e_i$$

is continuous. Therefore

$$\mathbb{K}^2 \to E \times E \to E$$
$$(\lambda_1, \lambda_2) \mapsto (\lambda_1 e_1, \lambda_2 e_2) \mapsto \lambda_1 e_1 + \lambda_2 e_2$$

[3]Felix Hausdorff 1868–1942, Germany.
[4]In a genaral top space this is false.
[5]Tihonov, Andrei Nikolajevitš 1906-1993, Venäjä

is continuous. Notice: the product topology is $\mathbb{K}^n$: is the Euclidean topology. IBy induction:

$$\mathbb{K}^n \to E \times E \to E$$

$$(\lambda_1, \dots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i e_i$$

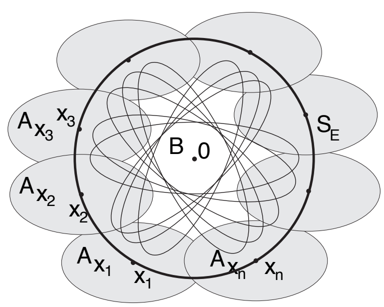
is continuous. Also it is a linear isomorhism, so we can take $E = \mathbb{K}^n$ and $\mathcal{T}$ The claim i such that at $\mathcal{T}$ is equal tot the Euclidean topology $\mathcal{T}_e$. We only have to prove that the identical mapping $(\mathbb{K}^n, \mathcal{T}_e) \to (\mathbb{K}^n, \mathcal{T})$ is a homeomorphism. We know already that it is continuous. Therefore, the Euclidean unit sphere $S = S_E$ is $\mathcal{T}_e$ is compact not only in the Euclicean topology but also in $\mathcal{T}$. Cover S by choosing for each $x \in S$ a $\mathcal{T}$ - open neiighbourhood $A_x \in \mathcal{U}_x$ and at the same time use T_2 to choose a neighbourhood of the origin ö $B_x \in \mathcal{U}_0$ such that $A_x \cap B_x = \emptyset$. There exists a finite sub-cover

$$A_{x_1} \cup \cdots \cup A_{x_n}$$

and a neighbourhood of the origin not intersectiing the cover:

$$B := B_{x_1} \cap \cdots \cap B_{x_n}.$$

B contains a balanced neighbourhood of the origin $C \in \mathcal{U}_0$, which is connected, and therefore conained in the Euclidean ball $B_{\|\cdot\|}$. the icdntical mapping $(E, \mathcal{T}) \to (E, \mathcal{T}_e)$ i such that erefore continuous. □



Kuva 3. Cover

2. Locally convex spaces

2.1. **Seminorms and semiballs.**

Definition 2.1. In a vector space E a *seminorm* eli *puolinorm* is a mapping $p : E \to \mathbb{R}$, for which $\forall\, x, y \in E$ and $\lambda \in \mathbb{K}$

$$p(x) \geq 0 \tag{2.1}$$

$$p(\lambda x) = |\lambda| p(x) \tag{2.2}$$

$$p(x + y) \leq p(x) + p(y). \tag{2.3}$$

Remark: also:

$$|p(x) - p(y)| \leq p(x - y).$$

If $p(x) = 0 \implies x = 0$, then p is *norm*. If p is a seminorm, $x \in E$ and $r > 0$, the set $B_p(x,r) = \{y \in E \mid p(x-y) < r\}$ is called a (x-centerd, r–radius) open *semiball* and $\bar{B}_p(x,r) = \{y \in E \mid p(x-y) \leq r\}$ the corresponding *closed semiball*. 0-centered semiballs are denoted $B_p(r)$ and $\bar{B}_p(r)$, unit 0-centered semiballs B_p and $\bar{B}_p$.

There is a natural lordering among seminorms in E namely : $p \leq qifandonlyifp(x) \leq (x) \forall x \in E$. Sufficient for this is $p(x) \leq (x) \forall x \in E$, for which $q(x) \leq 1$ or just $q(x) \leq 1 \implies p(x) \leq 1$, eli $\bar{B}_q \subset \bar{B}_p$.

A locally convex space is a vector space E with a family of seminorms $\mathcal{N}$. The family of seminorms $\mathcal{N}$ induces a *locally convex topology* which is the vector space topology, jdefined by choosing all finite intersections of $\mathcal{N}$ semiballs as beighbourhoods of the origin.

One can – of course – use closed semiballs as well.

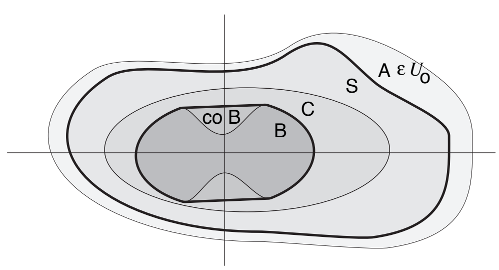
Theorem 2.2. *A topological vector space E is locally convex if and only if it has a neighbourhood basis of the origin consisting of convex sets. If this is the case, there even is a neighbourhood basis of the origin consisting of convex, absorbing, balanced and closed sets. Such sets are called* barrels.

Proof. On a locally convex space, consider the 0-centered closed semiballs.

$$\overline{B}_p(o, \varepsilon) = \{x \in E \mid p(x) \leq \epsilon\}$$

They are barrels! So are their finite intersections, and these form by a 1.14 a neighbourhood basis of the origin in some vector space topology in E. lso, they are closed in this topology, so they are barrels.

To construct the seminorms from the topology, consider a neighbourhood basis $\mathcal{K}_o$ of the origin, consisting of convex sets, first construct a neighbourhood basis of the origin, consisting of barrels. Take $A \in \mathcal{U}_0$. By theorem 1.12, there exists a closed neighbourhood of the origin, $S \subset A$. By assumption, S contains a convex neighbourhood of the origin, call it C and again by 1.12 there exists a balanced neighbourhood of the origin $B \subset C$.



Kuva 4. A barrel- neighbourhood

The closed, convex hull of B, denoted $\overline{\text{co}\, B}$ has the properties we want:

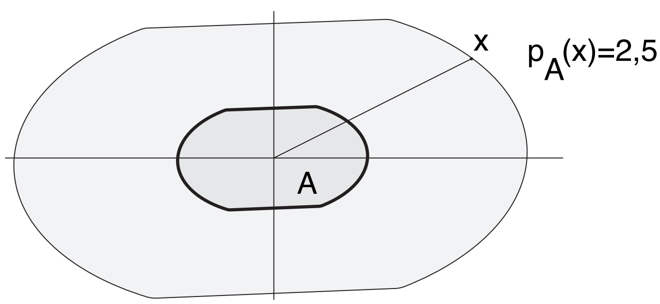
(1) The convex hull of (any) balanced set $\text{co}\, B$ is balanced[6]. Of course, it is convex, and $B \subset \text{co}\, B \subset C \subset S$.

[6]A little drawing in $\mathbb{R}^2$ can prove the balanced hull of a convex set is not convex in general. be careful!

(2) As the closure of a balanced set $\overline{\operatorname{co} B} \subset S \subset A$ is balanced by the proof of theorem 1.12 and of course closed and absorbing. We prove that it is convex: Let $x, y \in \overline{\operatorname{co} B}$ and $z = \alpha x + (1-\alpha)y$, where $0 < \alpha < 1$. Prove, that $z \in \overline{\operatorname{co} B}$. Consider a neighbourhood of the origin V such that $V + V \subset U$. Because $x, y \in \overline{\operatorname{co} B}$, there are v_x and $v_y \in V$ such that $x + v_x$ and $y + v_y \in \operatorname{co} B$, so $z + (\alpha v_x + (1-\alpha)v_y) = \alpha(x+v_x) + (1-\alpha)(y+v_y) \in \operatorname{co} B$ and $\alpha v_x + (1-\alpha)v_y \in V + V \subset U$, so inside $z + U$ we have found a point belonging to $\operatorname{co} B$.

The next — and last — step of the proof is the construction of a seminorm beginning with a barrel. The barrel will become the closed unit semiball of the seminorm: Let $A \in \mathcal{U}_0$ be a barrel, that is absorbing, balanced, closed and convex. Define its *gauge* p_A by

$$p_A(x) = \inf \{\lambda > 0 \mid x \in \lambda A\}.$$



KUVA 5. The gauge

We verify that it is a seminorm and

$$x \in A \text{ if and only if } p_A(x) \leq 1.$$

Al is easy, since

$$\begin{aligned} A \text{ absorbing} &\implies P_A(x) < \infty. \\ A \text{ balanced} &\implies P_A \text{ is homogenous.} \\ A \text{ convex} &\implies P_A \text{ is subadditive (ie. satisfies the triangle inequality)} \\ A \text{ closed} &\implies \quad (x \in A \text{ if and only if } p_A(x) \leq 1.) \qquad \square \end{aligned}$$

Next we characterize continuous seminorms in a locally convex space $(E, \mathcal{N})$. At least the members of $\mathcal{N}$ are continuous, of course, and adding continuous semonorms to $\mathcal{N}$ will not change the topology as $\mathcal{N}$.

Theorem 2.3. *let $(E, \mathcal{T})$ be a topological vector space and p a seminorm in E. The following are equivaklent:*

(1) *p is continuous.*
(2) *$B_p := \{x \in E \mid p(x) < 1\}$ is open.*
(3) $B_p := \{x \in E \mid p(x) < 1\} \in \mathcal{U}_0$.
(4) $\overline{B}_p := \{x \in E \mid p(x) \leq 1\} \in \mathcal{U}_0$.

(5) *p is bounded in some neighbourhood of the origin.* $A \in \mathcal{U}_0$.
(6) *p is continuous at* 0.

If $(E, \mathcal{N})$*is locally convex, then also the following are equivalent to the above:*
(7) $\exists \varepsilon > 0$ *and seminorms* $q_1, \ldots, q_n \in \mathcal{N}$ *such that*

$$\varepsilon(B_{q_1} \cap \cdots \cap B_{q_n}) \subset B_p$$

(8) $\exists \varepsilon > 0$ *ja* $\exists q_1, \ldots, q_n \in \mathcal{N}$ *such that* $\forall x \in E$

$$\varepsilon\, p(x) \leq \max\{q_1(x), \ldots, q_n(x)\}.$$

(9) $\exists \varepsilon > 0$ *ja* $\exists q_1, \ldots, q_n \in \mathcal{N}$ *such that* $\forall x \in E$

$$\varepsilon\, p(x) \leq (q_1(x) + \cdots + q_n(x)).$$

Proof. It is easy to check that (1)—(6) are equivalent, and in the locally convex case tapauksessa (3) and (7) are equivalent. Prove tahat (7),(8) and (9) are equivalent:
(i) For 2 seminorms in E:, say p and q we have

$$p(x) \leq q(x) \forall x \in E, \text{ eli } p \leq q if and only if B_p \supset B_q$$

(ii) for seminorms $q_1, \ldots q_n$ also $\max\{q_1, \ldots q_n\}$ is a seminorm and

$$B_{\max\{q_1, \ldots q_n\}} = B_{q_1} \cap \cdots \cap B_{q_n}.$$

(iii) for seminorms $q_1, \ldots q_n$

$$\max\{q_1, \ldots q_n\} \leq q_1 + \cdots + q_n \leq n \max\{q_1, \ldots q_n\}.$$

□

2.2. **Continuous linear mappings in locally convex spaces.**

Theorem 2.4. *Let* $(E, \mathcal{T})$ *be a topological vector space and* $(F, \mathcal{N}_F)$ *a locally convex space and* $T : E \to F$ *a linear mapping. The following are equivalent:*
(1) T *is continuous*
(2) *For all* $p \in \mathcal{N}_F$ *the mapping* $p \circ T$ *is a continuous seminorm in* E.

In particular, if also $(E, \mathcal{N}_E)$ *is lokcally convex, then also equivalent:*
(3) $\forall p \in \mathcal{N}_F$ $\exists \varepsilon > 0$ *ja* $\exists q_1, \ldots, q_n \in \mathcal{N}_E$ *such that* $\forall x \in E$

$$\varepsilon\, p(Tx) \leq (q_1(x) + \cdots + q_n(x)).$$

Proof. (1) $\Longrightarrow$ (2): If T is continuous, then $p \circ T$ is continuous, since [7] every seminorm spanning a locally convex topology $p \in \mathcal{N}_F$ is continuous. Also, it is easy to check that $p \circ T$ is a seminorm.

(2) $\Longrightarrow$ (1): Let $U \in \mathcal{U}_{0,F}$. By definition of a locally cojnvex space, there exist $p_1, \ldots, p_n \in \mathcal{N}_F$ and $r > 0$ such that $r \bigcap_{i=1}^n B_{p_i} \subset U$. By assumption (2) every $p \circ T$ is continuous, so there exist $V_i \in U_{0,E}$ such that $T(V_i) \subset B_{p_i}$. Now $T(\bigcap_{i=1}^n rV_i) \subset r \bigcap_{i=1}^n B_{p_i} \subset U$. We have found a neoighbourhood of the origin in E which is mapped into the given $U \in \mathcal{U}_{0,F}$, so T is continuous at the origin, hence everywhere.

[7] By 2.3

(2) $\implies$ (3): Let $(E; \mathcal{N}_E)$ be locally convex and $p \in \mathcal{N}_F$. By (2) the mapping $p \circ T$ is a continuous seminorm, so by (9) of the previous theorem it satisfies (3).

(3) $\implies$ (1): Let us prove that for any open $A \in \mathcal{U}_{0,F}$ there exista a neighbourhood of the origin in $(E, \mathcal{N}_E)$:stä which is mapped into A. By the definition of a locally convex topology, we can assume $A = B_p$ for some $p \in \mathcal{N}_F$. By (3) $\exists \varepsilon > 0$ and $\exists q_1, \dots, q_n \in \mathcal{N}_E$ such that $\forall x \in E$

$$\varepsilon\, p(Tx) \leq (q_1(x) + \cdots + q_n(x)).$$

In particular for points $x \in B_{q_1}(0, \frac{\varepsilon}{n}) \cap \cdots \cap B_{q_n}(0, \frac{\varepsilon}{n})$ is

$$\varepsilon\, p(Tx) \leq \varepsilon$$

which mean such that at $p(Tx) \leq 1$, in other words $Tx \in B_p = A$. □

Theorem 2.5. *For a non-zero linear form $f \in E'$ (same as alinear mapping $f : E \to \mathbb{K}$) the following are equivalent* [8]*:*

(1) *$f \in E^*$ meaning f is continuous*
(2) Ker f *is closed*
(3) Ker f *is not a dense subset $\neq E$*
(4) *f is bounded in some neighbourhood $A \in \mathcal{U}_0$.*

Proof. Implications (1) $\implies$ (2) $\implies$ (3) are easy. Let us prove (3) $\implies$ (4) $\implies$ (1).

To begin with, notice the following easy facts:

- Linear mappings preserve balancedness: If $A \subset E$ is balanced then its image in a linear mapping is also balanced.
- in one dimensional space $\mathbb{K}$ the only balanbced sets are balls around the origin, $\emptyset$ and $\mathbb{K}$. So all balanced sets except $\mathbb{K}$ itself are bounded.

In the theorem's setting f maps into $\mathbb{K}$. The given neighbourhood of the origin A conatins a closed balanced neighbourhood of the origin whose image $T(A) \subset \mathbb{K}$ contains a balanced set, which is either bounded or $\mathbb{K}$.

To prove (4) assume the contrary: No $f(A)$ is bounded. So

$$f(A) = \mathbb{K} \quad \forall A \in \mathcal{U}_0.$$

So for all $x \in E$ and $A \in \mathcal{U}_0$ $f(x + A) = f(x) + f(A) = f(x) + \mathbb{K} = \mathbb{K}$, and therefore $0 \in f(x + A)$ and Ker $f \cap (x + A) \neq \emptyset$. So the kernel of f is dense and by (3) it is all of $\mathbb{K}$. So $f = 0$, which is impossible, since no neighbourhood of the origin is mapped to a bounded set, let alone $\{0\}$.

The last implication (4) $\implies$ (1) follows directly from (2) in teh previous theorem and ehdosta (5) in 11.2.4., since $|\cdot|$ is a seminorm defining the topology of the locally convex space $\mathbb{K}$. □

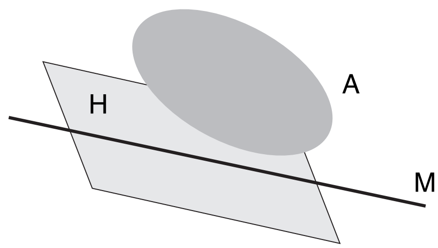
Remark 2.6. The kernel of a nonzero linear form is always either closed or dense depending on continuity. Inventing a noncontinuous linear form is nontrivial. (An example was constructed in the lectures using a Hamel basis in Hilbert space.)

[8] NOTATION VARIES, UNCLEAR, $E^* \leftrightarrow F'$.

3. Generalizations of theorems known from normed spaces

3.1. **Separation theorems.** We begin with the separation theorems by Mazur[9] and Banach[10] Thewse hold in any topological vector space with no extra conditions.

Theorem 3.1. Mazur's extension theorem[11]
Let E be a topological vector space, $\emptyset \neq A \subset E$ a convex, open set, and $M \subset E$ an affine subspace such that $M \cap A = \emptyset$. Tnen there exists a closed hyper space $H \subset E$ such that $M \subset H$ and $H \cap A = \emptyset$.



Proof. The commonly known proof from normed spaces can be copied. One first has to check some easy lemmas:

(3.1.1.) In any topological vector space a convex set is pathwise connected, hence connected.
(3.1.2.) In any topological vector space the closure of a convex set is convex.
(3.1.3.) In any topological vector space the interior of a convex set is convex (The interior of C even contains the intervals between the interior points and boundary points or points of C).
(3.1.5.) A linear algebraic projection from a topological vector space onto its subspace (with the subspace topology) is not only continuous but also an open mapping; images of open sets are open. To check openness: it is sufficient that open neighbourhoods of points are mapped to neighbourhoods of the image points. By translation invariances, it is sufficient to check this at the origin. This is easy, since the image of an open neighbourhood of the origin contains the intersection of the subspace and the open neighbourhood, which is open in the subspace.

The main proof uses a Hamel bases (which exists by the axiom of choice):
By translation invariance, we may assume $0 \in M$, so M is a linear subspace and $0 \notin A$. First also assume $\mathbb{K} = \mathbb{R}$.
For H we take a maximal element of the family of subspaces

$$\mathcal{A} = \{N \subset E \mid N \text{ is a linear subspace },\ M \subset N,\ N \cap A = \emptyset\}.$$

ordered by inclusion"$\subset$". A maximal element exists by *Zorn's lemma* which is a variant of the axiom of choice, and guarantees the existence of a maximal element,

[9]Stanisław Mazur, 1905 – 1981, Puola.
[10]Stefan Banach, 1992 – 1945, Puola.
[11]Vrt.XX

if every totally ordered subset $\mathcal{B} \subset \mathcal{A}$ has an upper bound in $\mathcal{A}$. And it has — obviously the union of all its elements $\bigcup \mathcal{B}$ is in $\mathcal{A}$ and is an upper bound of $\mathcal{B}$.

The maximal subspace H satisfies all our wishes — we just have to check that it is a hyperplane.

The linear subspace $H \subset E$ has a Hamel-basis K. Extend it to become a Hamel-basis of E, call it $L \supset K$. Prove that the subspace $F = \langle L \setminus K \rangle$ spanned by the "new" basis vectors is one dimensional: Consider the projection onto the subspace:

$$\varphi : E \to F = \langle L \smallsetminus K \rangle : \sum_{x \in L} \alpha_x x \mapsto \sum_{x \in L \smallsetminus K} \alpha_x x$$

This is an open mapping (lemma!) so it maps A: to an open, convex subset of F – obviously not containing the origin. If F were at least 2-dimensional, we would consider a suitable 2-dimensional subspace of F, which would contain a 2-dimenional subspace S, not intersectiing the image set $\varphi(A)$. (Easy in dimension 2. Just draw a picture!) In that case we would have

$$M \subset H = \varphi^{-1}(\{0\}) \subsetneq \varphi^{-1}(S)$$

and

$$\varphi^{-1}(S) \cap A = \emptyset$$

so the subspace $\varphi^{-1}(S)$would be in conflict with the maximality of H. Therefore $\dim F = 1$ and H is a hyperplane. Of course H cannot be dense since it does not intersect A. So it must be closed.

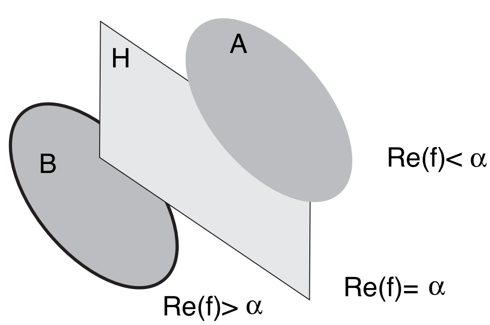
Thgis was the reeal version. For the complex version take a real hyperplane $H \supset M$ not intersectiing A Now

$$H \cap iH$$

is a complex subspace, obviously a hyperplane, not intersecting A, so closed. □

Theorem 3.2. (Banach's separation theorem) *Let A and B be convex, disjoint, A open. Then there exists a continuous linear form f and a real (!) number $\alpha \in \mathbb{R}$ such that*

$$Re\, f(x) < \alpha \quad \forall x \in A \text{ ja}$$
$$Re\, f(x) \geq \alpha \quad \forall x \in B.$$



Proof. Begin with the **real version**:

If E is a real topological vector space, and A and B be convex, disjoint, A open. Then there exists a continuous (real) linear form f $f : E \to \mathbb{R}$, which separates A *and B. By this we mean*

$$f(A) \cap f(B) = \emptyset.$$

Solution: We can assume $A, B \neq \emptyset$. The set

$$C = A - B = \{a - b \mid a \in A,\, b \in B\} = \bigcup_{b \in B} (A - b)$$

is open, convex, and does not contain the origin. Apply Mazur to C and the zero dimensional subspace $M = \{0\}$. So there exists a hyperplane H, not intersectiong C. Take f a linear form with $H = \operatorname{Ker} H$. this does it, since f is continuous and if $a \in A$ and$b \in B$ such that $f(a) = f(b)$, then $(a-b) \in (A-B) \cap \operatorname{Ker}(f) = (A-B) \cap H = \emptyset$.

The complex version: If $f : E \to \mathbb{C}$ is complex linear, then its real part

$$g : E \to \mathbb{R} : g(x) = \operatorname{Re} f(x) = \tfrac{1}{2}(f(x) + \overline{f(x)})$$

is real linear (in general not complex linear). On the other hand, every real linear form in a complex space $g : E \to \mathbb{R}$ is the real part of the complex linear

$$f : E \to \mathbb{C} : x \mapsto g(x) - ig(ix)$$

. So the real and complex linear forms of E can be identified with each other. Also, it is clear that f and g are both continuous or both discontinuous. □

Hahn and Banach theorem. A reminder from Functional Analysis The two theorems above are close to being equivalent to the famous Hahjn- Banach theorems and are often proved as corollaries of Haahn-Banach. Let us (almost) do the converse:

Theorem 3.3. *(Hahn and Banach 1927-29.)*[12] *Let $F \subset E$ be a subspace in a normed space and $f : F \to \mathbb{K}$ a linear form such that*

$$|f(x)| \leq \|x\| \quad \forall x \in F.$$

Then there exists a linear form $g : E \to \mathbb{K}$, such that

$$g(x) = f(x) \quad \forall x \in F \qquad \textit{and}$$
$$|g(x)| \leq \|x\| \quad \forall x \in E.$$

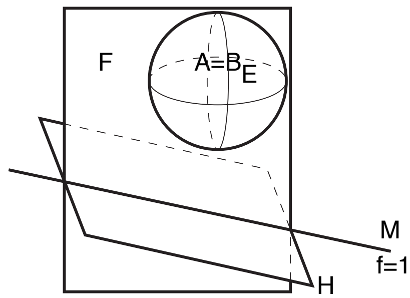
Proof. We can assume $f \neq 0$. Use Mazur is the open unit ball $A = B_E$ and the F-hyperplane

$$M = \{x \in F \mid f(x) = 1\}.$$

By Mazurwe extend M to a closed E-hyperplane $H \supset M$, not intersecting B_E. Because $H \cap F$ contains the F-hyperplane M, but is distinct of all of F (The subspace F of Econtains the origin!) we get $H \cap F = M$. define g to bethe linear form in E: with value 1 in H. That works! □

The assumption in HB theorem mean such that at $f : F \to \mathbb{K}$ is continuous in $F$and has norm $\|f\| = \sup_{\|x\| \leq 1} |f(x)| \leq 1$. So Hahn– Banach tells us, such a form has an extension g to all of E with norm 1. In particular, g is continuous.

[12]Hans Hahn 1879–1934, Austria.



Kuva 6. Corollary of Hahn and Banach theorem

Remark 3.4. Generalizations. In a normed space, every continuous linear forma is a subspace can be extended to a continuous form is the whole space having the same norm.

The image space was K so it can be any 1-dimensional space - and (going via coordiantes) in fact any finite dimensional space (how about the norm - I have forgotten. Be careful!) The theorem fails for infinite dimensional image space.

Hahn-Banachin lause holds in any *seminormed space* E. (one seminorm) — of course. In fact, in the real case it is sufficient to assume that E is a vector space and there exists a positive *sublinear mapping* in other words a mapping like a seminorm but we assume homogeneity only for positive coefficients: $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda > 0$.

This is the best known version of HB, and generally proven directly by applying Zorn in the real case and then reducing the complex to the real case.

So Mazur extension, Banachin separation and Hahn–Banach do work in (lc) topological vector spaces, BUT THEY FAIL IN GENERAL topological vector space . Counterexample . There are no nonzero continuous linear forms at all in ℓ^p for $0 < p < 1$.

Important consequences. The theorem has several important consequences, some of which are also sometimes called "Hahn–Banach theorem":

* If V is a normed vector space with linear subspace U (not necessarily closed) and if $T : U \to K$ is continuous and linear, then there exists an extension $T' : V \to K$ of T which is also continuous and linear and which has the same norm as T (see Banach space for a discussion of the norm of a linear map). In other words, in the category of normed vector spaces, the space K is an injective object. * If V is a normed vector space with linear subspace U (not necessarily closed) and if z is an element of V not in the closure of U, then there exists a continuous linear map $T' : V \to K$ with $T'(x) = 0$ for all x in U, $T'(z) = 1$, and $\|T'\| = 1?\,\mathrm{dist}(z, U)$. * In particular, if V is a normed vector space and if z is any element of V, then there exists a continuous linear map $T' : V \to K$ with $T'(z) = \|z\|$ and $\|T'\| \leq 1$. This

implie such that at the natural injection J from a normed space V into its double dual V^{**} is isometric.

Hahn-Banach separation theorem. Another version of Hahn–Banach theorem is known as Hahn-Banach separation theorem.[2] It has numerous uses in convex geometry [3] and it is derived from the original form of the theorem.

Theorem: Let V be a topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and A, B convex, non-empty subsets of V. Assume that $A \cap B = \varnothing$. Then

(i) If A is open, then there exists a continuous linear map $\lambda\colon V \to \mathbb{K}$ and $t \in \mathbb{R}$ such that Re $\lambda(a) < t \leq$ Re $\lambda(b)$ for all $a \in A$, $b \in B$

(ii) If V is locally convex, A is compact, and B *closed*, then there exists a continuous linear map $\lambda\colon V \to \mathbb{K}$ and $s, t \in \mathbb{R}$ such that Re $\lambda(a) < t < s <$ Re $\lambda(b)$ for all $a \in A$, $b \in B$.

Another consequence of HB:

Theorem 3.5. *Let E be a locally convex space and F its closed subspace and $x \in E \setminus F$. then there exists a continuous lin form $x^* \in E^*$ such that*

$$\langle x, x^* \rangle = 1 \text{ and}$$
$$\langle y, x^* \rangle = 0 \quad \forall y \in F.$$

Proof. Use Mazur choosing for A an open, convex neighbourhood of x not intersecting F. □

4. Metrizability and completeness (Frèchet spaces)

Remember from (functional) analysis:

Theorem 4.1. Baire category theorem: *Any complete metric space is of the 2. Baire category. In particular, A topological vector space is 2. Baire category, if its topology is given by some metric and it is complete in this metric.*

We will define the concepts (category) and sketch a proof (known from (functional) analysis courses). But i such that is useful? Not yet. We will find out how to check metrizability and completeness? At least for locally convex space such that ere is nice theory. The main applications — the *open mapping theorem* and the *closed graph theorem* will be proven as consequences of the *Barrel theorem*.

4.1. **Baire's category theorem.**

Definition 4.2. Consider a topological space X.

(1) A subset $M \subset X$ is *dense* in X, if $\overline{M} = X$.
(2) A set $M \subset X$is *nowhere dense*, in X, if its closure has no interior points, same as if the complement $X \setminus \overline{M}$ of the closure is dense $\overline{X \setminus \overline{M}} = X$ In particular a closed set is nowhere dense in X when it has no interior points.
(3) A set $M \subset X$ belongs to the 1. Bairen category in X, if it is the union of countably many nowhere dense sets.
(4) All oher sets o $M \subset X$ belong to the 2. Bairen category in X.

Example 4.3. a) $\mathbb{Q}$ in $\mathbb{R}$ is a 1 category set.
b) Baire's theorem will prove that $\mathbb{R}$ in $\mathbb{R}$ is a 2. category set.

Theorem 4.4. (Cantor's lemma) *A metric space (X,d) is complete if and only if for any closed sets $X \supset S_1 \supset S_2 \supset \dots$ with $\operatorname{diam}(S_n) \to 0$, we have*

$$\bigcap_{n\in\mathbb{N}} S_n \neq \emptyset.$$

Here $\operatorname{diam}(S)$ is the diameter of the set S

$$\operatorname{diam}(S) = \sup\{d(x,y) \;\big|\; x, y \in S\}.$$

PROOF. Consider first X complete and the closed sets as above By choosing for each $n \in \mathbb{N}$ an element $x_n \in S_n$ we get a Cauchy-sequence[13] , whose limit onis in the intersectionof the sets S_n.

Consider the case when the condition is true. Prove that any Cauchy–sequence in XBy choosing

$$S_n = \overline{\{x_m \;\big|\; m \geq n\}}$$

one can use the assumption and it is easy to verify that the element in the intersection is the limit. □

Theorem 4.5. *Consider a complete metric space X and a sequence A_n of open dense subsets. Then the intersection*

$$A = \bigcap_{n\in\mathbb{N}} A_n$$

is dense, in particular nonempty.

PROOF. Consider an open ball $B(x,r)$in X. Prove that

$$B(x,r) \cap A \neq \emptyset.$$

Since A_1 is dense and $B(x,r)$ open, the intersection $A_1 \cap B(x,r)$ is nonempty — and also open. So there exists a ball $B(x_1,r_1)$, whose closure S_1 is contained in the set $A_1 \cap B(x,r)$. Since also A_2 is dense and $B(x_1,r_1)$ open, the intersection $A_1 \cap B(x_1,r_1)$ is nonempty — and open. So there exists a ball $B(x_2,r_2)$, whose closure S_2 is included in $A_2 \cap B(x_1,r_1)$. Repeat this to find a sequence sisäkkäisiä of nested sets S_n. The radii r_n can be chosen such that $r_n \to 0$. By Cantor,

$$\bigcap_{n\in\mathbb{N}} S_n \neq \emptyset.$$

This proves it. □

Theorem 4.6. (Baire's category theorem) *No complete metric space X is of the first category but all are of the 2.*

[13]Augustin Louis Cauchy 1789–1857, Ranska.

PROOF. This is the lemma resatated. Consider X 1 cat.,

$$X = \bigcup_{n\in\mathbb{N}} M_n = \bigcup_{n\in\mathbb{N}} \overline{M_n}, \text{ ts.}$$

$$\emptyset = X \setminus X = \bigcap_{n\in\mathbb{N}} (X \setminus \overline{M_n}).$$

The sets $X \setminus \overline{M_n}$ are open and dense. By the lemma, their intersection is nonempty
. □

4.2. **Complete topological vector spaces.** In a nonmetrical space "Cauchy" must be redefined. We will also replace sequences by filters for the general case.

Definition 4.7. .

(1) A filter or filter basis $\mathcal{F}$ in a topological space $A \subset E$ is a *Cauchy-filter(basis)*, if for all neighbourhoods of the origin $U \in \mathcal{U}_0$ there exists a set $M \in \mathcal{F}$ such that $M - M \subset U$. **Remark:** Often $A = E$.
(2) A subset $A \subset E$ of a topological vector space is *complete*, if its every Cauchy-filter (or filterbasis) $\mathcal{F}$ converges to some point in A:. (For a filter $\mathcal{F} \to x if and only if \mathcal{U}_x \subset \mathcal{F}$, for a filter basis : $\mathcal{F} \to x if and only if \forall U \in \mathcal{U}_x \,\exists A \subset \mathcal{F}$, $A \subset U$.
(3) In a topological vector space $(E, \mathcal{T})$, a sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy-sequence*, if for every neighbourhood of the origin $A \in \mathcal{U}_0$ there exists a number $n_A \in \mathbb{N}$, for which
$$n, m > n_A \implies (x_n - x_m) \in A.$$
(4) A subset $A \subset E$ of a topological vector space is *sequentially complete*, if its every Cauchy-sequence $\mathcal{F}$ converges to some point in A.

Remark 4.8. i) A sequence is Cauchy-sequence if and only if the corresponding lter is a Cauchy-filter.
ii) The neighbourhood filter is a Cauchy-filter.
iii) If $\mathcal{F}$ contains a Cauchy-filter, then $\mathcal{F}$ is a Cauchy-filter. In particular every convergent filter is a Cauchy-filter.
iv) In a Hausdorff-topological vector space every complete set is closed.
v) In a complete space, every closed subset is complete.
vi) Continuous linear mappings map Cauchy-filterto Cauchy-filters.
vii) The *trace* of a Cauchy-filter $\mathcal{F}$ *jälki* in a subset $A \subset E$ is the set family $\mathcal{F}_A = \{A \cap B \mid B \in \mathcal{F}\}$. It is either a Cauchy– filter or $\mathcal{F}_A \ni \emptyset$.

PROOF. Only iv) and v) are slightly nontrivial

iv) Consider a complete subset $A \subset E$ and an element $x \in \bar{A}$. Let $\mathcal{B} = \{U \cap A \mid U \in \mathcal{U}_x\}$. By assumption no element of $\mathcal{B}$ is the empty set, j so $\mathcal{B}$ is a filter in A, in fact a Cauchy-filter: Check it: for each $U' \in \mathcal{U}_0$ there exists $M = U \cap A \in \mathcal{B}$ such that $M - M = (U \cap A) - (U \cap A) \subset U - U \subset U'$. Since A is complete, $\mathcal{B}$ converges to some point in Y in A. As a filter basis $\mathcal{B} \to x$ and $\mathcal{B} \to y$, so by Hausdorff $x = y$. (In a Hausdorffspace all limits are unique (well known and easy)).

v) Assume now, that E is complete and $A \subset E$ is closed and $\mathcal{B}$ is a Cauchy-filter in A. Then $\mathcal{B}$ is a Cauchy-filterbasis in avaruuden E, and converges to some $x \in E$.

of course $x \in \bar{A} = A$, since for all $U \in \mathcal{U}_x$ there exists a subset $B \in \mathcal{B}$, consisting of nonempty subsets of A, so $U \cap A \neq \emptyset$.

4.3. **Metrizable locally convex spaces.**

Theorem 4.9. *A topological vector space is of 2 category if its topology comes from some metric, and the space is complete in that metric.*

Proof. Baire $\square$

This theorem is almost useless unless we find a way to check metrizability and completeness in the metric. Fortunately, this works at least for locally convex spaces.

Theorem 4.10. *Consider $(E, \mathcal{T})$, a topological vector space, locally convex and Hausdorff. The following are equivalent:*

(i) *There exist(s) a neighbourhood basis $\mathcal{U}_0$, of $\mathcal{T}$ which is countable.*
(ii) *There exist(s) a defining (countable) sequence $\mathcal{N}$ of seminorms defining the topology $\mathcal{T}$.*
(iii) *There exist(s) a basis $\mathcal{P}$ of continuous seminorms, which is not only countable but also ordered increasingly $p_1 \leq p_2 \leq \dots$.*
(iv) *In E there exists a metric d, which is* translation invariantt

$$d(x, y) = d(x + z, y + z) \quad \forall\, x, y, z \in E,$$

and who defines the topology $\mathcal{T}$.
(v) *In E there exists metric d, who defines the topology $\mathcal{T}$.*

Inthis case we call E a matrizable locally convex space .

Proof. The main step is *iii)→ iv).* : (By Banach himself): If $\mathcal{N} = \{p_1, p_2, \dots\}$, then, then this is the metric:

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x - y)}{1 + p_k(x - y)}.$$

(Check it!)

If E is metrizable, then take balls around the origin

$$B_n = B_d(0, \frac{1}{n}).$$

Each of these contains a neighbourhood which is barrel $B_n \supset A_n \in \mathcal{U}_0$. The gauges of the barrels A_n are a countable seminorm family giving the metric's topology $\mathcal{T}$. Check it! $\square$

Example 4.11. *Banach's sequence space* $E = \{x = (x_n)_{\mathbb{N}} \mid x_n \in \mathbb{K}\} = \mathbb{K}^{\mathbb{N}}$ with seminorms $p_k(x) = |x_k|$ is metrizable.

4.4. **Fréchet spaces.** Remeber that in a topological space E a sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy-sequence, if for all $A \in \mathcal{U}_0$ there exists a number $n_A \in \mathbb{N}$, for which

$$n, m > n_A \implies (x_n - x_m) \in A,$$

and that a topological vector space E is sequentially complete, if its every Cauchy-sequence converges.

Theorem 4.12. *A sequence in a metrizable locally convex space is a Cauchy-sequence if and only if it is Cauchy-sequence in Banach's metric d.*

Proof. Possible exercise. □

A metrizable locally convex space $(E, \mathcal{N})$ is sequentially complete as a topological vector space if and only if it is sequentially complete in Banach's metric d.

Theorem 4.13. *A metrizable locally convex space E is complete in the filter sense as a topological vector space if and only if it is sequentially complete.*

Proof. Completeness of course implies sequential completeness. [1.15].

Assume next, that the space $(E, \mathcal{N})$ has a countable neighbourhood basis of the origin $U_1 \supset U_2 \supset \dots$ and that every Cauchy-sequence in E converges. Consider a Cauchy–filter $\mathcal{F}$. We prove, that $\mathcal{F}$ converges.

By assumption, for all $k \in \mathbb{N}$ there exists $M_k \in \mathcal{F}$ such that $M_k - M_k \subset U_k$. Define a sequence by choosing $x_n \in M_1 \cap M_2 \cap \dots \cap M_n$. In this way we get a Cauchy–sequence: $m, m' \geq n \implies x_m - x_{m'} \in M_n - M_n \subset U_n$. JBy sq compl there exists a limit $x = \lim_{n\to\infty} x_n$. We prove, that $\mathcal{F} \to x$ same as $x + \mathcal{U}_0 \subset \mathcal{F}$. This mean such that at for every point x the basis neighbourhood $x + U_n$ belongs to the filter $\mathcal{F}$, so for each n there exists $M \subset x + U_n$ belonging to the filter $\mathcal{F}$. So $M - x \subset U_n$. Now use the information $x_n \to x$, guaranteeing that for every k there exists p_k such that $x_{p_k} \in x + U_k$ equivalently $x \in x_{p_k} - U_k$. So for all k

$$M - x \subset (M - x_{p_k}) + U_k,$$

Next we have to select a number k and a suitable p_k and $M \in \mathcal{F}$ such that

$$(M - x_{p_k}) + U_k \subset U_n.$$

This works: In the topological vector space E we can choose $U_k \in \mathcal{U}_0$ such that

$$U_k + U_k \subset U_n.$$

Now try to arrange $(M - x_{p_k}) \subset U_k$. By the choice of the sequence $(x_n)_{\mathbb{N}}$ we have $x_p \in M_k$, if $p \geq k$. The number p_k can we get the reason to choose $M = M_k$, so

$$(M - x_{p_k}) + U_k = (M_k - x_{p_k}) + U_k \subset (M_k - M_k) + U_k \subset U_k + U_k \subset U_n.$$

□

Definition 4.14. A complete metrizable locally convex space is called a *Fréchet space* .

Theorem 4.15. *The isomorphic image of a Fréchet space is a Fréchet space,*

Proof. [Why?] Easy. But notice, completeness is not preserved under general homeomorphisms! [14] □

[14] The standard counterexample: ℝ:n metrics $|x - y|$ and $|\overline{\arctan}\, x - \overline{\arctan}\, y|$ give diffferent Cauchy-sequences, but the same topology. .

4.5. **Corollaries of Baire.**

Theorem 4.16. Barrel theorem *In a Fréchet-avaruudessa every barrel is a neighbourhood of the origin.*

Proof. Let $T \subset E$ be a barrel. Since T is absorbing, $E = \bigcup_{n \in \mathbb{N}}$ and so by Baire and by being closed, T has an interior point x. Since T is balanced, also $-x \in \mathrm{int} T$ and so by convexity of the interior $0 \in \mathrm{int} T$. □

Proof. Clever, short, by Baire!

Theorem 4.17. Open mapping theorem *A continuous linear mapping between Fréchet spaces is always an open mapping (ie. images of open sets are open)*

Proof. We presented a short proof using the barrel theorem

Corollary 4.18. *A continuous linear mapping between Fréchet spaces is open as a mapping to its image if and only if the image is a closed subspace.*

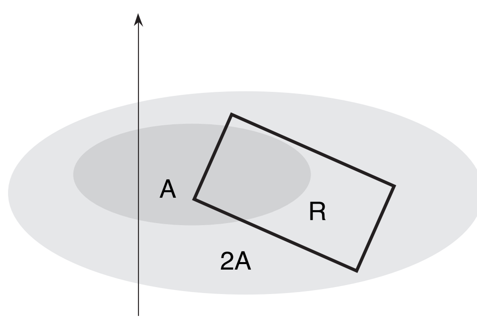
Proof. Corollary of open mapping theorem

Theorem 4.19. Closed graph theorem. *A linear mapping $T : E \to F$ between Fréchet spaces is continuous if and only if its graph $Gr\,T = \{(x, Tx) \mid x \in E\}$ is a closed set in $E \times F$.*

Proof. Also a corollary of open mapping theorem

Definition 4.20. Consider a topological vector space E and a subset $R \subset E$. We say that R is *bounded*, if every neighbourhood of the origin *absorbs* it:

$$\forall A \in \mathcal{U}_0 \quad \exists \lambda > 0 : \quad R \subset \lambda A.$$



Example 4.21. Any finite set is bounded. The bal hull, the closure any subset and a continuous linear image of a bounded set is bounded. In a locally convex space also the convex hull is bounded.

Mostly semiballs are unbounded. By a theorem by *Kolmogorov in 1935*, a locally convex Hausdorff-space is finite dimensional if there exists a bounded set with an interior point. (Am easy proof in hand written Finnish was given)

Definition 4.22. Let E and F be topological vector spaces and

$$\mathcal{Y} \subset L(E, F)$$

a family of linear mappings.

(a) We say that all mappings in $\mathcal{Y}$ are *equicontinuous*, or call the family $\mathcal{Y}$ itself *equicontinuous* if:

$$\forall A \in \mathcal{U}_{0,F} : \exists B \in \mathcal{U}_{0,E} \quad \text{such that}$$
$$\forall T \in \mathcal{Y} : T(B) \subset A.$$

(b) $\mathcal{Y}$ is *pointwise bounded*, , if:

$$\forall x \in E : \mathcal{Y}(x) := \{Tx \mid T \in \mathcal{Y}\} \text{ is bounded.}$$

Theorem 4.23. Banach-Steinhaus[15] *Let E and F be Fréchet spaces and*

$$\mathcal{Y} \subset L(E, F)$$

a family of linear mappings.

The following conditions are equivalent:

(1) *$\mathcal{Y}$ is equicontinuous.*
(2) *$\mathcal{Y}$ is pointwise bounded.*

Proof. (2) $\Longrightarrow$ (1): Consider $A \in \mathcal{U}_{0,F}$. Prove that

$$\bigcap_{T\in\mathcal{Y}} T^{-1}(A) \in \mathcal{U}_{0,E}. \qquad (*)$$

As a lc space, F has a neighbourhood basis of barrels, so we may assume that A is a barrel. Then the set $\bigcap_{T\in\mathcal{Y}} T^{-1}(A)$ is a barrel, since it is obviuously balanced, closed and convex, and also absorbing, because the family $\mathcal{Y}$ is pointwise bounded. By the barrel theorem $(*)$ is true. □

5. Constructions of spaces from each other

5.1. **Introduction.** There are many ways to construct new spaces from old. Always it is interesting to see which properities are preeserved — or possibly improved! Exx

- subspace
- factor space
- product space
- direct sum
- completion
- projective limit (inverse image object)
- inductive locally convex imit (image object)
- direct inductive limit
- dual (and more general Hom)

5.2. **Subspaces.**

Definition 5.1. A *subspace* of a topological vector space E is its linear subspace M with the subspace topology induced ny E.

Theorem 5.2. *The following are éasy to check:*

(i) *M is tva.*
(ii) *M inhrits from E the following properties*
 (a) *Hausdorff*

[15]Władysław Hugo Dionizy Steinhaus 1887–1972, Puola

(b) *metrizable*
(c) *locally convex (seminorms restricted to subspace give topology)*
(d) *normable (has norm giving topology)*

(iii) *every continuous seminorm in $M$is the restriction of some continuous seminorm in E (Exercise - hands on, but not trivial)*
(iv) *if E is locally convex, then every continujous linear form on M is the restrictionof some continuous linear form aon E. (This is Hahn and Banach)*
(v) *the corresponding statement for linear mappings to infinite dimensional spaces is false.*
(vi) *A subspace (except E itself) has no interior points, but may be dense.*

5.3. **Factor spaces.**

Definition 5.3. A *factor space of a topological vestor space $E$with respect to a subspace $H \subset E$* is linear algebraic factor space E/H with the *factor spacetopology* τ, defined by the following equivalent conditions:

(i) τ is the finest topology, in which the canonical surjection $\phi : E \to E/H$ is continuous.
(ii) τ is the image of the topology in $E$by the canonical surjection $\phi : E \to E/H$, this meaning that a subset $A \subset E/H$ is open if and only if $\phi^{-1}(A) \subset E$ is open.
(iii) τ is the tvs-topology with a 0-neighbourhood basis consisting of the images of the 0 neighbourhoods of E in the canonical surjection $\phi : E \to E/H$, ie.

$$\mathcal{U}_{E/H} = \{\phi(U) \bigm| U \in \mathcal{U}_E\}.$$

Theorem 5.4. *The following are easy to verify:*

(i) *The canonical surjection $\phi : E \to E/H$ is continuous and open, but not even in 2-dimensionall space $E = \mathbb{R}^2$ is it a closed mapping (closed to closed).*
(ii) *E/H is Hausdorff if and only if $H \subset E$ is closed.*

Theorem 5.5. *A factor space E/H of a locally convex space E is locally convex. The seminormas are constructed in the following way:*

(1) *Choose a seminorm family $\mathcal{P}$ defining the topology in E such that for all $p, q \in \mathcal{P}$ there exists $r \in \mathcal{P}$ such that $r \geq \max p, q$. (This can be done and gives a basis of origin-neighbourhoods defined by one seminorm each!)*
(2) *define for each $p \in \mathcal{P}$ a seminorm in E/H by*

$$\hat{p}(x + H) = \inf_{y \in x+H} p(y).$$

(3) *Notice that this gives a seminorm family with the same property as originally: it gives a basis of origin-neighbourhoods defined by one seminorm $\hat{p}$ each in the topology of E/H.*
(4) *Now it is not difficult to check tht you got the factor topology. In particular*
(5) *$\hat{p}$ is a normi in the factor space $E/Kerp$.*

5.4. **Product spaces.**

Definition 5.6. In the product of topological spaces $\prod_{i\in I} X_i$ the *product topology* is defined by taking as a basis of open sets all products $\prod_{i\in I} U_i$, where every $U_i \subset X_i$ is open, and for all indices except finitely many we have $U_i = E_i$. All other open sets are unions of the basis sets. The product topology is the coarsest topology in which the *projections* $\pi_j : \prod_{i\in I} X_i \to X_j : (x_i)_I \mapsto x_j$ are all continuous.

A *product space* of topological vector spaces $\prod_{i\in I} E_i$ generally has the product topology. (Exceptions exist,in particular some "interior" open sums) More details are given in the exercises.

Theorem 5.7. *Every locally convex Hausdorff-space is isomorphic to (linear and homeomorhpic) to some subspace of some product of Banach-spaces.*

PROOF. Not difficult. Given in lecture and the handwritten. To come later here. XXXX. □

5.5. **Direct sums.**

Remark 5.8. Details of the "inner direct sum" of two subspaces were discussed in the exx. in partricular the *topological direct sum.*

Definition 5.9. The "outer" direct sum of (possibly infinitely many) distinct topological vector spaces is the topological vector sunspace

$$\bigoplus_{i\in I} E_i = \big\{(x_i)_I \in \prod_{i\in I} \mid x_i \neq 0 \text{ only for finitely many } i\big\}$$

5.6. **The completion.**

Remark 5.10. Completeness was already discussed. Here we just remark that a product space (or an "outer" direct sum) is complete if and only if each "factor" is complete Byt fadtror spaces are generally not complete except in the metric case. (Counterexample still missing XXXXX.)

Theorem 5.11. *Let E and F be topological Hausdorff-spaces, F complete. Let $A \subset E$ be a dense subspace and $T : A \to F$ a continuous linear mapping. Then there exists exactly one continuous linear mapping $S : E \to F$, whose restriction to the space A is T.*

PROOF. Uniqueness follows from the fact that continuous functions coinciding in a dense set coincide everywhere. Existence is easily proven in the metrizable case by using Cauchy-sequences. The general construction must use Cauchy filters. Not too bad either - done in the lectrures and the hand wsrtittten hand out. □

Theorem 5.12. Existence of completion. *Let E be a topological vector space and Hausdorff. Then there exists a* completion*, of E, that is a complete Hausdorff-tvs $\hat{E}$, whjose some dense subspace E_1 is isomorphic to E. All completions of E are isomorphic to each other.*

PROOF. [Proof idea] Thegeneral proof is involved and is omitted.[16] In the locally convex case, we can use 5.7, by which E is isomorphic to a subspace of some product

[16] Ks. Cf. Köthe §5.

of complete spaces — whichis complete — so its closure will work as completion. Yniqueness is proven by using theorem 5.11. (Do it, or look at the hand written text.)

5.7. **Projektive limits.**

Definition 5.13. Let X be a set and $\{f_i : X \to X_i \mid i \in I\}$ a family of mappings from X to some topological spaces (X_i, τ_i). The *projective topology* spanned by the mappings f_i $(i \in I)$ is the coarsest topology in E where every f_i is continuous. A sub-basis of this topology conists of the inverse images of open sets: $f^{-1}(U_i)$. The other open sets are finite intersections of the su-basis sets and all unions of these intersections.

Example 5.14. (1) The product topology is — by definition — an example of projective topology.
(2) The subspace topology is — by definition — an example of projective topology (with respect ti the inclusion mapping).
(3) The weak topology is the projective topology of the mappings $|\langle \cdot, x^* \rangle|$. (So completeness is not inherited, since we will prove (and it is well known) that an infinite dimmensional Hilbert spadce is not weakly complete)
(4) If E is a vector space and every f_i is a linear mapping into the topological vector space F_i, then the projective topology makes Einto a tvs.
(5) If every F_i is locally convex, then the projective topology is also locally convex with seminorms $p_{ij} \circ f_i$, where $(p_{ij})_j$ defines the topology of F_i.
(6) Warnimg: A locally convx $(E, \mathcal{P})$ has the coarsest locally convex topology, where every seminorm $p \in \mathcal{P}$ is continuous, but this is generally not the projective topology induced by the family $\mathcal{P}$. A counterexample is given by the usual absolute value norm $\mathbb{R}$, in whose projective topology the interval $]0, 1[$ is not open. (all open sets are symmetric)

Theorem 5.15. Projective topology heorem *Let E be a vector space, with the projective topology defined by the linear mappings $f_i : E \to F_i$ (F_i tvs). Then a linear mapping T from any topological vector space to E is continuous if and only if every $f_i \circ T$ is continuous.*

Proof. Directly form the definitions! □

5.8. **Inductive locally convex limits.**

Definition 5.16. Let E be a vector space and $\{T_i : F_i \to E_i \mid i \in I\}$ a family of linear mappings from some lc spaces to E. The *locally convex inductive (limit) topology* T_i $(i \in I)$ is the finest locally convex (!) topology in E, where every T_i is continuous. A bsis of 0 neighbourhoods is

$$\mathcal{B}_E = \{U \subset E \mid U \text{ is bal, konv, and abs and } T_I^{-1}(U) \in \mathcal{U}_{E_i} \,\forall\, i \in I\}.$$

Example 5.17. (1) WARNING: In general the inductive locally convex topology differs from *image topology* in E, which is the finest topology, in which the mappings T_i are all continuous. The reason for this i such that at the image topology is usually not a lc tvs topology.
(2) The factor space topology is the lc with respect to the canonical surjection.

(3)

Theorem 5.18. *The inductive limes E of some barreled spaces F_i is barreled.*

Proof. Directly form the definitions! □

Theorem 5.19. *A space is called is called* bornological, *if every convex, balanced set, which absorbs all bounded sets, is a neighbourhood of the origin .)*

An inductive limit of bornological spaces is bornological.

Proof. Directly form the definitions! □

Theorem 5.20. Inductive lc topology theorem *Let E be a vector space, and equip it with the inductive lc topology with respect to some linear meppings $f_i : F_i \to E$ (F_i tva). Then*

(i) *any linear mapping T from E to any locally convex space E is continuous if and only if every $T \circ T_i$ is continuous. In particular this is true for linear forms $E \to \mathbb{K}$.*
(ii) *a seminorm p in E is continuous if and only if every $p \circ T_i$ is continuous (always a seminorm).*

Proof. Directly form the definitions, but begin with ii).

□

5.9. **Direct inductive limits.**

Definition 5.21. Let $E_1 \subset E_2 \subset \dots$ be a sequence of nested, closed lc Hausdorff spaces. The union $E = \bigcup_{\mathbb{N}} E_n$ with the inductive topology of the inclusion mappings is called the *direct inductive limit* $\underset{\longrightarrow}{\lim} E_n$ of the spaces E_n.

Example 5.22. Main example: {Test functions for Schwarzin distributions.} Discussion comes later.

6. Bounded sets

6.1. **Bounded sets.** We have alredy defined the concept of a bounded set in a tvs. (At 4.20) To repeat: Let E be a topological vector space and $R \subset E$. A set $R \subset E$ is *bounded*, if every neighbourhood of the origin absorbs it:

$$\forall U \in \mathcal{U}_0 \quad \exists \lambda > 0 : \quad R \subset \lambda U.$$

Theorem 6.1. *A subset $A \subset E$ of a lc space $(E, \mathcal{P})$ is bounded if and only if every seminorm $p \in \mathcal{P}$ is a bounded function in the set joukossa A. This mean such that at every $p(A)$ is a bounded set of numbers.*

Proof. Let A be bounded and $p \in \mathcal{P}$. Now the semiball U_p is aneighbourhood of the origin, so $\exists \lambda > 0 : \quad R \subset \lambda U_p$, and therefore $p(A) \subset [-\lambda, \lambda]$.

Let every p be bounded and $U \in \mathcal{U}_0$. Choose a number $\epsilon > 0$ and seminorms $p_i \in \mathcal{P}$ (i=1,2,...n) such that $U \supset \epsilon \bigcap_{i=1}^n U_{p_i}$. Since every p_i is bounded in A, there exists numbers $\lambda_i > 0$ such that $p_i(A) \subset [-\lambda, \lambda]$, so $A \subset \bigcap_{i=1}^n \lambda_i U_{p_i} \subset \lambda \bigcap_{i=1}^n U_{p_i}$, where $\lambda = \max_i \lambda_i$. So $A \subset \lambda \bigcap_{i=1}^n U_{p_i} \subset \frac{\lambda}{\epsilon} U$. □

These examples of bounded sets weree already mentioned before at 4.20 (?) .

Example 6.2. Bounded:

(1) finite set,
(2) compact set,
(3) balanced hull of bounded set,
(4) closure of bounded set,
(5) subset of bounded set
(6) continuous image of bounded set
(7) finite union of bounded sets
(8) finite sum of bounded sets
(9) in a locally convex space the convex hull of a bounded set.

PROOF. Easy. □

Remark 6.3 (Warning)**.** In other metrizable topological vector space such that an normed spaces, metric balls are not bounded in this sense. of definition 4.20 mukaisessa topologisessa mielessä, by the Kolmogorov theorem :: j

Theorem 6.4. *Any lc Hausdorff space with a bounded set having an interior point is normable.*

PROOF. Let E locally convex. By the translation inbvariance of the topology, one can take the interior point ti be the originm so there exists a bounded neighbourhood $U \in \mathcal{U}_0$. There exists a barrel $V \in \mathcal{U}_0$, such that $U \supset V$. Let p be its gauge which is a seminorm. By assumption p is bounded in U, so $V \subset U \subset \lambda V$ for some $\lambda > 0$. So p defines already by itself the topology of E. By Hausdorff, seminorms is a norm. □

Definition 6.5. A subset of a topological space $A \subset E$ is *totally bounded*, if for all environments $U \in \mathcal{U}_0$ there exist finitely many points $x_1, \ldots, x_n$ such that

$$A \subset \bigcup_{i=1}^{n} (x_i + U).$$

Example 6.6. he following are totally bounded:
(1) finite sets,
(2) compact sets,
(3) Cauchy-sequences
(4) a balanced hull of a totally bounded set
(5) a closure of a totally bounded set
(6) a subset of a totally bounded set
(7) a continuous image of a totally bounded set
(8) a finite union of a totally bounded set
(9) a finite sum of totally bounded set
(10) in a locally convex the convex hull of a totally bounded set

Every totally bounded set is bounded.

PROOF. Mostly easy exercises. The convex hull is more difficult (I have to look for a proof in books?)

Remark 6.7 (Warning)**.** In a normed spacem the unit ball is bounded but never totally bounded (except in finite dimension).

Definition 6.8. A set absorbing all bounded sets, is called a *bornivorous* set.

6.2. **Bounded linear mappings.**

Definition 6.9. A linear mapping $L : E \to F$ is called*bounded*, if it maps all bounded sets to bounded sets.

Remark 6.10. A continuous linear mapping is bounded, but in some space such that ere are also other bounded lin mappings. The next theorem gives a clue on to find an example.

Theorem 6.11. *A lc space E is bornological (Ks. kohta 5.19 tai alla), if and only if for every locally convex space F every bounded linear mapping $T : E \to F$ is continuous.*

Proof. Let the locally convex space E be bornological. By the definition in 5.19 its every convex, balanced set, absorbing all bounded sets (so every convex, balanced, bornivorous set) is a neighbourhood of the origin. Let $T : E \to F$ be a bounded linear mapping, where F is locally convex. Let $U \in \mathcal{U}_F$. We can assume that U is a barrel. Now $T^{-1}U$ is balanced and convex, so it is sufficient to prove that it is bornivorous. Let $A \subset E$ be bounded. Then $T(A)$ is by assuption bounded in F, so $T(A) \subset \lambda U$ for some $\lambda > 0$. Obviously $A \subset T^{-1}(\lambda U) = \lambda T^{-1}(U)$. So $T^{-1}(U)$ is a neighbourhood of the origin, so T is continuous.

To prove the inverse, assume that every bounded linear mapping $T : E \to F$ is continuous. Apply this ti the identical mapping $T : E \to E$, where the image E is equipped with a locally convex topology τ, where a basis of neighbourhoods of the origin are all bounded, balanced, convex, bornivorous sets in E. Every originally bounded set in E is bounded also in thei topology. So T is a bounded mapping, so T is continuous, and so every balanced, convex, bornivorous set is a neighbourhood of the origin in the original topology. □

Example 6.12. Every metrizable locally convex space E, in particular every normed space is bornological.

Proof. Consider a countable basis $U_1 \supset U_2 \supset \dots$ of neighbourhoods of the origin in E. Let $A \subset E$ be a convex, balanced, bornivorous set, so A absorbs all bounded sets. Let us prove that A is aneighbourhood of the origin. It is sufficient to prove that $nA \supset U_n$ for some $n \in \mathbb{N}$. If not, then $U_n \setminus nA \neq \emptyset$ for all n, so there exists a sequence of points $x_n \in U_n \setminus nA$. Then $x_n \to 0$, so $(x_n)_{\mathbb{N}}$ is bounded and so A absorbs it and there exists $\lambda > 0$, for which every $x_n \in \lambda A$. Choose $m \in \mathbb{N}$ larger than λ, so every $x_n \in mA$. In particular $x_m \in mA$ in contradiction to the construction of the sequence (x_n). □.

Corollary 6.13. *A linear mapping from a normed space to a locally convex spacde is continuous if and only if it maps the unit sphere to a bounded set.*

7. Duals, dual pairs and dual topologies

7.1. **Duals.**

Definition 7.1. The *(algebraic) dual* of a vector space E is the vector space $E' = \{f : E \to \mathbb{K} \bigm| f \text{ is a linear mapping } \}$. clearly $E' \subset \mathbb{K}^E$. For $x \in E$ and $x' \in E'$, we often write

$$x'(x) = \langle x, x' \rangle.$$

The *(topological) dual* of a topological vector space E is the vector space $E^* = \{f : E \to \mathbb{K} \mid f \text{ is a continuous linear mapping }\}$. Obviously $E^* \subset E'$.

The weak topology is the subspace topology from the product topology in $\mathbb{K}^E$. In E' it is denoted $\sigma(E', E)$ and in E^* $\sigma(E, E^*)$. The reason for this notaion comes from the generalizzations considered below.

7.2. **Dual pairs.**

Definition 7.2. (a) Let E and F be vector spaces. A mapping

$$\langle \cdot, \cdot \rangle : E \times F \to \mathbb{K}$$

is called a

(1) *bilinear form* or a
(2) *duality*; we also say that
(3) $((E, F), \langle \cdot, \cdot \rangle)$ is a *dual pair*,

if all partial mappings

$$\langle \cdot, y \rangle : E \to \mathbb{K} : x \mapsto \langle x, y \rangle \quad \text{and}$$
$$\langle : F \to \mathbb{K} : y \mapsto \langle x, y \rangle$$

are linear. This mean such that at $\langle \cdot, y \rangle \in E'$ and $\langle \cdot, y \rangle \in F'$.

Example 7.3. Examples of dualities.

a) Any vector space and its algebraic dual have a *canonical duality*
$$E \times E' \to \mathbb{K} : (x, x') \mapsto \langle x, x' \ \rangle = x'(x).$$
b) If E is a topological vector space, then also the restriction of the canonival duality to the pair (E, E^*) is called a *canonical duality*.
c) Inner products are (separable) dualities.more later

Remark 7.4. The mappings $E \to F' : x \mapsto \langle x, \cdot \rangle$ and $F \to E' : x \mapsto \cdot, y \rangle$ are linear, generally neither in- nor surjective. Injektiity is called "separation" and surjectivity has to do with "reflexivity".

Definition 7.5. A duality $\langle \cdot, \cdot \rangle$ is *separating* or *separates* (points in) F, if for all $x \in E \setminus \{0\}$ there exists $x \in E$, such that $\langle x, y \rangle \neq 0$. In the same way for E. The duality is *separable* if it separates both sides.

Remark 7.6. A duality $\langle \cdot, \cdot \rangle$ separates F, if it satisfies the following equivalent conditions:

(1) If $\langle x, y \rangle = 0$ for all $x \in E$, then $y = 0$.
(2) $\operatorname{Ker}(E \to F' : x \mapsto \langle x, \cdot \rangle) = \{0\}$.
(3) The mapping $E \to F' : x \mapsto \langle x, \cdot \rangle$ is an injection, so as a vector spadce we cdan interprete $E \subset F'$. (No topology here, the, algebraic dual!)

similarly for E avaruuden E.

Example 7.7. The canonical algebraic duality (E, E') obviously separates E' and so does (E, E^*) for E^*. The canonical algebraic duality (E, E') also separates E, which can be checked using a Hamel basis, since the Hamel coordinates do it. But the topological canonical duality (E, E^*) does NOT always separate E, in fact the dual E^* coulod be $\{0\}$. For locally convex space such that is problem does not arise:

Theorem 7.8. *For a locally vonvex Hausdorff–space, in particular a normed space, the canonical duality (E, E^*) separates E^*:n.*

Proof. Apply Hahn–Banach.s corollary 3.5. (XXX)

7.3. **Weak topologies in dualities.**

Definition 7.9. *The weak topology* in E from a duality $((E, F), \langle\cdot,\cdot\rangle)$ is denoted by

$$\sigma(E, F)$$

and defined as the lc topology in E by the seminorm family

$$\{p_y = |\langle\cdot, y\rangle| \ \big| \quad y \in F\}.$$

Sinilarly for F the weak topology $\sigma(F, E)$ comes from the seminorms

$$\{p_x = |\langle x, \cdot\rangle|, \quad x \in E\}.$$

By **??** the weak topology $\sigma(E, F)$ is Hausdorff if and only if the eduality separates E.

Remark 7.10. The weak topology is defined such that, for all $y \in F$, any linear mapping $\langle\cdot, y\rangle \in E'$ is continuous $(E, \sigma(E, F)) \to \mathbb{K}$, same as

$$\langle\cdot, y\rangle \in E^*_{\sigma(E,F)}.$$

This defines a linear mapping

$$F \to E^*_{\sigma(E,F)} : \quad y \mapsto \langle\cdot, y\rangle.$$

This mapping is — by definition – an injection if and only if the duality separates E. By the following theorem 7.12 it is ALWAYS surjective. We need a linear algebraic lemma:

Lemma 7.11. *Let E be a vector space and E' its algebraic dual, and $y, y_1, \ldots, y_n \in E'$. The following conditions are equivalent to each other:*

(7.1) *y is a linear combination of the linear forms$y_1, \ldots, y_n$.*

(7.2) $$\mathrm{Ker}\,(y) \supset \mathrm{Ker}\,(y_1) \cap \cdots \cap \mathrm{Ker}\,(y_n)$$

Proof. Assume that $\mathrm{Ker}\,(y) \supset \mathrm{Ker}\,(y_1) \cap \cdots \cap \mathrm{Ker}\,(y_n)$. Osoitetaan induktiolla n:n suhteen, että joillakin $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ is $y = \sum_{i=1}^n \lambda_i y_i$.

Tapaus $n = 1$: Assume that $\mathrm{Ker}\,(y) \supset \mathrm{Ker}\,(y_1)$. We can assume that neither form is 0, so their kernels are hyperplanes. Therefore $\mathrm{Ker}\,(y) = \mathrm{Ker}\,(y_1)$. The forms y and y_1 kcan be factored as

$$\begin{array}{ccccc} & & E/\mathrm{Ker}\,y & & \\ & \varphi \nearrow & & \searrow I & \\ E & & & & \mathbb{K} \\ & \varphi_1 \searrow & & \nearrow I_1 & \\ & & E/\mathrm{Ker}\,y_1 & & \end{array}$$

Since $\mathrm{Ker}\,(y) = \mathrm{Ker}\,(y_1)$, then $E/\mathrm{Ker}\,y_1 = E/\mathrm{Ker}\,y$ and so $\varphi_1 = \varphi$. Since $E/\mathrm{Ker}\,y \sim \mathbb{K}$, then $E/\mathrm{Ker}\,y$ is one dimensional, so the isomorfisms I and I_1 differ by a constant multiple only.

Step $n \to n+1$: Let $H = \operatorname{Ker} y_{n+1}$ and $g_k = y_k \big|_H$, $g = y \big|_H$ and $g = y \big|_H$ for all $k = 1, \dots, n+1$. In $H = \operatorname{Ker} y_{n+1}$ by assumption $\operatorname{Ker}(g) \supset \operatorname{Ker}(g_1) \cap \dots \cap \operatorname{Ker}(g_n)$, so for some $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ is $g = \sum_{i=1}^n \lambda_i g_i$. Since

$$H = \operatorname{Ker} y_{n+1} \subset \operatorname{Ker} \big(y - \sum_{i=1}^n \lambda_i y_i\big),$$

there exists $\lambda_{n+1} \in \mathbb{K}$, such that $y - \sum_{i=1}^n \lambda_i y_i = \lambda_{n+1} y_{n+1}$. □

Theorem 7.12. *Let $((E, F), \langle \cdot, \cdot \rangle)$ be a duality. The natural mapping*

$$F \to E^*_{\sigma(E,F)} : y \mapsto \langle \cdot, y \rangle,$$

is a surjection.

Proof. Let $x^* \in E^* := (E, \sigma(E, F))^*$. The seminorm $|\langle \cdot, x^* \rangle|$ is continuous in the seminorm spae $(E, \sigma(E, F))$, so bythe characdterization 2.3(9) there exists $\varepsilon > 0$ and topology $\sigma(E, F)$ defining seminorms

$$|\langle \cdot, y_1 \rangle|, \dots, |\langle \cdot, y_n \rangle|,$$

with $y_1, \dots, y_n \in F$, such that for all $x \in E$

$$\varepsilon |\langle \cdot, x^* \rangle| < |\langle \cdot, y_1 \rangle| + \dots + |\langle \cdot, y_n \rangle|.$$

In particular

$$\operatorname{Ker} \langle \cdot, y_1 \rangle \cap \dots \cap \operatorname{Ker} \langle \cdot, y_n \rangle \subset \operatorname{Ker} \langle \cdot, x^* \rangle.$$

By the previous lemma 7.11 x^* is a linear combination lineaarimuodoista of the liner forms $\langle \cdot, y_n \rangle$, so there exist numbers $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, such that

$$\forall x \in E : \quad \langle x, x^* \rangle = \sum_{i=1}^n \langle x, \alpha_i y_i \rangle$$

and so $x^* = \langle \cdot, \sum_{i=1}^n \alpha_i y_i \rangle$ is an eloemt of the image of F. □

Corollary 7.13. (FAN I XX.) *Let E be locally convex Hausdorff-space — for example normed — and $x^* \in E'$ its linear form. The following are equivalent:*

(i) *x^* is continuous, eli $x^* \in E^*$.*
(ii) *the seminorm $|\langle \cdot, x^* \rangle|$ is continuous.*
(iii) *x^* is* weakly continuous *same as continuous as a mapping from E withthe weqk topology $\sigma(E, E^*)$ varustetusta avaruudesta to $\mathbb{K}$.*

Remark 7.14. BE CAREFUL: The dual E^* in the sense of the topolgy $\sigma(E^*, E)$ mielessä — we coulod call it $(E^*_{\sigma(E^*,E)})^*$ — is in the case 7.13 the original space E, and not E^{**}, unless E happens to be reflexive.

Theorem 7.15. *Let $((E, F), \langle \cdot, \cdot \rangle)$ hbe a duality,separating E. The following are equivalent:*

(i) *$\langle \cdot, \cdot \rangle$ separates also F.*
(ii) *E is dense in $F'_{\sigma(F',E)}$.*

Proof. Let (i) be true. Let f cbe a ontinuous linear form in $F'_{\sigma(F',E)}$ such that $E \subset \operatorname{Ker} f$. Since f is continuous, $\operatorname{Ker} f$ is closed and so also $\bar{E} \subset \operatorname{Ker} f$. By the surjectivity theorem 7.12 there exists $y \in F$ such that $f(y') = \langle y', y \rangle$ for all $y' \in F'$. In particular $\langle y', y \rangle = 0$ for all $y' \in E$. Since theduality (E, F) separates E we conclude $y = 0$. So $f = 0$.

We know now that the only extensionof the zero form $\bar{E} \to \mathbb{K}$ to a continuous linear form in $F'_{\sigma(F',E)}$ is 0, so by Hahn-Banach 3.5 we must have $\bar{E} = F'$.

The proof of the inverse is a straight-forward verification. □

Theorem 7.16. *Let $((E, F), \langle\cdot,\cdot\rangle)$ be a duality, separating F and let $d \subset F$ be a proper subspace. The weak topology $\sigma(E, F)$ is strictly finer than $\sigma(E, G)$.*

Proof. HT Use 7.12. □

Theorem 7.17. *Let E be a vector space. $(E', \sigma(E', E))$ is algebraically and topologically isomorphic to $\mathbb{K}^K$, where K is a Hamel basis of E.*

Proof. Every linear mapping $x' : E \to \mathbb{K}$ is determined by the values on the basis vectors, and these can be any numbers. So x' can be interpered as a mapping $K \to \mathbb{K}$ same as an element of $\mathbb{K}^K$the topology is given by *the seminorms of pointwise convergence* same as the *absolute values of the evaluation functionals* $p_x(x') = |x'(x)|$, but in the product space $\mathbb{K}^K$ the vector x goe such that roughonly a basisa and in E' all of E. But this makes no difference, since all vectors are finite linear combinations of the basis vectors. □

Theorem 7.18. *Let $((E, F), \langle\cdot,\cdot\rangle)$ be a duality, separating E, so ä $E \subset F'$. Equip the algebraic dual F' with the topology $\sigma = \sigma(F', F)$ and E with the topology $\sigma = \sigma(E, F)$, (so $E \subset \sigma(F', F)$ with induced subspace topology.) Now:*

a) The completion of E_σ is the closure of E:in a F'_σ.

b) In particular, if the duality separates also F, then the completion of E_σ is F'_σ itself.

Proof. a) Since $\mathbb{K}$ is complete, also every product space $\mathbb{K}^K$ is complete, and by the previous theorem 7.17 mukaan $F'_{\sigma(F',E)}$ is isomorphic to such a product, so it is complete. So the completion of E is the closure in F'_σ.

b) Assume that the duality is separates both sides, and remember that in this case E is dense in F'_σ [7.15]. Therefore its closure is F'_σ. □

Corollary 7.19. *A locllay convex Hausdorff-space E never is "weakly complete" unless it is finite dimensional.*

Proof. Since E is a locally convex Hausdorff-space,the duality (E, E^*) is separating. (Ks. 7.8.) Therefore the completion of E_σ:n is the algebraic dual $(E^*)'$, which in the infinite dimensional case is not only E. (HT: verifioi. XXX)

Theorem 7.20. *Let (E, F) be a separating dual pair. For a set $A \subset E$ the following are equivalent in the weak topology $\sigma(E, F)$:*

(i) *A is bounded.*
(ii) *In the completion $\tilde{E}_\sigma$ the closure $\bar{A}$ is compact.*

Sometimes (not in all books) a subset, whose closure is compact, is called *relatively compact*, and a set set, whose closure in the completion of the original space is compact, is called *precompact*. In these words, the previous theorem expresse such that at in the weak topology of a separating dual pair every bounded set is precompact.

(Obviously compact $\implies$ relatively compact $\implies$ precompact $\implies$ totally bounded $\implies$ bounded.)

Proof. Every compact set is totally bounded, so also every relatively compact be bounded. By **??** the completion of E_σ: is $F'_\sigma \sim \mathbb{K}^K$ and so A is bounded also in the product topology of $\mathbb{K}^K$. Since the projections are continuous, every $\pi_\alpha(A) \subset \mathbb{K}$ is bounded and so its closure is compact. So $A \subset \prod_\alpha \pi_\alpha(A) \subset \prod_\alpha \overline{\pi_\alpha(A)}$, jwhich by Tihonov's famous theorem is compact. So $\bar{A}$ is compact in the completion $\tilde{E}_\sigma$. $\square$

7.4. **Polars.**

Remark 7.21. Remember form Functional analysis *Alaoglu's theorem* by which the unit ball of a normed space is weakly compact. We generalize it.

Definition 7.22. Let $((E, F), \langle\cdot,\cdot\rangle)$ be a dual pair .

1. The *polar* of a set $A \subset E$ is the set $A^\circ = \{y \in F \mid \langle x, y\rangle| \le 1 \quad \forall x \in A\}$.
2. The polar of the polar of A is denoted $(A^\circ)^\circ = A^{\circ\circ}$ and called the *bipolar* of A.
3. Two sets $A \subset E$ and $B \subset F$ are *orthogonal to each other*, if $\langle x, y\rangle = 0$ eli $x \perp y$ for all $x \in A$ and $y \in B$.
4. The *orthogonal complement* of a set $A \subset E$ is the set $A^\perp = \{y \in F \mid \langle x, y\rangle = 0 \ \forall x \in A\}$.

Remark 7.23. The polar A° and bipolar $A^{\circ\circ} = (A^\circ)^\circ$ have the following properties:

(i) $A \subset B \implies B^\circ \subset A^\circ$.
(ii) $(\operatorname{bal} A)^\circ = A^\circ$
(iii) $A \subset A^{\circ\circ}$
(iv) $A^{\circ\circ\circ} = A^\circ$
(v) A° is balanced, convex and $\sigma(F, E)$–closed.
(vi) $\left(\bigcup_{i\in I} A_i\right)^\circ = \bigcap_{i\in I} A_i^\circ$.
(vii) $(\alpha A)^\circ = \frac{1}{\alpha} A^\circ$
(viii) If A is a subspace, then $A^\circ = A^\perp$.
(ix) $E^\circ = \{0\}$ if and only if the duality separates F.
(x) A° is absorbing if and only if A is $\sigma(E, F)$ − bounded

Proof. All directly from the definitions. In particular, a polar is weakly closed since A° is the intersection of closed unit semiballs in the weak topology, $\{y \in F \mid |\langle x, \cdot\rangle| \le 1\}$ $(x \in A)$. Proving (x) iremember that in a lc space (in particular in weak topologies) a set is bounded tasan silloin, kun every defining (or any continuous) seminorm is bounded in the set.

Theorem 7.24. Banach–Alaoglu–Bourbaki *In a tvs E, the polar of a neighbourhood of the origin polar U° is weakly (same as $\sigma(E^*, E)$) compact.*

Proof.

The topological dual l $E^*_\sigma = E'_{\sigma(E',E)}$ is a topological subspace of the algebraic dual $E'_{\sigma(E',E)}$, so we only have to prove that U° is compact in the topology induced by E'_σa. One may notice that U° is defined in the duality (E, E^*) But in fact the polars of U are the same in both dualities (E, E^*) and (E, E'), since if $x' \in E'$ such that $|\langle \cdot, x' \rangle| \leq 1$ in the neighbourhood of the origin $U \subset E$, then x is continuous so $x' \in E^*$. Since $U \subset U^{\circ\circ}$, the bipolar $U^{\circ\circ}$ is a neighbourhood of the origin and so absorbing. By 7.23 (x) U° is bounded in E'_σ. Since the dualityu $/E.E')$ separates, we can use 7.20 to see that U° is compact in the completion of E'_σ, but E'_σ is already complete. □

Corollary 7.25. *llllll„ ,*

Proof. „ , □

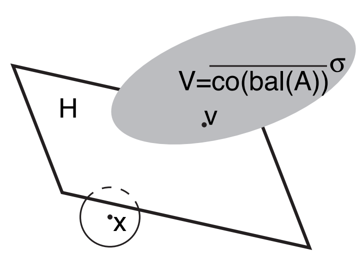
Theorem 7.26. *The bipolar of any set A is its balanced, convex $\sigma(E, F)-$closed hull:*

$$A^{\circ\circ} = \overline{\mathrm{co}(\mathrm{bal}(A))}^\sigma.$$

Proof. Since the convex hull of a balanced set is balanced, and the closure of a convex, balanced set is convex and balanced, the set $V := \overline{\mathrm{co}(\mathrm{bal}(A))}^\sigma$ on is the smallest balanced, convex, $\sigma(E, F)-$closed set, containing A. By the previou such that erem the bipolar $A^{\circ\circ}$ is balanced, convex and $\sigma(E, F)-$closed, so it contains V.

The main result is the inverse inclusion. If it is false, then there exists

$$x \in A^{\circ\circ} \setminus V.$$



To be able to apply Banach's separation theorem 3.2, take a convex open neighbourhood of x not intersectiong V. There exists an $x^* \in E^*$ and a number $\alpha > 0$ such that

$$\mathrm{Re}\langle x^*, x\rangle > 1 \text{ and}$$
$$\mathrm{Re}\langle x^*, v\rangle \leq 1 \quad \forall v \in V.$$

In particular

$$|\langle x^*, x\rangle| \quad > 1, s \text{ and} \tag{7.3}$$
$$\langle x^*, v\rangle \quad \neq \alpha \quad \forall \alpha > 1,\, v \in V. \tag{7.4}$$

Since the hull V is balanced, (2) implies

$$|\langle x^*, v\rangle| \leq 1 \quad \forall\ v \in V.$$

On the other hand, the weak topology gives the dual (by 3.24 ?)

$$E^* = F.$$

So $x^* \in V^\circ$. And by (1) : $x^* \notin \{x\}^\circ \supset A^{\circ\circ\circ} = A^\circ$. This is a contradiction.

Therefore

$$A \subset V \implies A^\circ \supset V^\circ.$$

□

Corollary 7.27. *If B is balanced, convex and $\sigma(E, F)$-closed, then $B^{\circ\circ} = B$.*

Corollary 7.28. *If the sets A_i $(i \in I)$ are balanced, convex and $\sigma(E, F)$-closed, then $(\bigcap_{i\in I} A)^\circ$ is the balanced, convex and $\sigma(E, F)$-closed hull of $\bigcup_{i\in I} A^\circ$.*

Theorem 7.29. *A convex subset $S \subset E$ of a locally convex space is closed exactly when it is weakly closed.*

Proof. Exercise. Hint: Hahn-Banach-tyype theorem it is an intersection of closed hyperplanes. □

Corollary 7.30. *Let E be locally convex and $\emptyset \neq A \subset E$. in the sense of the duality (E, E^*) $A^{\circ\circ}$ is the balanced, convex, closed hull of A.*

Proof. By the bipolar theorem **??** $A^{\circ\circ}$ is the balanced, convex, weakly closed hull of the set A. By 7.29 it is closed also in the original topology of E. □

Theorem 7.31. Mackey's theorem [17] *A subset $A \subset E$ of a locally convex Hausdorff-space is bounded exactly when it is weakly, same as $\sigma(E, E^*)$ –bounded.*

Proof. Evidently every bounded set is weakly bounded, since the weak topology $\sigma = \sigma(E, E^*)$ is coarser than the original topology τ of E.

Let A be weakly eli $\sigma(E, E^*)-$r bounded. We have to prove, that every $\tau-$ neighbourhood U of the origin absorbs A:n.

(i) Since E is locally convex we can assume, that U is a barrel.
(ii) By the bipolar theorem **??** $U^{\circ\circ} = U$.
(iii) U° is balanced and convex.
(iv) By Alaoglu-Bourbaki theorem 7.26 U° is $\sigma(E^*, E)$ –compact.
(v) So U° is $\sigma(E^*, E)$ – complete, since it is a compact subset of the complete space $(E', \sigma(E', E))$. (Ks also Treves p.53??)
(vi) As a $\sigma(E^*, E)$ –compact set U° is evidently $\sigma(E^*, E)$–bounded.
(vii) Let us define $E'_{U^\circ} = \bigcup_{n\in\mathbb{N}} nU^\circ \subset E$ and equip it with the gauge of the balanced, convex and absorbing (!) set U°, which turns out to be a norm and is denoted $\|\cdot\|$, since U° is bounded in the Hausdorff- topology $\sigma(E^*, E)$.
(viii) $(E^*_{U^\circ}, \|\cdot\|)$ is complete, therefore a Banach-space, see the lemma below(and exercise set 7 or 6).
(ix) A is weakly $= \sigma(E, E^*)-$bounded means, that for all $x^* \in E^*$ the linear form $\langle \cdot, x^*\rangle$ is bounded in the set A, so $\sup_{x\in A} |\langle \cdot, x^*\rangle| < \infty$ at every point $x^* \in E^*$.

[17] George Whitelaw Mackey (February 1, 1916 in St. Louis, Missouri – March 15, 2006 in Belmont, Massachusetts)

(x) For all $x \in E$ the linear form $\langle x, \cdot \rangle$ is evidently continuous in the space $E^*_{\sigma(E^*,E)}$, so in particular $A \subset E^*_{\sigma(E^*,E)}$.

(xi) Since in the space $(E^*_{U^\circ}, \| \cdot \|)$ clearly is a finer topology than the one induced by the weark topology $\sigma(E^*, E)$, the linear form $\langle x, \cdot \rangle$ is continuous also in the also Banach space $(E^*_{U^\circ}, \| \cdot \|)$.

(xii) The family of linear mappings $\{\langle x, \cdot \rangle \mid x \in A\} \subset (E^*_{U^\circ}, \| \cdot \|)^*$ satisfies the assumptions in the principle of uniform boundedness (Banach-Steinhus) **??**: it is pointwise bounded (ix) jfamily of continuous (xi) liner mappings betweeen Fréchet spaces.

(xiii) Therefore the family $\{\langle x, \cdot \rangle \mid x \in A\}$ is equicontinuous

$$\sup_{x \in A,\, x^* \in U^\circ} = \lambda < \infty,$$

which means, that

$$A \subset \lambda U^{\circ\circ} = \lambda U.$$

So A is bounded. □

Theorem 7.32 (Lemma)**.** *Let E be a locally convex Hausdorff-space and B its balanced, convex, bounded subset . Let us define $E_B = \bigcup_{n \in \mathbb{N}} nB \subset E$ and equip it with the gauge of the bal, convex and absorbing (!) set $B \subset E_B$ mittausfunctionlla, which is a norm $\| \cdot \|$. Let us assume, that B is complete in the topology induced by (E, τ) . Claim: $(E_B, \| \cdot \|)$ is a Banach-space.*

Proof. Let $(x_n)_{\mathbb{N}}$ be a Cauchy-sequence in the space $(E_B, \| \cdot \|)$. Since B is complete, it is also closed and therefore a barrel in the topology τ of E_B. So it is the closed unit ball of its gauge = the unit ball inthe normed space $(E_B, \| \cdot \|)$. The Cauchy-sequence $(x_n)_{\mathbb{N}}$ in the space $(E_B, \| \cdot \|)$ is bounded, so it is contained in some ball λB, which is complete by assumption in the topology induced by (E, τ). Now $(x_n)_{\mathbb{N}}$ is a Cauchy-sequence also in the space (E, τ), which follows from, that B to some vector $y \in \lambda B \subset E_B$. But by assumption this implies convergence in the norm topology, since for all $\epsilon > 0$ we have $\|x_n - y\| \leq \epsilon$ for large enough n, since the norm $\| \cdot \|$ is $\tau-$continuous . (Check the proof once more)

7.5. **Compatible topologies.**

Definition 7.33. A locally convex topology τ in E *is compatible with a duality* (E, F), if

$$E^*_\tau = F.$$

Example 7.34. The original topology of a locally convex space is compatible with $\sigma(E, E^*)$. There are others!

a) If E is a locally convex Hausdorff-space, then the weak topology $\sigma(E, E^*)$ is the weakest topology, which is compatible with the duality (E, E^*).
b) If E is a ocally convex Hausdorff-space and τ is a locally convex topology, which is finer than $\sigma(E, E^*)$ and coarser than the original, then τ is compatible with the duality (E, E^*).

Theorem 7.35. *Let (E, F) be a separable duality .*

(i) *A convex set $A \subset E$ has the same closure in all compatible topologies in (E, F).*
(ii) *E ha sthe same barrels in all topologies compatible with (E, F) .*

Proof. Exercise . 7.29.

7.6. **Polar topologies.**

Definition 7.36. Let (E, F) be a separable duality and $\mathfrak{S}$ a family of
weakly bounded sets in E set ja. The *polar topology* (spanned by $\mathfrak{S}$) is the locally convex topology in F, who has a neighbourhood basis of the origin consisting of the finite intersections of polars of the sets belonging to $\mathfrak{S}$, ie. :

$$\mathcal{U}_{\mathfrak{S}} = \{\epsilon \bigcap_I A_i^\circ \bigm| A_i \in \mathfrak{S},\ \epsilon > 0\ I \text{ finite}\}.$$

Synonyms: $\mathfrak{S}$– *topology, topology of* $\mathfrak{S}$*-konvergecen, uniform convergence topology in* $\mathfrak{S}$*–sets.*

Example 7.37. a) The first example is any weak topology $\sigma(F, E)$, which one gets by choosing $\mathfrak{S} = \{A \subset E \bigm| A \text{ is finite}\}$, $\mathfrak{S} = \{A \subset E \bigm| A \text{ has only one point }\}$ or $\mathfrak{S} = \{A \subset E \bigm| A \text{ is the balanced, convex hull of a finite set}\}$. The weak topology is the weakest $\mathfrak{S}-$ topology for (E, F).

b) The second example is the *strong topology* $b(E, F)$, which comes from $\mathfrak{S} = \{A \subset E \bigm| A \text{ is } \sigma(E, F) \text{ bounded}\}$. Yhe strong topology is the finest $\mathfrak{S}-$ topology for (E, F).

c) The usual topology in the dual of a normed space is a polar topology, which one gets from a one element set $\mathfrak{S} = \{\text{1-ball}\}$.

d) The topology of *compact convergence* $c(E, F)$, comes from $\mathfrak{S} = \{A \subset E \bigm| A$ is $\sigma(E, F)$ compact$\}$.

Remark 7.38. a) An $\mathfrak{S}-$topology is locally convex and defined by the gauges of the polars A°, $A \in \mathfrak{S}$, namely $p_A(y) = \sup_{x \in A} |\langle x, y\rangle|$.

b) If $\mathfrak{S}$ satisfies the conditions

(i) $A, B \in \mathfrak{S} \implies \exists\, C \in \mathfrak{S}$ such, that $A \cup B \subset C$ and
(ii) $A \in \mathfrak{S}$, $\lambda \in \mathbb{K} \implies \exists\, B \in \mathfrak{S}$ such, thatt $\lambda A \subset B$,

then $\mathcal{U}_{\mathfrak{S}} = \{A^\circ \bigm| A \in \mathfrak{S}\}$.

c) If $\mathfrak{S}$ satisfies the condition

$$\bigcup_{\mathfrak{S}} A = E,$$

then the $\mathfrak{S}$-topology is finer than the weak topology $\sigma(F, E)$ and therefore Hausdorff.

d) A necessary and sufficient condition for an $\mathfrak{S}$-topology to be Hausdorff is the following:

(i) the duality (E, F) separates F:n and
(ii) the linear hull of the union $\bigcup \mathfrak{S}$ is weakly $= \sigma(E, F)-$dense in the space E.(such sets are sometimes called total sets)

Proof. a)-c): Exercise . The following theorem proves the statements above: 7.39.

d) A locally convex topology is Hausdorff if and only if for everu nonzero vector there exists a defining seminorm with value $\neq 0$ at$x$. In particular a $\mathfrak{S}-$topology is

Hausdorff iff, that if $p_A(y) = 0$ for all $A \in \mathfrak{S}$, then $y = 0$, same as $A^\perp = \{0\}$ for all $A \in \mathfrak{S}$, even shorter $\bigcap_{A\in\mathfrak{S}} A^\perp = \{0\}$.

Assume first that the $\mathfrak{S}-$topology is Hausdorff so $\bigcap_{A\in\mathfrak{S}} A^\perp = \{0\}$. If the duality doesn't separate F then there exists $y \in F \setminus \{0\}$, such that $|\langle x, y\rangle| = 0$ for all $x \in E$. In particular $p_A(y) = \sup_{x\in A} |\langle x, y\rangle| = 0$ for all $A \in \mathfrak{S}$ so $y \in \bigcap_{A\in\mathfrak{S}} A^\perp$ in contradiction to the assumption. If again we assume that the linear hull of the union $\bigcup \mathfrak{S}$, call it V, is not weakly$= \sigma(E, F)-$dense, then there exists $x \in E$ not belonging to $\bar{V}$. Using Mazur **??** there exists a continuous linear form $f \in E^*$, which has the value 1 at x and vanishes in the space $\bar{V}$. By the surjectivity in **??** we can choose $y \in F$ such, that $f(x) = \langle y, x\rangle$ for all $x \in E$, so $y \in \bigcap_{A\in\mathfrak{S}} A^\perp \setminus \{0\}$ in contradictionti the assumption.

Assume next that thet (i) and (ii) are true. The linear hull V of the union $\bigcup \mathfrak{S}$ is i $\sigma(E, F)-$dense, so the only linear form vanishing in all $A \in \mathfrak{S}$ is $f = 0$. By the separation proof (11) there exists only one $y \in F$ such that $f(x) = \langle y, x\rangle = 0$ for all $x \in E$, and that particular y is evidently 0. So if $y \in F$ such, that $P_A(y) = 0$ for all $A \in \mathfrak{S}$, then $y = 0$, so $E_{\sigma(E,F)}$ is Hausdorff. □

Theorem 7.39. *The $\mathfrak{S}$-topology does not change, if the sets in $\mathfrak{S}$ are replaced in the following ways:*

a) *Add subsets of sets in $\mathfrak{S}$.*
b) *Add finite unions of sets in $\mathfrak{S}$.*
c) *Add sets $\lambda A,\ A \in \mathfrak{S},\ \lambda \in \mathbb{K}$.*
d) *Add balanced hulls of sets in $\mathfrak{S}$.*
e) *Add $\sigma(E, F)-$ closures of sets in $\mathfrak{S}$.*
f) *Add bipolars of sets in $\mathfrak{S}$.*

Proof. .

a) $B \subset A \in \mathfrak{S} \implies A^\circ \subset B^\circ \implies A \in \mathcal{U}_{\mathfrak{S}}$.
b) $A, B \in \mathfrak{S} \implies (A \cup B)^\circ = A^\circ \cap B^\circ \in \mathcal{U}_{\mathfrak{S}}$
c) Kun $\lambda \neq 0$, then $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ \in \mathcal{U}_{\mathfrak{S}} tai$. Kun $\lambda = 0$, then $(\lambda A)^\circ = \{0\}^\circ = F \in \mathcal{U}_{\mathfrak{S}}$.
d) $(\operatorname{bal} A)^\circ = A^\circ \in \mathfrak{S}$.
e) $(\bar{A}^\sigma)^\circ = A^\circ \in \mathfrak{S}$, since by the bipolar theorem **??** $A \subset \bar{A}^\sigma \subset A^{\circ\circ}$, so $A^\circ = A^{\circ\circ\circ} \subset (\bar{A}^\sigma)^\circ \subset A^\circ$.
f) Bipolars are by the bipolar theorem the same thing as weakly closed, balanceds, convex hulls.

□

Theorem 7.40. *In every locally convex Hausdorff-space E the topology τ is an $\mathfrak{S}-$ topology, where $\mathfrak{S} = \{A \subset E^* \mid A$ is equicontinuous $\}$.*

Proof. By definition 4.23 mukaan $A \subset E^*$, is equicontinuous, if

$$\forall U \in \mathcal{U}_{\mathbb{K}}\, \exists V \in \mathcal{U}_\tau \text{ such, that } f(V) \subset U \text{ for all } f \in A.$$

Ccoosing $U = \{\lambda \in \mathbb{K} \mid |\lambda| \leq 1\}$ huomaa, that $A \subset E^*$, is exactly then equicontinuous, when $A \subset U^\circ$ for some $U \in \mathcal{U}_\tau$. Every equicontinuous set $A \subset E^*$ is by Alaoglu-Bourbaki weakly compact, so weakly bounded, so at least the $\mathfrak{S}-$ topology exists.

Take a neighbourhood basis $\mathcal{K}$ for the original topology τ in E, consisting of barrels. By the bipolar theorem **??** and the theorem by which the barrels are the same in all compatible topologies, $T = T^{\circ\circ}$ for all $T \in \mathcal{K}$. Since $T \in U_\tau$, we see that $T^\circ \subset E^*$ is equicontinuous. So we have proven, that T is the polar of some equicontinuous set and therefore a neighbourhood of the origin in the $\mathfrak{S}-$topology. The original topology is therefore coarser than the $\mathfrak{S}-$ topology. The reverse implication goes easily: Let $A \in \mathfrak{S}$, so it is equicontinuous. This means $A \subset U^circ$ for some $U \in \mathcal{U}_\tau$. We can take U to be a barrel,and use the bipolar theorem to find out that a $U = U^{\circ\circ}$ and therefore $A^\circ \supset U^{\circ\circ} = U \in \mathcal{U}_\tau$. So the $\mathfrak{S}-$topology is coarser than the original topology. □

Remark 7.41 (Remember: 7.20)**.** Let (E,F) be a separable duality . Then in the wea topology $\sigma(E,F)$ a set $A \subset E$ is bounded exactly when it is totally bounded.

Theorem 7.42. Mackeyn and Arensin theorem[18] *Let (E,F) be a separable duality and τ ja locally convex topology in E. Necessary and sufficient for t $E^*_\tau = F$ (ie τ compatible with (E,F)) is, that τ is an $\mathfrak{S}$- topology, where $\mathfrak{S}$ can be taken such, that the following 2 conditions hold:*

(i) $\bigcup \mathfrak{S} = E$
(ii) *Every $A \in \mathfrak{S}$ is balanced, convex and $\sigma(F,E)$-compact.*

Proof. Later— not really bad . □

Definition 7.43. Let (E,F) be a duality, which separates F. *The Mackey topology* $\tau(E,F)$ is the $\mathfrak{S}-$topology in the space E (this way!) where $\mathfrak{S} = \{A \subset F \mid A$ is balanced, convex and $\sigma(E,F)$-compact$\}$.

A locally convex space E, with the Mackey topology of the duality $(E.E^*)$ is called a *Mackey space.*

Theorem 7.44. *Let (E,F) be a dual pair separating F. The Mackey topology is the finest topology compatible with the duality (E,F).*

Proof. Corollary of Mackeys and Arens's theorem (HOW???) □

Theorem 7.45. *A locally convex space E is a barreled space exactly when its topology coincides with the strong topology of the duality dualiteetin (E,E^*) (ie with $b(E,f)$).*

Theorem 7.46. *Every barreled space is a Mackey space. (So is every bornological space.)*

Corollary 7.47. *Since every metrisable locally convex space is bornological, every metrisable locally convex space is a Mackey space.*

PART II Distributions

1. The idea of distributions

1.1. Schwartz test functions and distributions.

[18] Richard Friederich Arens 24 April 1919 – 3 May 2000 Saksa → USA.

Remark 1.1. Distributions make ist possible to define a derivative for almost any function - but the derivative will not be a function, it is a measure and higher derivatives are —- distributions!. The theory works fine in any dimension or even Banach spaces, but we do it in $\mathbb{R}^1$ for clarity. There is no real difference to the general case.

Integrals are Lebesgue

Let us begin by a heuristic description of what we want.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a *locally inegrable* function, ie we assume, that the integral $\int_K f$ exists always, when $K \subset \mathbb{R}$ is compact. Notice that f defines for each " test function" φ a number

$$\langle \varphi, f \rangle := \int f\varphi$$

and the mapping

$$\varphi \mapsto \langle \varphi, f \rangle := \int f\varphi$$

is linear and continuous wrt φ. The function f can be identified withthis mapping, if we have enough test functions to have different values for at least some $\langle f, \varphi \rangle$ if f aris altered in a positive measure set. This is true for instance in the case, when all infinitely differentiable functions with compact support are taken into the set of test functions.

If f is infinitely differentiable, then

$$\int f'\varphi = -\int f\varphi',$$

where the other summand is left out, since the product function has compact support. This gives a reason to define the derivative of a distribution

$$f \,:\, \varphi \mapsto \langle \varphi, f \rangle := \int f\varphi$$

to be the "distributioni"

$$f' \,:\, \varphi \mapsto \langle \varphi, f' \rangle := -\int f\varphi'.$$

One could be tempted to to define general distributions by repeating this procedure. This is not the fulltruth. Also,one has to be careful withthe choice of the space of test functions and its topology – these is room for alternatives. In particular it would be nice to define the topology such that differentiation becomes a continuous linear operator – something not comon in usual topological function spaces.

2. Schwartz's test function space

[19]

[19]Laurent-Moïse Schwartz 1915–2002, France. Book Théorie des Distributions 1950-1951.

2.1. **The spaces $\mathcal{C}^\infty(\Omega)$ and $\mathcal{D}_K$.**

Definition 2.1. Denot differentiationby D, so the k:th derivative of a function $\varphi : \mathbb{R} \to \mathbb{R}$ is $D^k\varphi = \varphi^{(k)}$.[20]

The φ *support* $\operatorname{supp}(\varphi)$ of a function φ is the closure of the set $\{x \mid \varphi(x) \neq 0\}$.

In the following we **always have $\Omega \subset \mathbb{R}$ open and $K \subset \Omega$ compact.** Define

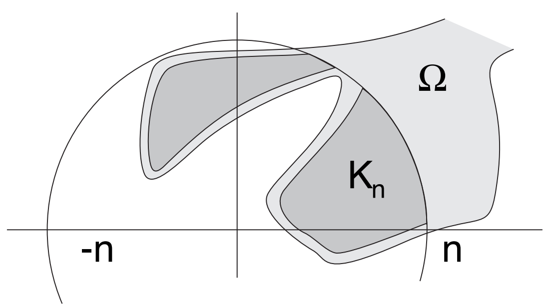
$$\mathcal{C}^\infty(\Omega) := \{\varphi : \Omega \to \mathbb{R} \mid \forall k : \exists D^k\varphi\}$$
$$\mathcal{D}_K := \{\varphi \in \mathcal{C}^\infty(\Omega) \mid \operatorname{supp}(\varphi) \subset K\}.$$

Equip $\mathcal{C}^\infty(\Omega)$ with its *metric standard topology* also called *compact* $\mathcal{C}^\infty$*−convergence*. The space $\mathcal{D}_K$ i such that en its linearand topological subspace.

Definition 2.2. Let $K_1, K_2 \dots$ be compact sets such, that

$$K_1 \subset \operatorname{int} K_2, \quad K_2 \subset \operatorname{int} K_3, \dots \subset \Omega, \quad \text{and}$$
$$\Omega = \bigcup_{i \in \mathbb{N}} K_i,$$

for example [21]



$$K_i := \{x \in \Omega \mid \|x\| \leq i \quad \text{and} \quad \mathrm{d}(x, \partial\Omega) \leq \frac{1}{i}\}.$$

The seminorms — they are in fact norms —

$$p_i(\varphi) := \sup\{\, |D^k\varphi(x)| \mid k \leq i,\, x \in K_i\} \quad i \in \mathbb{N}$$

define in $\mathcal{C}^\infty(\Omega)$ a locally convexn topology, which is independent of the choice of the sequence $K_1, K_2, \dots$. We call it the topology of *compact* C^∞*–convergence* topology i, since $\varphi_n \to \varphi$ means, that $\varphi_n(x) \to \varphi(x)$ with all derivatives uniformly in any compact set $K \subset \Omega$.[22]

Remark 2.3. The topology of compact C^∞–convergence in the spaces $\mathcal{C}^\infty(\Omega)$ and $\mathcal{D}_K$ has the following properties:

(1) $\mathcal{C}^\infty(\Omega)$ and $\mathcal{D}_K$ are metrizable.
(2) The *Evaluation functionals* $\varphi \mapsto \varphi(x)$ are continuous. ($x \in \Omega$ or K)

[20] Here k isa natural number. Multi-indices k tare needed for higher dimensions.

[21] The drawing is 2- dimensional and not 1–dimensional - hope to make clear the general case. $\mathbb{R}^n$.

[22] By definitionin the K_i, but really in all compact $K \subset \Omega$, since all these have positive distance from the boundary of Ω.

(3) $\mathcal{D}_K$ is a closed subspace of $\mathcal{C}^\infty(\Omega)$.
(4) $\mathcal{C}^\infty(\Omega)$ is complete, so it si a Fréchet–space.
(5) The subspace $\mathcal{D}_K \subset \mathcal{C}^\infty(\Omega)$ is closed. So also every $\mathcal{D}_K$ is a Fréchet–space.

Proof. Metrizability is evident since there are denumerably many defining seminorms. Continuity of the evaluation functionals is clear by 6.2.4.

$$\mathcal{D}_K = \bigcap_{x \notin K} \text{Ker } (\varphi \mapsto \varphi(x))$$

is a closed subspace, since it is an intersection of closed hyperplanes.

To prove completeness of the spaces $\mathcal{C}^\infty(\Omega)$ ttake a Cauchy-sequence $(\varphi_n)_{n\in\mathbb{N}}$. Let $i \in \mathbb{N}$. in the sense of the seminorm p_i the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy, so in particular the sequence by assumption converges uniformly in the compact set K_i. The same applies to its derivatives up to degree i. By a teheorem from alalysis, the liit function g is differentiabne equally many times and $D^k\varphi_n \to D^k g$ uniformly in compact sets . By definition, this happens C^∞- in every $x \in \Omega$ neighbourhood and all degrees. That is what we wanted .

Remark 2.4. The seminorms p_j are norms in $\mathcal{D}_{K_i}$, $i \le j$, and the sequence $(p_j)_{j\in N}$ is even increasing, but the spaces $\mathcal{D}_{K_i}$ are not normed spaces. This can be seen by noticing that they are infinite dimensional but nevertheless have the *Heine-Borel property*, [23] stating that every closed set is contained in some compact set.

Proof. Exercise 8

2.2. **Schwartz's test function space $\mathcal{D}(\Omega)$.**

Definition 2.5.

$$\mathcal{D}(\Omega) = \{\varphi \in \mathcal{C}^\infty(\Omega) \;\big|\; \text{supp}(\varphi) \text{ on compact} \subset \Omega\} = \bigcup_{i=1}^{\infty} \mathcal{D}_{K_i}.$$

Remark 2.6. $\mathcal{D}(\Omega) \subset \mathcal{C}^\infty(\Omega)$. For compactly supported functions, the standard norms[24]

$$p_i(\varphi) := \sup\{\, |D^k\varphi(x)| \;\big|\; k \le i,\, x \in K_i\} \quad i \in \mathbb{N}$$

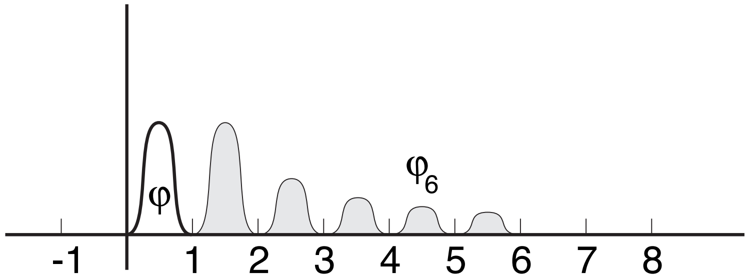
of

$\mathcal{C}^\infty(\Omega)$ can be replaced by the norms

$$\|\varphi\|_i := \max\{|D^k\varphi(x)| \;\big|\; k \le i,\, x \in \Omega\,\}$$

and these define the $\mathcal{C}^\infty-$convergence topology $\mathcal{C}^\infty(\Omega)$, which is metrisable and induces into the subspaces $\mathcal{D}_K$ the topology used already above. One is tempted to use the same topology for test functions as well, but there is a drawback: the topology of $\mathcal{C}^\infty-$ does not make the space $\mathcal{D}(\Omega)$ complete — not even sequentially. A countrexaple is easily found: : Take $\Omega = \mathbb{R}$. There exists a $\mathcal{C}^\infty$–function $\varphi : \mathbb{R} \to [0,1]$, which takes on $[\frac{1}{4}, \frac{3}{4}]$ the value 1 and outside $[0,1]$ the value 0. The sequence $(\varphi_n)_{n\in\mathbb{N}}$: turns out to be Cauchy but not convergent.

[23] Use *Ascolin (Giulio Ascoli 1843–1896, Italia) theorem* — **Exercise set 8** .
[24] In particular the uniform convergence - "sup"norm$\|\cdot\|_o$.



$$\varphi_n(x) = \varphi(x-1) + \frac{1}{2}\varphi(x-2) + \cdots + \frac{1}{n}\varphi(x-n).$$

This gives reason to modify the topology.

Definition 2.7. *The standard topology of the test function space $\mathcal{D}(\Omega)$ y τ* is defined like this:

(1) Basic neighbourhoods of the origin in τ are all such balanced, convex sets $A \subset \mathcal{D}(\Omega)$, for which $A \cap \mathcal{D}_K$ is a neighbourhood of the origin in $\mathcal{D}_K$:ssa for for all compact $K \subset \Omega$.

(2) Translation — of course — gives neighbourhoods of other points.

$$A \in \mathcal{U}_x \iff A - x \in \mathcal{U}_0.$$

Remark 2.8. We use this topology when not stated otherwise.

Remark 2.9. The topology is locally convex. It also has the following properties:

Theorem 2.10. (1) *A balanced, convex set $A \subset \mathcal{D}(\Omega)$ is open if and ony if $A \cap \mathcal{D}_K$ is open for all compact $K \subset \Omega$.*

(2) *τ induces to each $\mathcal{D}_K$ their original topology.*

(3) *A set $R \subset \mathcal{D}(\Omega)$ is bounded if and only if it is contained in some D_K and i such that ere bounded.*

(4) *Every bounded, closed set in the space $D(\Omega)$ is compact. so it has the Heine–Borel–property.*

(5) *A sequence of test functions $(\varphi_i)_{i\in\mathbb{N}}$ is Cauchy if and only if it it is contained in some D_K and i such that ere Cauchy.*

(6) *A sequence of test functions $(\varphi_i)_{i\in\mathbb{N}}$ converges to a test function ψ if and only if it it is contained in some D_K and converge such that ere. kohti functionta ψ.*

(7) *$\mathcal{D}(\Omega)$ is sequentially complete.*

(8) *$\mathcal{D}(\Omega)$ is complete.*

(9) *$\mathcal{D}(\Omega)$ not metrizable.*

(10) *A linear mapping T from $\mathcal{D}(\Omega)$ to any locally convex E – in particular any linear form – is continuous if and only if its restrictions to the spaces $\mathcal{D}_K$ are continuous.*

(11) *In particular the derivation opertor D is continuous in $\mathcal{D}(\Omega)$.*

Proof. We will prove two general theorems implying all these statements. 2.11 and 2.12 implying all these statements.

On the other hand, most of these (full completeness is complicated) are not too difficult to prove directly – in this order. The FINNISH text below gives arguments for (3), (8) and (9). I have deleted them here, since there come theoretical ones later on.

2.3. $\mathcal{D}(\Omega)$ as a direct inductive limit.

Remark 2.11. In fact $\mathcal{D}(\Omega)$ is the direct inductive limit of the spaces $\mathcal{D}_K$. Remember 5.16: Let $\{T_i : F_i \to E_i \mid i \in I\}$ be a family of linear mappings froma some locally convex spaces to a vector space E. The mappings T_i $(i \in I)$ create a locally convex topology, called thelocally convex inductive limit topology in E which is the finest locally convex topology, where every T_i is continuous . Its neighbourhood basis of the origin can be taken to be

$$\mathcal{B}_E = \{U \subset E \mid U \text{ is bal, konv, and abs and } T_I^{-1}(U) \in \mathcal{U}_{E_i} \, \forall \, i \in I\}.$$

A locally convex ind limit has the following properties t:

(i) Any lineaar mapping T from E to any locally convex n vector space E is continuous if and only if every $T \circ T_i$ is continuous . In particular this applies to linear forms $E \to \mathbb{K}$.
(ii) In the space E, a seminorm p is continuous if and only if every $p \circ T_i$ is (a) continuous (seminorm).
(iii) The inductive limit of barreled spaces is a barreled space.
(iv) The inductive limit of bornological spaces is a bornological space. (Definition: A space is *bornological*, if every convex, balanced set, which absorbs all boundd sets, is a neighbourhood of the origin.)

A special case is the direct inductive limit (**English name still to be checked XXX)** Let $E_1 \subsetneq E_2 \subsetneq \dots$ be a secuence of closed subspaces, all locally convex and Hausdorff. Their union $E = \bigcup_{\mathbb{N}} E_n$ with the inductive limit topology of the inclusions $i_n : E_i \to E$ is called the *direct inductive limit of the spaces*, and denoted $\varinjlim E_n$.

Theorem 2.12. *The direct inductive limit $E = \varinjlim E_n$ ihas the following properties*

(i) *In E a seminorm p is continuous if and only if its restriction to each E_n is continuous.*
(ii) *A linear mapping from E to any locally convex F (in particular linear form) T is continuous, if and only if its restriction to each E_n is continuous*
(iii) *The original topology τ_n of each E_n is induced by the topology τ in $E = \varinjlim E_n$ topologyn τ indusoima subspacetopology τ.*
(iv) *E_n is a closed subspace of E.*
(v) *E is Hausdorff.*
(vi) *subset $A \subset E$ is bounded exactly whenit is ja bounded subset of some E_n.*
(vii) *A sequence $(x_n) \subset E$ is convergent (or Cauchy) exactly when it is convergent (or Cauchy) in some n E_n so E is sequentially complete exactly when each E_n is sequentially complete.*
(viii) *E is complete exactly when each E_n is complete.*

(ix) *No $E_n \subset E$ has interior points. So if every E_n is complete, then E cannot be metrizable.*
(x) *If every E_n is a barreled space, then E is barreled space.*
(xi) *If every E_n is a bornological space, then E is bornological space.*
(xii) *If in E_n every bounded closed set is compact, then in E every bounded closed set is compact. (Cf. warning* **??***)*

Proof.

(i) The restriction of a seminorm p in E to E_n is $p \circ i_n$. Since Ecarries the induktiivinen locally convex topology of the injections i_m, then by **??** p is continuous in E, if and only if every such mapping is continuous .
(ii) The restriction of a linear mapping $T : E \to F$ r to E_n is $T \circ i_n$. Since Ecarries the induktiivinen locally convex topology of the injections i_m, then by T is continuous in E if and only if every such mapping is continuous .
(iii) To prove, that the original topology τ_n of E_n: is the same as the subspace topology τ induced by the direct ind limit $E = \varinjlim E_n$ we have to check that both have the same continuous seminorms:

Let p be a seminorm of E_n, continuous in τ . It has (by an older exercise) a continuation to become a continuous semnorm in E. This continuation to E has restrictions, that are continuous in the E_n in their original topologies. So p is continuous in the original topology τ_n.

On the other hand, if we assume, that p is a τ_n-continuous seminorm, then it can be extended to become a continuous seminorm in E_{n+1}, since $E_n \subset E_{n+1}$ is a topological and linear subspace. Repeating this, we construct a semnorm p in E. Its restricitions are the ones just constructed, so they are τ_n–continuous. Therefore p is τ-continuous, and so are its restrictions, among themth eoriginal one.
(iv) Prove, that E_n is a closed subspace of E. Let $x_0 \in E \setminus E_n$. Then for some m we have $x_0 \in E_m \setminus E_n$, so, since E_n is by assumption closed in E_{m+1} and therefore by induction in later E_m, there exists a continuous seminorm in E_m such, that $x_0 + B_p \cap E_n = \emptyset$. We just noticed that p van be extended to a continuous seminorm p in E. This has the property $x_0 + B_p \cap E_n = \emptyset$, so x_0 is an interior point of the set $E \setminus E_n$.
(v) Prove, that E is Hausdorff. Let $x \in E$. There exists $x \in E_n$ for some n, and E_n is by assumption Hausdorff, so there exists a continuous seminorm in E, for which $p(x) \neq 0$. This - –continued to all of E has at x a nonzero value.
(vi) Since any continuous linear mapping maps bounded sets to bounded sets, every bounded set in some E_n is bounded set also a a subset of E . Let us prove the converse: Let $A \subset E$ be bounded. If A is not in cluded in any E_n, then choose strictly increasing sequence of numbers (n_k) and vectors $x_{n_k} \in (E_{n_{k+1}} \setminus E_{n_k}) \cap A$. Then construct continuous seminorms $p_k : E_k \to \mathbb{R}$ such, that they are restrictions of each other and are such that $\frac{1}{k} x_k$ lies outside the open unit ball of p_{n_k} (Exercise or lemma 2.15) Since n_k is strictly increasing, then $E = \bigcup_{k \in \mathbb{N}} E_{n_k}$. As before, construct a continuous seminorm p in E, whose restrictions to the E_{n_k} are the seminorms p_{n_k}. Its open unit ball is a neighbourhood of the origin in the space E. Since $x_k \in E_{n_{k+1}}$, then

evidently also $\frac{1}{k}x_k \in E_{n_{k+1}}$. But $\frac{1}{k}x_k$ lies outside the open unit ball of $p_{n_{k+1}}$, so $p_{n_{k+1}}(\frac{1}{k}x_k) \geq 1$. In other words $p(\frac{1}{k}x_k) \geq 1$ for all $k \in \mathbb{N}^*$. So $(\frac{1}{k}x_k)$ does not converge to 0. But since $(x_k) \subset A$ is bounded, then easily $(\frac{1}{k}x_k) \to 0$. This contradiction proves the statement.

(vii) A sequence $(x_n) \subset E$ is convergent (or Cauchy) exactly when it is convergent (or Cauchy) in some n E_n. This is evident, since such sequences are bounded, hence inside some E_n, and the E_n are topological subspaces of E.

(viii) Completeness is tricky. Make it a theorem 2.18

(ix) Every $E_n \subsetneq E$ is a subspace, so it has no interior points. E is the denumerable union of the subspaces E_n, so it is Baire 1. category. If every E_n is complete, then E is complete and Bairen 1. category, so by Baire's theorem **??** not metrizable.

(x) If every $E_n \subset E$ is a barreled space, then E is barreled by 5.16

(xi) If every $E_n \subset E$ is bornological space, then E is bornological by 5.16

(xii) Let for each E_n every bounded closed set be compact. Let $A \subset E$ be bounded and closed. We rove, that A is compact: Since A is bounded, then it is contained in some E_n and is bounded and closed there, so it is compact.

Definition 2.13. An barreled space where every closed, bounded set is compact, is called a *Montel space*.

Remark 2.14. By what we just proved 2.12, a direct inductive limit of Montel spaces is Montel.

Theorem 2.15 (Lemma)**. Here I omitted the Finnish construction of the sequences above – not very illuminating.**

Definition 2.16. a) A direct inductive limit of Banach spaces is called an $\mathcal{LB}-$*space.*
b) A direct inductive limit ofFréchet spaces is called an $\mathcal{LF}-$*space.*

Remark 2.17. In particular every $\mathcal{LB}-$space is a $\mathcal{LF}-$space. Every $\mathcal{LF}-$space is l by 2.12 bornological. So it is a complete Montel-space, but never metrisable.

Theorem 2.18. (Köthe's theorem)*A direct inductive limit of complete spaces is is complete.*

Proof. Let $\mathcal{F}$ be a Cauchy filter in the space E. The family of sets

$$\mathcal{G}_0 = \mathcal{F} + \mathcal{U}_0 = \{M + V \bigm| M \in \mathcal{F}, V \in \mathcal{U}_0\}$$

is a filter basis in the space E. It spans a filter call it $\mathcal{G} \subset \mathcal{F}$. This is aCauchy- filter: Let $U \in \mathcal{U}_E$. Choose a balanced $V \in \mathcal{U}_E$, such that $V + V + V \subset U$. Since $\mathcal{F}$ is a Cauchy filter, there exists $M \in \mathcal{F}$ such that $M - M \subset V$. Now

$$(M + V) - (M + V) = (M - M) + (V - V) \subset V + V + V \subset U.$$

Prove next, that the trace of the filter $\mathcal{G}$ in some subspace E_n is a filter. We have to verify, that some E_n intersects each $A \in \mathcal{G}$. Antitheesis: no E_n intersects each $A \in \mathcal{G}$, **but for all** $n \in \mathbb{N}$ **there exists an** $A_n \in \mathcal{G}$. not intersecting E_n. In particular there exists a filter basis set of this kind: $\forall n \in \mathbb{N} \exists M_n \in \mathcal{F}$ and a barrel $V_n \in \mathcal{U}_E$ such, that $(M_n + V_n) \cap E_n = \emptyset$, and we evidently can arrange $V_0 \supset V_1 \supset V_2 \dots$. Next we "make V_n independent of n", Let $Y = \mathrm{co}(\bigcup_{n\in\mathbb{N}}(V_n \cap E_n))$. Now $Y \in \mathcal{U}_E$, since Y is convex, balanced and absorbing and for all n is $Y \cap E_n \in \mathcal{U}_{E_n}$, which mean such that at Y is a neighbourhood of the origin in the direct inductive limit topology.

- We prove that for all n on $(M_n + Y) \cap E_n = \emptyset$. Another antithesis: (We have not finished the previous indirect proof yet): $\exists n \in \mathbb{N} : \ (M_n + Y) \cap E_n = \emptyset$ so there exists $y \in (M_n + Y) \cap E_n$. This mean such that at for some n there exists $y \in E_n$ such that

$$y = z_n + \sum_{k=1}^{r} \lambda_k x_k,$$

where

$$z_n \in M_n, \tag{2.1}$$

$$x_k \in E_k \cap V_k \tag{2.2}$$

$$\lambda_k \geq 0 \text{ ja } \sum_{k=1}^{r} \lambda_k = 1. \tag{2.3}$$

If $r \leq n$, niin summassa $\sum_{k=1}^{r} \lambda_k x_k$ jokainen x_k kuuluu avaruuteen E_n, onhan $E_k \subset E_n$ kaikilla $k \leq n$. Tällöin $\sum_{k=1}^{r} \lambda_k x_k \in E_n$, joten, koska tietenkin $M_n \subset M_n + V_n$,

$$z = y - \sum_{k=1}^{r} \lambda_k x_k \in E_n \cap M_n \subset E_n \cap (M_n + V_n) = \emptyset,$$

ja ristiriita on saatu.

If $r > n$, we divide the sum $\sum_{k=1}^{r} \lambda_k x_k$ in two parts:

$$\sum_{k=1}^{r} \lambda_k x_k = \sum_{k=1}^{n} \lambda_k x_k + \sum_{k=n+1}^{r} \lambda_k x_k$$

The first part belongs to E_n, since $x_k \in E_k \subset E_n$ for $k \leq n$. In the second sum again $x_k \in V_k \subset V_n$, since $V_1 \supset V_2 \supset \ldots$. because the coefficients λ_k are positive and sum up at most to 1 the second sum belongs to the convex set V_n. So

$$y - \sum_{k=1}^{n} \lambda_k x_k = z_n + \sum_{k=n+1}^{r} \lambda_k x_k \ \in (M_n + V_n) \cap E_n = \emptyset.$$

This contradiction proves, that for all $n \in \mathbb{N}$ we have $(M_n + Y) \cap E_n = \emptyset$.

By assumption, $\mathcal{F}$ is a Cauchy filter. Since $Y \in \mathcal{U}_E$ there exists $M \in \mathcal{F}$ such that $M - M \subset Y$. **Let $x \in M$. Since $x \in E = \bigcap_{n\in\mathbb{N}} E_n$, there exists $n \in \mathbb{N}$, such that $x \in E_n$. So $x \in E_n \cap M$.** Since M and M_n belong to the filter $\mathcal{F}$, their intersection $M \cap M_n \neq \emptyset$ so we can choose an $y' \in M \cap M_n$. Now

$$x = y' + (x - y') \in y' + (M - M) \subset y' + Y \subset M_n + Y.$$

But $x \in E_n$. Siis $x \in E_n \cap (M_n + Y) = \emptyset$, whichis impossible. This contradiction proves, that $\mathcal{G}_n$ is a filter in the space E_n. It also is Cauchy: If $U_n \in \mathcal{U}_{E_n}$, then there exists $U \in \mathcal{U}_E$ such that $U_n = U \cap E_n$, and since $\mathcal{G}$ is Cauchy, there exists $A \in \mathcal{G}$ such, that $A - A \subset U$,so $(A \cap E_n) - (A \cap E_n) \subset U \cap E_n = U_n$.

Since we assumed, that E_n is complete $\mathcal{G}_n$ converges to someä $x_0 \in E_n$. So every neighbourhood of x_0 has a subset belonging to $\mathcal{G}$.

Prove i, that $\mathcal{F} \to x_0$. Since $\mathcal{G} \subset \mathcal{F}$, it is sufficient to prove, that $\mathcal{G} \to x_0$. Let $U \in \mathcal{U}_E$. Try to find $A \in G$ such that $A \subset x_0 + U$. Take a balanced $V \subset U_E$ such that $V + V \subset U$ and $A \in G$ such that $A - A \subset V$. Since $G_n \to x_0$, there exists $B \in G$ such that $B \cap E_n \subset x_0 + (V \cap E_n)$. Now $A \cap B \in \mathcal{G}$, so $A \cap B \cap E_n \neq \emptyset$ and so also $A \cap (x_0 + V) \neq \emptyset$. Let $x \in A$ and $y \in A \cap (x_0 + V)$. Now $x - x_0 = x - y + y - x_0 \in A - A + V \subset V + V \subset U$. So $A \subset x_0 + U$, so $\mathcal{G} \to x_0$. □

2.4. **Continuity of linear mappings in the test function space $\mathcal{D}(\Omega)$.**

Remark 2.19. Any continuous linear mapping $T : E \to F$ maps bounded sets to bounded sets. In particular sequences converging to the origin are mapped to bounded sets. If E is metrizable and locally convex, then this criterion is also sufficient for continuity. [25]

Proof. If E is metrizable, then any sequentially continuous mapping from E to a topological space is continuous. For a linear mapping this has to be checked at the origin only. So consider a sequence $x_n \to 0$ in a metrizable a locally convex space E. There exists a sequence of numbers $c_n > 0$ such, that

$$c_n \to \infty \quad \text{and} \tag{2.4}$$

$$c_n x_n \to 0. \tag{2.5}$$

One can take

$$c_n := \begin{cases} \frac{1}{\sqrt{d(x_n,0)}}, & \text{kun } x_n \neq 0 \\ n, & \text{kun } x_n = 0, \end{cases}$$

where d is the metric

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)}.$$

To see this, notice that $c_n \to \infty$, so for large n we have $c_n > 1$ and therefore also

$$d(c_n x_n, 0) = \sum_{k=1}^{\infty} \tfrac{1}{2^k} \tfrac{p_k(c_n x_n)}{1+p_k(c_n x_n)} \tag{2.6}$$

$$= c_n \sum_{k=1}^{\infty} \tfrac{1}{2^k} \tfrac{p_k(x_n)}{1+c_n p_k(x_n)} \tag{2.7}$$

$$\leq c_n d(x_n, 0) \leq \sqrt{d(x_n,0)} \to 0. \tag{2.8}$$

By our assumption T maps $c_n x_n$ to a bounded seqquence

$$c_n T x_n \subset F.$$

finally, it is easy to check that the product of a bounded sequence and a sequence converging to 0 converges to 0. □

Corollary 2.20. *A linear mapping $T : \mathcal{D}(\Omega) \to E$, where E is a locally convex space, is continuous if and only if it satisfies the equivalent conditions:*

[25] Actually this holds without assuming local convexity.

a) T *is* bounded : $R \overset{bounded}{\subset} \mathcal{D}(\Omega) \implies T(R) \overset{bounded}{\subset} E$.
b) $\varphi_i \to 0 \implies \{T\varphi_i \mid i \in \mathbb{N}\}$ *is bounded* .
c) T *is* sequentially continuous *at the origin* : $\varphi_i \to 0 \implies T\varphi_i \to 0$.

PROOF. Let $T : \mathcal{D}(\Omega) \to E$ be bounded. In particular bounded sets in a subspace $\mathcal{D}_K$ are mapped to bounded sets, so byy the previous remark, tha mapping is continuous in the (metrizable) subspace and so it is by 2.19 continuous in every $\mathcal{D}_K$ and by **??** also in the whole space $\mathcal{D}(\Omega) = \varinjlim \mathcal{D}(K)$.

Similarly (b)or (c) gives continuity in the subspaces $\mathcal{D}_K$. □

3. Distributions and measures

3.1. **Distributions and their degrees.**

Definition 3.1. *Distributions* are elements of the space

$$\mathcal{D}^* = \mathcal{D}(\Omega)^*,$$

the topological dual of their *test function space* $\mathcal{D}(\Omega)$.

Remark 3.2. $\mathcal{D}(\Omega)$ carries the inductive limit $\tau-$ topology. Therefore, among other things, a linear form

$$\Lambda : \mathcal{D}(\Omega) \to \mathbb{R}$$

is a distribution if and only if for each compact $K \subset \Omega$ there exist numbers $C > 0$ and $n \in \mathbb{N}$ such, that for all $\varphi \in \mathcal{D}_K$:

$$|\langle \varphi, \Lambda \rangle| \leq C \|\varphi\|_n. \tag{3.1}$$

the norm $\|\varphi\|_n$ is what we defined above at 2.6.

Definition 3.3. The distribution Λ has *finite degree*, if 3.1 for all K with the same n. The smallest such n called the *degree* . of Λ.

Example 3.4. (a) Evaluation functionals t also called *Dirac δ–measures*

$$\langle \varphi, \delta_x \rangle := \varphi(x), \quad x \in \Omega,$$

are distributions of degree 0.

(b) Let $f : \Omega \to \mathbb{R}$be locally integrable, so for all compact $K \subset \Omega$

$$\int_K |f| < \infty.$$

Now f defines a degree 0 distribution Λ_f by

$$\langle \varphi, \Lambda_f \rangle := \int_\Omega f\varphi,$$

since

$$|\langle \varphi, \Lambda_f \rangle| \leq \|\varphi\|_0 \int_K |f| \quad \forall \varphi \in \mathcal{D}_K.$$

One can identify Λ_f with the function f (almost everywhere!) .

(c) Similarly, every (Radon ?) measureμ on Ω,for which $0 \leq \mu(K) < \infty$ for compact $K \subset \Omega$, defines and is identidied with the zero degree distribution Λ_μ:

$$\langle \varphi, \Lambda_\mu \rangle := \int_\Omega f \, d\mu.$$

By derivation, one can create higher degree distributions

3.2. **Derivatives of distributions.**

Definition 3.5. Let $\Lambda \in D^*$. Its *derivative* is the distribution $D\Lambda := -\Lambda \circ D$, ie.

$$\langle \varphi, D\Lambda \rangle := -\langle \Lambda, D\varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

This linear form is continuous in the test function space, since differntiation D is a continuous operator in the test function space $\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$. We can also verify directly:

$$\text{If } |\langle \varphi, \Lambda \rangle| \leq C\|\varphi\|_n \quad \forall \varphi \in \mathcal{D}_K,$$
$$\text{niin } |\langle \varphi, D\Lambda \rangle| \leq C\|D\varphi\|_n \leq C\|\varphi\|_{n+1} \quad \forall \varphi \in \mathcal{D}_K,$$

and find out that differentiation rises the degree by at most one. By calculating the derivative of the Dirac deltan one finds an example where the degree is changed.

Higher derivatives are defined by repeating this procedure, so a distribution has all derivatives.

Remark 3.6. There is a slight problem in the definition above. If Λ corresponds to a function a.e. there are 2 derivatives, the usual one and the one derined above. They might indeed be different: An example is given by the step function $f(x) = 0,$ for $x < 0, f(x) = 1$ elsewhere. Its derivative is ae. 0, but its distribution derivative is the Dirac δ_0. But this is not very harmful:

Theorem 3.7. *Lause 8.6 If Df is continuous in Ω, then*

$$D\Lambda_f = \Lambda_{Df}.$$

Proof. Partial integration / exercise. □

More examples of ditstributions can be constructed by multiplying distributions and functions:

3.3. **Products of distributions and functions.**

Example 3.8. Let $\Lambda \in \mathcal{D}^*$ and $g \in \mathcal{C}^\infty$. The *product* $g\Lambda$ is the distribution

$$\langle \varphi, g\Lambda \rangle := \langle g\varphi, \Lambda \rangle$$

Motivation: Two things have to be verified: There should be no contradiction with the usual definition,if $\Lambda = \Lambda_f$ is a function in the sense of 3.4 (b). This is true — and motivates our definition:

$$\int_\Omega (fg)\varphi = \int_\Omega f(g\varphi).$$

The other thing to be verified i such that at $f\Lambda$ is really a distribution. Evidently it is a linear mapping, so only continuity remains to be checked. Use 3.2. Let $K \subset \Omega$ be compact.

$$\begin{aligned}
\text{Assume: } & \exists C, n : \ |\langle \varphi, \Lambda\rangle| \le C\|\varphi\|_n \quad \forall \varphi \in \mathcal{D}(\Omega). \\
\text{Claim: } & \exists C', n' : \ |\langle g\varphi, \Lambda\rangle| \le C'\|\varphi\|_{n'} \quad \forall \varphi \in \mathcal{D}(\Omega). \\
\text{Calculate: } & |\langle g\varphi, \Lambda\rangle| \le C\|g\varphi\|_n = C \max\left\{ |D^k(g\varphi)| \;\middle|\; k \le n,\, x \in \Omega \right\} \\
& \le C \max\left\{ \sum_{j=0}^{k} \binom{k}{j} \left|(D^j g)(D^{k-j}\varphi)\right| \;\middle|\; \dots \right\} \\
& \le C \max\left\{ \sum_{j=0}^{k} \binom{k}{j} \left|(D^j g)\right| \;\middle|\; \dots \right\} \max\left\{ |D^{k-j}\varphi| \;\middle|\; \dots \right\} \\
& = C'\|\varphi\|_n,
\end{aligned}$$

where

$$C' := C \max\left\{ \sum_{j=0}^{k} \binom{k}{j} \left|(D^j g)\right| \;\middle|\; k \le n,\, x \in \Omega \right\}$$

□

The crucial phase in the proof was Leibnitz's formula for higher derivatives — containing the binomial coefficients.

Remark 3.9. By calculation, one can verify Leibnitz's formula for derivatives of distributions as well. There are higher dimensional versions also.

3.4. **Topology in the space of distributions.**

Definition 3.10. $\mathcal{D}^*(\Omega)$ is the topological dual of the test function space, so we can intriduce polar topologies, in particular the weak topology $w^* := \sigma(\mathcal{D}^*(\Omega), \mathcal{D}(\Omega))$. This is almost always used, so convergence of a sequence of distributions means:

$$\Lambda_n \to \Lambda \iff \langle \varphi, \Lambda_n \rangle \to \langle \varphi, \Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Being the transpose of a continuous linear mapping, derivation is continuous in the w^*–topologies.

Theorem 3.11. *Let $\Lambda_i \in \mathcal{D}(\Omega)^*$ be a sequence of distributions such that*

$$\langle \varphi, \Lambda_n \rangle \to \langle \varphi, \Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.2}$$

equivalently $\Lambda_n \to \Lambda$ The derived sequence converges in the same topology:

$$\langle \varphi, D\Lambda_n \rangle \to \langle \varphi, D\Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Proof. Immediate from **??**. □

3.5. **Radon-measures are distributions.**

Example 3.12 (**General Radon measures).** When X is a locally compact Hausdorff-space — for example an open set $X \subset \mathbb{R}^n$ — we write $\mathcal{C} = (X) = \{f : X \to \mathbb{K} \mid f$ is continuous$\}$ and $\mathcal{C}_c(X) = \{f \in \mathcal{C}(X) \mid f$ is compactly supported$\}$. For copact $K \subset X$ we write $\mathcal{C}_K(X) = \{f \in \mathcal{C}(X) : \operatorname{supp} f \subset K\}$. The sup-norm makes $\mathcal{C}_c(X)$ a Banach-space and $C_K(X)$ its closed subspaces.

Since $\mathcal{C}_c(X) = \bigcup_{K \overset{komp}{\subset} X} C_K(X)$, we can equip $\mathcal{C}_c(X)$ with the locally convex limit topology τ wrt. the inclusion mappings $C_K(X) \to \mathcal{C}_c(X)$. We denote it by $\mathcal{C}_c(X)_\tau$.

A Radon measure is a continuous linear form $\mathcal{C}_c(X)_\tau \to \mathbb{K}$ ie. an element of the topolgical dual $\mathcal{C}_c(X)_\tau{}^*$. If also $\mu(f) \geq 0$ for all positive $f \in \mathcal{C}_c(X)$, then μ is a *positive Radon–measure* .

A linear form $\mu \in \mathcal{C}_c(X)'$ is by **??** a Radon–measure exactly if it is continuous in the sup-norm in every subspace $\mathcal{C}_K(X)$ equivalently:

$$\forall K \overset{komp}{\subset} X \,\exists\, \lambda_K > 0 \text{ such that } |\langle f, \mu\rangle| \leq \lambda_K \|f\|_\infty \forall\, f \in \mathcal{C}_K(X).$$

Definition 3.13. If a topological space X is a union of countably many sets $X = \bigcup_{n\in\mathbb{N}} K_i$, then we say, that the sequence $(K_n)_{n\in\mathbb{N}}$ is a *tyhjennys* of X by compact sets, if

(i) $K_1 \subset K_2 \subset \dots$, and
(ii) for every compact set $K \subset X$ there exists $n \in \mathbb{N}$ such that $K \subset K_n$.

For example for every open set $X \subset \mathbb{R}^n$ there exists tyhjennys by a sequence of compact seie. **??**

Example 3.14 (**Radon measures and tyhjennys).** If $(K_n)_{n\in\mathbb{N}}$ is a tyhjennys of a locally compact space X by a sequence of compact sets, then $\mathcal{C}_c(X)_\tau$ is the direct inductive limit $\varinjlim \mathcal{C}_{K_n}(X)$.

Proof. Every compact set $K \subset X$ is included in some K_n, so the Banach-space $\mathcal{C}_K(X)$ is a closed topological and linear subspace of $\mathcal{C}_{K_n}(X)$. So the locally convex inductive topology τ of $\mathcal{C}_c(X)$ coincides with the direct inductive limit topology in $\varinjlim \mathcal{C}_{K_n}$. So the linear form $\mu \in \mathcal{C}_c(X)'$ is a Radon–measure exactly, when it is sup-norm-continuous in each $\mathcal{C}_{K_n}(X)$ ie.

$$\forall n \in \mathbb{N} \,\exists\, \lambda_n > 0 \text{ such that } |\langle \varphi, \mu\rangle| \leq \lambda_n \|\varphi\|_\infty \forall\, \varphi \in \mathcal{C}_{K_n}(X).$$

BNouded sets in $\mathcal{C}_c(X)_\tau$ always are included in some $\mathcal{C}_{K_n}(X)$ and are bounded there,

$$A = \{\varphi \in \mathcal{C}_c(X) \mid \exists n \in \mathbb{N} \text{ such that } \operatorname{supp}\varphi \subset K_n \text{ and } \|\varphi\|_\infty \leq n\}.$$

Example 3.15. (Lebesgue measure) Let $\Omega \subset \mathbb{R}^n$ be open. The mapping $\mathcal{C}_c(\Omega) \to \mathbb{K}$: $\varphi \mapsto \int_\Omega \varphi$ is a positive positive Radon measure, called the *Lebesgue measure m*.

Proof. For all $\varphi \in \mathcal{C}_{K_n}(\Omega)$ is

$$|\langle \varphi, m\rangle| = |\int_\Omega \varphi| = \int_{K_n} \varphi \leq \|\varphi\|_\infty \int_{K_n} 1.$$

Remark 3.16. Some measure theory

a) In an open set $X = \Omega \subset \mathbb{R}^n$ a positive linear form $\mu \in \mathcal{C}_c(\Omega)'$ is always continuous, so a Radon-measure .

PROOF. We prove, that a positive linear form μ is continuous in every space $\mathcal{C}_K(\Omega)$. Take a continuous non-negative compactly supported function $\psi \in \mathcal{C}_c(\Omega)$ such that $\psi(x) = 1$ for all $x \in K$. (Sellainen exists!) For all $\varphi \in \mathcal{C}_K(\Omega)$ by positivity of μ:n

$$\begin{aligned}|\langle \varphi, \mu \rangle| &\le \langle |\varphi|, \mu \rangle = \langle |\varphi|\psi, \mu \rangle \\ &\le \langle \sup_{x\in K} |\varphi(x)|\psi, \mu \rangle \\ &= \langle \psi, \mu \rangle \sup_{x\in K} |\varphi(x)|.\end{aligned}$$

b)Remember *Rieszrepresentation theorem*, by which every Radon-measure $\mu \in \mathcal{C}_c(\Omega)^*$ can be represented as

$$\langle \varphi, \mu \rangle = \int_\Omega \varphi \, d\bar{\mu},$$

where $\bar{\mu}$ is a regular Borel-measure [26] in Ω.

Example 3.17. (Functions as Radon–measures) Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{K}$ locally integrable ie. for each compact set $K \subset \Omega$ is $\left| \int_K f \right| < \infty$. Then $\varphi \mapsto \int_\Omega \varphi f$ ia a Radon measure called *Lebesgue measure with density* f. The measure corresponding to f is positive if and only if $f \ge 0$ ae.

Perustelu. for all $\varphi \in \mathcal{C}_{K_n}(\Omega)$ we haave

$$|\langle \varphi, f \rangle| = \Big| \int_\Omega \varphi f \Big| = \int_{K_n} \varphi f \le \|\varphi\|_\infty \Big| \int_{K_n} f \Big|.$$

Example 3.18. (Dirac δ Radon–measure) Let $\Omega \subset \mathbb{R}^n$ be open and $x \in \Omega$. The evaluation functional $\delta_x : \mathcal{C}_c(\Omega) \to \mathbb{K} : \delta_x(\varphi) = \varphi(x)$ isalso called the *Dirac δ-measure* at x. It It is a positive Radon–measure .

Remark 3.19. The linear form $f : \mathcal{D}(\Omega) \to \mathbb{K}$ is by 2.20 continuous, so a distribution, exactly when it is sequentially continuous, And that happens, if and only if the following condition is true:

If $(\varphi_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\Omega)$ and there exists acompact $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for all n and for all $k \in \mathbb{N}$

$$D^k \varphi_k(x) \to 0 \text{ uniformly in } K,$$

then $f(\varphi_k) \to 0$.

PROOF. Exercise

Theorem 3.20. *Let $\Omega \subset \mathbb{R}^m$ be open.*

a) *The restriction of a Radon-measure $\mu \in {\mathcal{C}_c(\omega)_\tau}^*$ to $\mathcal{D}(\Omega)$ is a distribution.*
b) *Two distinct Radon-mesures $\mu \in {\mathcal{C}_c(\omega)_\tau}^*$ have distinct restrictions to $\mathcal{D}(\Omega)$.*

---

[26]Borel-measure is Borel-set jen σ–algebrassa defined measure . Reguilarity: $\forall K \overset{\text{komp}}{\subset} \Omega \, \forall \epsilon > 0 \, \exists \overset{\text{open}}{\subset} \Omega$ siten, such that $\bar{V}$ is compact, $K \subset V$ and $\forall W \overset{\text{open}}{\subset} \Omega$, for which $K \subset W \subset V$, is $|\mu(W) - \mu(K)| < \epsilon$. In a metric space all Borel-measures are regular.

Proof. (a) follows from the fact that the topology induced by $\mathcal{C}_c(\Omega)$ to its subspace $\mathcal{D}(\Omega)$ is coarser than the strandard topology of $\mathcal{D}(\Omega)$, since for all $n \in \mathbb{N}$ we have $\mathcal{D}_K \subset \mathcal{C}(K)$ and the inclusion is continuous.

(b) follows from the fact that $\mathcal{D}(\Omega)$ is obviously dense in $\mathcal{C}_c(\omega)_\tau$.

Remark 3.21. We can identify Radon-measures with the corresponding distributions. They obviously have degree 0. [27]

3.6. The product of a distribution and a function.

Example 3.22. Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$ ja $g \in \mathcal{C}^\infty$. Then the *product* $g\Lambda$ is defined to be the distribution

$$\langle \varphi, g\Lambda \rangle := \langle g\varphi, \Lambda \rangle$$

Conistency. When $\Lambda = \Lambda_f$ is a function, the definition should coincide with the older one in 3.4 (b). TThis is OK:

$$\int_\Omega (fg)\varphi = \int_\Omega f(g\varphi).$$

Check that $f\Lambda$ os a distribution ie continuous: Use 3.2. Let $K \subset \Omega$ be compact.

$$\text{Assume: } \exists C, n : \; |\langle \varphi, \Lambda \rangle| \le C\|\varphi\|_n \quad \forall \varphi \in \mathcal{D}(\Omega).$$

$$\text{Claim: } \exists C', n' : \; |\langle g\varphi, \Lambda \rangle| \le C'\|\varphi\|_{n'} \quad \forall \varphi \in \mathcal{D}(\Omega).$$

$$\begin{aligned}\text{Calculation: } |\langle g\varphi, \Lambda \rangle| &\le C\|g\varphi\|_n = C \max\left\{ |D^k(g\varphi)| \;\middle|\; k \le n,\, x \in \Omega \right\} \\ &\le C \max\left\{ \sum_{j=0}^k \binom{k}{j} \left|(D^j g)(D^{k-j}\varphi)\right| \;\middle|\; \dots \right\} \\ &\le C \max\left\{ \sum_{j=0}^k \binom{k}{j} \left|(D^j g)\right| \;\middle|\; \dots \right\} \max\left\{ |D^{k-j}\varphi| \;\middle|\; \dots \right\} \\ &= C'\|\varphi\|_n,\end{aligned}$$

where

$$C' := C \max\left\{ \sum_{j=0}^k \binom{k}{j} \left|(D^j g)\right| \;\middle|\; k \le n,\, x \in \Omega \right\}$$

□

Multiplication of a distribution by a function is the transpose of multiplying test functions by the same function. In fact we just verified the continuity of that operation.

Remark 3.23. The core formuola above was Leibnitz formula for higher derivatives of a product.

Remark 3.24. By direct calculation ome can check Leibnitz formula for higher derivatives of distributions as well.

$$D(f\Lambda) = Df\,\Lambda + f\,D\lambda.$$

[27] Exercise: Are there others?

There is a version for partial derivatives and derivatives in higher dimensions .
PROOF. Exercises. □

4. The support of a distribution

4.1. **The support of a distribution.**

Definition 4.1. Two distributions Λ_1 and Λ_2 *coincide in an open set* $\omega \subset \Omega$, if
$$\langle \varphi, \Lambda_1 \rangle = \langle \varphi, \Lambda_2 \rangle \quad \forall \varphi \in \mathcal{D}(\omega).$$
In particular $\Lambda = 0$ ω, if Λ takes to 0 all test functions whose support is included in j ω.

Example 4.2. (a) A locally integrable function f vanishes ae. in a set ω, if the corresponding distribution $\Lambda_f = 0$ vanishes in the set ω.

(b) A Radon–measure gives measure 0 to all Borel sets $B \subset \omega$ exactly when the corresponding distribution is 0 in the set ω.

Definition 4.3. The support of a distribution Λ *support* $\operatorname{supp} \Lambda$ is the closed set
$$\operatorname{supp}(\Lambda) = \Omega \setminus \bigcup_{\Lambda = 0 \ \omega\text{:ssa}} \omega = \Omega \setminus \bigcup \{\omega \overset{\text{open}}{\subset} \Omega \mid \langle f, \Lambda \rangle = 0, \ \text{When } \operatorname{supp} f \subset \omega\}.$$

Example 4.4. (i) The support of a distribution Λ is empty if and only if $\Lambda = 0$.

(ii) The support of the distribution δ is the set $\mathbb{R} \setminus \bigcup \{\omega \overset{\text{open}}{\subset} \mathbb{R} \mid f(0) = 0,$ When $\operatorname{supp} f \subset \omega\} = \{0\}$.

Remark 4.5. $\Lambda = 0$ in $\bigcup \{\omega \overset{\text{open}}{\subset} \Omega \mid \langle f, \Lambda \rangle = 0, \ \text{when } \operatorname{supp} f \subset \omega\}$.
PROOF. Exercise. Use partitions of unity.

Theorem 4.6. *Let $\Lambda \in \mathcal{D}(\Omega)^*$ be a distribution.*

a) *If $\varphi \in \mathcal{D}(\Omega)$ and $\operatorname{supp} \varphi \cup \operatorname{supp} \Lambda = \emptyset$, then $\langle f, \Lambda \rangle = 0$.*
b) *If $\psi \in \mathcal{C}^\infty(\Omega)$ and $\psi(x) = 1$ in an open set $A \supset \operatorname{supp} \Lambda$, then $\Lambda \psi = \Lambda$.*

PROOF. (i) by definition. (ii) easy exercise.

4.2. **Compact supports.**

Theorem 4.7. *The following are equivalent:*

a) *The support of the distribution Λ is compact*
b) *The distribution Λ is of finite degree and has a of finite degree and has a unique extension*
$$\Lambda \in C^\infty(\Omega)^*.$$

PROOF. Remember, in $C^\infty(\Omega)^*$[28] we use the locally convex metrizable topology, defined by the seminorms $q_n(f) = \|D^n f\big|_{K_n}\|\|_\infty$ over a compact countable tyhjennys of the open set Ω. This topology can also be defined by the increasing sequence of seminorms $p_n = q_0 + q_1 + \cdots + q_n$. Since $\mathcal{D}(\Omega) = \mathcal{C}_c^\infty(\Omega) \overset{\text{dense}}{\subset} \mathcal{C}^\infty(\Omega)$ and the inclusion is continuous $\mathcal{D}(\Omega)^* \supset \mathcal{C}^\infty(\Omega)^*$

[28] some books call it $\mathcal{E}(\Omega, \mathbb{R})$.

Let $\Lambda \in C^\infty(\Omega)^*$. Since Λ is continuous in $C^\infty(\Omega)$, there exists a seminorm p_n, such that $|\langle\cdot,\Lambda\rangle| \le p_n$. Take a test function $f \in D(\Omega)$,such that $\operatorname{supp} f \subset \Omega \setminus K_n$. Obviously $p_n(f) = 0$, so $\langle f,\Lambda\rangle = 0$. Therefore $\operatorname{supp}\Lambda \subset K_n$. Therefore every $\Lambda \in C^\infty(\Omega)^*$ is a compactly supported distribution.

For the inverse implication we need PARTITIONS OF UNITY: *ykkösen ositus-lemmaa*:

Lemma 4.8. *Let $(\omega_i)_{i\in I}$ be a family of open sets $\omega_i \subset \mathbb{R}$ and $\Omega = \bigcup_{i\in I}\omega_i$. Then there exists a sequence (!) of functions $(\psi_n)_{n\in\mathbb{N}}$ such that*

a) $\forall n \in \mathbb{N}\, \exists i \in I$ *such that* $\operatorname{supp}\psi_n \subset \omega_i$.
b) $\sum_{n\in\mathbb{N}}\psi_n = 1$ *in the set* Ω.
c) *To every compact $K \subset \Omega$ there exists an open $A \supset K$ and a number $m \in \mathbb{N}$ such that $\sum_{n=1}^{m}\psi_n = 1$ in the set A.*

PROOF. Classical analysis. (Rudin: Functional analysis p. 146.)

Proof of 4.7 inverse half: Let the distribution $\Lambda \in \mathcal{D}(\Omega)^*$ have compact support $\operatorname{supp}\Lambda \subset \Omega$. We construct an extension of Λ into $\mathcal{C}^\infty(\Omega)$. By the partitions of unity lemma 4.8 — as a special case – there exists $\psi \in \mathcal{D}(\Omega) = \mathcal{C}^\infty(\Omega)$ such that $\psi(x) = 1$ jossain avoimessa in set $A \supset \operatorname{supp}\Lambda$. Let $K = \operatorname{supp}\psi$. Since $\Lambda \in \mathcal{D}(\Omega)^*$, thenthere exists $C_1 > 0$ and $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_K(\Omega)$ we have

$$|\langle\varphi,\Lambda\rangle| \le C_1\|\varphi\|_N,$$

where $\|\varphi\|_N = \max_{j\le N}\|D^j\varphi\|_\infty$.

By repeating the derivation formula for products (Leibnitz) we get

$$\|\psi\varphi\|_N = \max_{j\le N}\|\sum_{\alpha=0}^{N}(D^{N-\alpha}\psi D^\alpha\varphi)\|_\infty \le C_2\|\varphi\|_N,$$

where the constant $C_2 > 0$ may depend on ψ. Therefore for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$|\langle\varphi,\Lambda\rangle| = |\langle\psi\varphi,\Lambda\rangle| \le C_1\|\psi\varphi\|_N \le C_1C_2\|\varphi\|_N,$$

so Λ is of finite degree.

The linear form $\Lambda \in \mathcal{D}(\Omega)^*$ can be extended to $\mathcal{C}^\infty(\Omega) \supsetneq \mathcal{D}(\Omega)$ by

$$\langle f,\Lambda\rangle = \langle\psi f,\Lambda\rangle,$$

since the right hand side is defined and linear with respect to f and for all $\varphi \in \mathcal{D}(\Omega)$ we have $\langle\varphi,\Lambda\rangle = \langle\psi\varphi,\Lambda\rangle$. We have to prove continuity for this extension of Λ and that will also prove uniqueness, since $\mathcal{D}(\Omega)$ is dense in $\mathcal{C}^\infty(\Omega)$. To check this, we only have to prove sequential continuity **??** . Let $f_j \to 0 \in \mathcal{C}^\infty(\Omega)$. This mean such that at for all $n \in \mathbb{N}$ and in all compact $C \in \Omega$ we have $\sup_{x\in C}|D^n f_j(x)| \to 0$. Differentiation of products prove such that at $\psi\, f_j \to 0$ in$\mathcal{D}(\Omega)$, so, since $\Lambda \in \mathcal{D}(\Omega)^*$,

$$\langle f_j,\Lambda\rangle = \langle\psi f_j,\Lambda\rangle \to 0.$$

□

Theorem 4.9. *Let $\Omega = \mathbb{R}$ and let the support of Λ be one point $\{x\}$. Then (and only then) Λ is a linear combination of the Dirac δ_x and it derivatives*

$$\Lambda = \sum_{j=0}^{m} \lambda_j \, D^j \delta_x.$$

PROOF. We can assume that $x = 0$. By the previous theorem Λ has finite degree, say N.

(i) Oletus $\operatorname{supp} \Lambda = \{0\}$ mean such that at $\langle \varphi, \Lambda \rangle = 0$ always when $0 \notin \operatorname{supp} \varphi$.

(ii) We prove, that if $\varphi \in \mathcal{D}(\Omega)$ and $D^k \varphi(0) = 0$ for all $k = 0, 1, \ldots, N$, then $\langle \varphi, \Lambda \rangle = 0$.

This is suffidcient, since the condition $\varphi \in \mathcal{D}(\Omega)$ and $D^k \varphi(0) = 0$ for all $k = 0, 1, \ldots N$ mean such that at

$$\operatorname{Ker}(\Lambda) \supset \operatorname{Ker}(\delta) \cap \operatorname{Ker}(D\delta) \cdots \cap \operatorname{Ker}(D^N \delta),$$

which by the linear algebra lemma 7.11prove such that at, Λ is a linear combination of the linear forms δ, $D\delta \ldots D^N \delta$.

(iii) To prove (ii) take for each $\eta > 0$ a compact interval $K = [-\rho, \rho]$ such that

$$\|D^N \varphi\big|_K\|_\infty \le \eta.$$

Then[29] for all $k \le N$ and $x \in K$

$$|D^k \varphi(x)| \le \eta |x|^{N-k}.$$

Choose a ψ, for which $\psi(x) = 1$ in some neighbourhood of the origin and whose support is included in the unit interval of $\mathbb{R}$. We define for all $r > 0$: $\psi_r(x) = \psi\big(\frac{x}{r}\big)\big)$. For small r we have $\operatorname{supp} \psi \subset K$. Differentiating products we find

$$D^k(\psi_r \varphi)(x) = \sum_{\beta \le k} c_{k,\beta} D^{k-\beta} \psi\big(\tfrac{x}{r}\big) D^\beta \varphi(x) r^{\beta - k},$$

so

$$\|\psi_r \varphi\|_N \le \eta C \|\psi\|_N,$$

whenever r is small enough.

Since the degree Λ is N, there exists a constant C_1 such that $|\langle \psi, \Lambda \rangle| \le C_1 \|\psi\|_N$ for all $\psi \in \mathcal{D}_K(\mathbb{R})$. Since $\Psi_r = 1$ in a neighbourhood of the origin

$$|\langle \varphi, \Lambda \rangle| \le C_1 \|\psi\|_N = |\langle \psi\varphi, \Lambda \rangle| \le C_1 \|\psi_r \varphi\|_N \le \eta C C_1 \|\psi\|_N.$$

Since η was arbitrary, this implies, that

$$|\langle \varphi, \Lambda \rangle| = 0.$$

□

[29] The n-dimensional version is : $\|D^k \varphi\big|_K\|_\infty \le \eta m^{N-k}$.

4.3. **All distributions as derivatives.** A distribution, whose support is one point, consits of derivatives of the Diracin delta. Dirac delta δ_x onis in itself the derivative of the step function (8.5.), and so it is the second derivative of the continuous function. $f(t)$ =max$\{t-x,0\}$. By induction, every distribution, whose support is a finite set, is a linear combination of various derivatives of functions, in fact a higher derivative of some function. In some sense all other distributions are such derivatives as well, at least locally. Therefore, distributions form a minimal mathematical sysem containing al derivatives of continuous functions. We will state 3 theorems on this subject:

Theorem 4.10. *Let $\Lambda \in \mathcal{D}(\Omega)^*$ and $K \subset \Omega$ a compact set in an open $\Omega \subset \mathbb{R}$.*

There exists a continuous function $f : \Omega \to \mathbb{R}$, and a number $\alpha \in \mathbb{N}$ such that for all [30] $\varphi \in \mathcal{D}_K(\Omega)$

$$\langle \varphi, \Lambda \rangle = (-1)^{\alpha} \int_{\Omega} D^{\alpha}\varphi\,(x)\, f(x)\, dx.$$

Proof. We may assume, that $K \subset [0,1]$. For all $\psi \in \mathcal{D}_{[0,1]}(\Omega)$ we can define the zero extension $\psi : \mathbb{R} \to \mathbb{R} : x \mapsto \psi(x)$, when $x \in \Omega$, and 0 elsewhere. N0w for all $x \in [0,1]$ and $\psi \in \mathcal{D}_{[0,1]}(\Omega)$ we have

$$\psi(x) = \psi(x) - 0 = \psi(x) - \psi(0) = \int_0^x D\psi(t)\, dt.$$

Since differentiation $D : \mathcal{D}_K(\Omega) \to \mathcal{D}_K(\Omega) : \varphi \mapsto D\varphi$ is a bijection, also higher derivatives are bijectionsm so themapping kuvauksella $D^{n+1} : \mathcal{D}_K(\Omega) \to \mathcal{D}_K(\Omega)$ has an inverse $(D^{n+1})^{-1}$, of course linear. So we can define a linear form $\Lambda_1 : \mathcal{D}(\Omega) \to \mathbb{R}$ by setting for all $\varphi \in \mathcal{D}_K(\Omega)$

$$\langle \varphi, \Lambda_1 \rangle = \langle (D^{n+1})^{-1}\varphi, \Lambda \rangle,$$

eli

$$\langle D^{n+1}\varphi, \Lambda_1 \rangle = \langle \varphi, \Lambda \rangle.$$

The mapping $\Lambda \mapsto \Lambda_1$ is the restriction of the algebraic transpose of $n+1$-fold integration $(D^{n+1})^{-1} : \mathcal{D}_K(\Omega) \to \mathcal{D}_K(\Omega)$ to the space $\mathcal{D}_K(\Omega)^*$.

We prove, that

(i) Λ_1 is continuous in the sense of the integral norm $\|\cdot\|_1$ in $\mathcal{D}_K(\Omega) \subset L^1(K, dx)$.

(ii) Λ_1 can be extended to a continuous linear form $\Lambda_1 \in L^1(K, dx)^* = L^\infty(K, dx)$, in the space $L^1(K, dx)$ ie.

$$\langle \varphi, \Lambda \rangle = \langle D^{n+1}\varphi, \Lambda_1 \rangle = \int_K D^{N+1}\varphi\, g(x)\, dx$$

; $g \in L^\infty(K, dx)$.

(iii)

$$\langle \varphi, \Lambda \rangle = \int_K D^{N+2}\varphi\, f(x)\, dx$$

; $f \in \mathcal{C}(\Omega)$.

[30]Remark: $\varphi \in \mathcal{D}_K(\Omega)$!

(i) We prove, that there exists a constant $C > 0$ such that for all $\varphi \in \mathcal{D}_K(\Omega)$ is

$$|\langle \varphi, \Lambda_1 \rangle| \leq C \int_K |\varphi(x)|\, dx.$$

Since for all $x \in [0,1]$ and $\psi \in \mathcal{D}_{[0,1]}(\Omega)$ $\psi(x) = \int_0^x D\psi(t)\, dt$, so $\|\psi(x)\|_\infty \leq \int_0^x |D\psi(t)|\, dt$. SApplying the same reasoning to the derivatives, one finds out that for all $n \in \mathbb{N}$

$$\|\psi\|_n = \|D^n \psi\|_\infty \leq \int_0^1 |D^{n+1}\psi(x)|\, dx.$$

Since $\Lambda \in \mathcal{D}(\Omega)^*$, there exists $N \in \mathbb{N}$ ja $C > 0$ such that for all $\varphi \in \mathcal{D}_K(\Omega)$

$$|\langle \varphi, \Lambda \rangle| \leq C \|\varphi\|_N,$$

so

$$|\langle D^{N+1}\varphi, \Lambda_1 \rangle| = |\langle \varphi, \Lambda \rangle| \leq C \int_0^1 |D^{N+1}\varphi(x)|\, dx.$$

and therefore for all $\varphi \in \mathcal{D}_K(\Omega)$

$$|\langle \varphi, \Lambda_1 \rangle| \leq C \int_K |\varphi(x)|\, dx,$$

as propoded.

(ii) By Hahn and Banachi corollary **??** there exists an extension of each $L^1(\mu)$-jcontinuous linear form $\Lambda_1 \in \mathcal{D}_K(\Omega)'$ to a continuous linear form in $L^1(K, dx)$.[31] $\Lambda_1 \in L^1(K, dx)^* = L^\infty(K, dx)$, ie. exists a Borel-measurable $g \in L^\infty(K, dx)$ such that

$$\langle \varphi, \Lambda \rangle = \langle D^{n+1}\varphi, \Lambda_1 \rangle = \int_K D^{N+1}\varphi\, g(x)\, dx.$$

(iii) Extend g as zero to the whole space $\mathbb{R}$ and integrate by parts:

$$\langle \varphi, \Lambda \rangle = \int_{-\infty}^{\infty} D^{N+1}\varphi\, g(x)\, dx = (-1)^N \int_{-\infty}^{\infty} D^{N+2}\varphi \left(\int_{-\infty}^{x} g(t)\, dt \right) dx.$$

Define $f(x) = \int_{-\infty}^x g(t)\, dt$, and now f is continuous and

$$\langle \varphi, \Lambda \rangle = (-1)^N \int_{-\infty}^{\infty} D^{N+2}\varphi \left(\int_{-\infty}^{x} g(t)\, dt \right) dx = (-1)^N \int_{-\Omega} D^{N+2}\varphi\, f(x)\, dx.$$

□

Theorem 4.11. *Let $\Lambda \in \mathcal{D}(\Omega)^*$ and $K \subset V \subset \Omega \subset \mathbb{R}$, where $K = \operatorname{supp}\Lambda$ is compact and V and Ω open. We denote the degree of Λ by N. ($N < \infty$, Ks.* **??***.)*

There exist continuous functions $f_1, \ldots, f_{N+2} : \Omega \to \mathbb{R}$, such that $\operatorname{supp} f_i \subset V$ and

$$\Lambda = \sum_{\beta=1}^{N+2} D^\beta f_\beta.$$

---

[31] If a measure μ is S is sigma-finite, then the dual of $L^1(\mu)$ is — by the well known mapping — isometrically isomorphic to $L^\infty(\mu)$. However, except in rather trivial cases, the dual of $L^\infty(\mu)$ is much bigger than $L^1(\mu)$. Elements of $L^\infty(\mu)^*$ can be identified with bounded signed finitely additive measures is S that are absolutely continuous with respect to μ.

Proof. Choose some open W such that $K \subset W \subset \overline{W} \subset V$. Apply theorem 4.10 Choosing for K $\overline{W}$, and fins a continuous $f : \Omega \to \mathbb{R}$ such that for all $\varphi \in \mathcal{D}_{\overline{W}}(\Omega)$

$$\langle \varphi, \Lambda \rangle = (-1)^N \int_\Omega D^{N+2}\varphi\, f(x)\, dx. \tag{4.1}$$

The equation 4.1 remains valid, even if f is multiplied by any continuous function h, for which the restriction $h\big|_{\overline{W}}$ is 1 and $\operatorname{supp} h \subset V$.

Choose for h a function $\psi \in \mathcal{D}(\Omega)$ such that $\operatorname{supp}\psi \subset W$ and $\psi = 1$ in some open set $U \supset K$. For every $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned}
\langle \varphi, \Lambda \rangle &= \langle \psi\varphi, \Lambda \rangle \\
&= (-1)^N \int_\Omega D^{N+2}(\psi\varphi)\, f(x)\, dx \\
&= (-1)^N \int_\Omega \sum_{m=0}^{N+2} \binom{N+2}{m} D^{N+2-m}\psi D^m \varphi\, f(x)\, dx
\end{aligned}$$

□

Theorem 4.12. *Let $\Lambda \in \mathcal{D}(\Omega)^*$.*

There exists a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that

a) *No compact set $K \subset \Omega$ intersects more than finitely many of the supports of the functions g_n and*
b) *$\Lambda = \sum_{n\in\mathbb{N}} D^n g_n$. (locally finite sum!)*

If Λ is of finite degree, then finitely many will do in the sum g_n.

Proof. Choose (Exercise!) compact intervals Q_i and open sets V_i (i=1,2,...) such that

(i) $Q_i \subset V_i \subset \Omega$,
(ii) $\bigcup_{i\in\mathbb{N}} V_i = \Omega$, ja
(iii) no compact set $K \subset \Omega$ intersects more than finitely many V_i.

Ny lemma 4.8 there exists a sequence of functions $\psi_n \in \mathcal{D}(\Omega)$ such that

a) $\forall n \in \mathbb{N}\, \exists i \in \mathbb{N}$ such that $\operatorname{supp}\psi_n \subset V_i$.
b) $\sum_{n\in\mathbb{N}} \psi_n = 1$ in Ω.
c) For each compact $K \subset \Omega$ there exists an open $A \supset K$ and a number $m \in \mathbb{N}$ such that $\sum_{n=1}^m \psi_n = 1$ in A.

Apply 4.11 to each product distribution $\Psi_i\Lambda$ and find finite sums:

$$\psi_i \Lambda = \sum_\alpha D^\alpha f_{i,\alpha}.$$

Define for all $\alpha \in \mathbb{N}$

$$g_\alpha = \sum_\alpha f_{i,\alpha}$$

niticing that there are only finitely many nonzero terms in the sum when we restrict to a compact set. Therefore every g_α is continuous in Ω and no compact set $K \subset \Omega$ intersects the support of more than finitely many functions g_n.

Since $1 = \sum_{n\in\mathbb{N}} \psi_n$, every $\varphi \in \mathcal{D}(\Omega)$ can in compact sets be expressed as a finite sum $\varphi = \sum_{n\in\mathbb{N}} \varphi\psi_n$, we have

$$\langle \varphi, \Lambda \rangle = \langle \sum_{n\in\mathbb{N}} \varphi\psi_n, \Lambda \rangle = \langle \varphi, \sum_{n\in\mathbb{N}} \Lambda\psi_n \rangle,$$

and siis

$$\begin{aligned} \Lambda &= \sum_{i\in\mathbb{N}} \psi_i \Lambda \\ &= \sum_{i\in\mathbb{N}} \sum_{\alpha} D^\alpha f_{i,\alpha} \\ &= \sum_{\alpha} D^\alpha \sum_{i\in\mathbb{N}} f_{i,\alpha} \\ &= \sum_{\alpha} D^\alpha g_\alpha. \end{aligned}$$

If finally Λ is of finite degree, then by the previous theorem finitelyy many functions are sufficient. □

5. Convolution

5.1. **Convolutions of functions.** Seuraavassa $\Omega = \mathbb{R}$.

Definition 5.1. The *Convolution* of two functions

$$u, v : \mathbb{R} \to \mathbb{C}$$

is the function

$$(u * v)(x) = \int_{\mathbb{R}} u(t)v(x-t)\,dt,$$

when the Lebesgue–integral exists.

Remark 5.2. If we denote by $\tau_x(v)$ the function v shifted by x: $\tau_x(v)(t) = v(t-x)$, and we denote by $\tilde{v}$ the function v reflected $\tilde{v}(t) = v(-t)$, then $\tau_x(\tilde{v})(t) = v(x-t)$ and the definition of convolution becomes

$$(u * v)(x) = \int_{\mathbb{R}} u(t)\,\tau_x(\tilde{v})(t)\,dt = \int_{\mathbb{R}} u\,\tau_x(\tilde{v})$$

Remark 5.3. Convolutions have some well known properties (real analysis!)

(1) GFenerally $u * v$ is as"smooth" as the smoother function u or v.
(2) $u * v = v * u$.
(3) In Fourier–transform convolution becomes product.

$$(uv)\hat{} = \hat{u} * \hat{v}.$$

(4) etc.
(5) ....
(6) ...

5.2. **Convolutions of distributions and functions.**

Definition 5.4. The *Convolution* of a function $\varphi \in \mathcal{D}$ and a distribution $\Lambda \in \mathcal{D}(\mathbb{R})^*$ is the $C^\infty-$function[32]

$$(\varphi * \Lambda)(x) = \langle \tau_x(\tilde{\varphi}), \Lambda \rangle,$$

which — of course — coincides with the previous definition 5.1, if $\Lambda = \Lambda_u$ is a function.[33]

Definition 5.5. The distribution $\Lambda \in \mathcal{D}(\mathbb{R})^*$ *is translated to the right* by x if it is replaced by $\tau_x\Lambda$, for which

$$\langle \varphi, \tau_x\Lambda \rangle = \langle \tau_{-x}\varphi, \Lambda \rangle.$$

Check sign! XXX

Theorem 5.6. *The convolution of a function and a distribution ha sthe following properties: for all $\Lambda \in \mathcal{D}(\mathbb{R})$, $\varphi, \psi \in Dr$, $n \in \mathbb{N}$, $x, y \in \mathbb{R}$:*

(i) $\tau_x(\Lambda * \varphi) = (\tau_x\Lambda) * \varphi = \Lambda * (\tau_x\varphi)$
(ii) $D^n(\Lambda * \varphi) = (D^n\Lambda) * \varphi = \Lambda * (D^n\varphi)$
(iii) $(\Lambda * \varphi) * \psi = \Lambda * (\varphi * \psi)$

Todistus. Obviously

(i) $\tau_x \circ \tau_x = \tau_{x+y}$ and $(\tau_x\varphi)^\sim = \tau_x\tilde{\varphi}$

(i):

$$\begin{aligned}(\tau_x(\Lambda * \varphi))(y) &= (\Lambda * \varphi)(y - x) = \langle \tau_{y-x}\tilde{\varphi}, \Lambda \rangle,\\ ((\tau_x\Lambda) * \varphi)(y) &= (\tau_x\Lambda)(\tau_y\tilde{\varphi}) = \langle \tau_{y-x}\tilde{\varphi}, \Lambda \rangle,\\ (\Lambda * (\tau_x\varphi))(y) &= (\Lambda(\tau_y(\tau_x\varphi)^\sim) = \langle \tau_{y-x}\tilde{\varphi}, \Lambda \rangle.\end{aligned}$$

(ii): Appy $\Lambda*$ to both sides of

$$\tau_x((D^n\varphi)^\sim) = (-1)^n D^n(\tau_x\tilde{\varphi})$$

and find

$$(\Lambda * (D^n\varphi))(x) = ((D^n\Lambda) * \varphi)(x),$$

which is the second equation in (ii). To prove the first we write for all $r > 0$

$$\eta_r = \frac{\tau_0 - \tau_r}{r}$$

and apply (i):

$$\eta_r(\Lambda * \varphi) = \Lambda * (\eta_r\varphi).$$

When $r \to 0$, then

$$\eta_r\varphi \to D\varphi \quad \in \mathcal{D}(\mathbb{R}),$$

so for all $x \in \mathbb{R}$

$$\tau_x((\eta_r\varphi)^\sim) \to \tau_x(D\varphi)^\sim \quad \in \mathcal{D}(\mathbb{R}),$$

and

$$\lim_{r\to 0}(\Lambda * (\eta_r\varphi))(x) = (\Lambda * (D\varphi))(x).$$

---

[32] A function like the smoother factor!

[33] Cf. 5.13 and 6.25.

Combining these we find

$$D(\Lambda * \varphi) = \Lambda * (D\varphi),$$

and by rewpetition (ii).

(iii): For every ψ and $\varphi \in \mathcal{D}(\mathbb{R})$:

$$(\varphi * \psi)^{\sim}(t) = \int_{\mathbb{R}} (\tilde{\psi}(s))\,(\tau_s\tilde{\varphi})(t)\,ds. \tag{5.1}$$

We write $\operatorname{supp}\tilde{\psi} = K_{\tilde{\psi}}$ and $\operatorname{supp}\tilde{\varphi} = K_{\tilde{\varphi}}$ amd $K = K_{\tilde{\psi}} + K_{\tilde{\varphi}}$. In 5.1 the right hand side can be interpreted as an integral of the $\mathcal{D}(\mathbb{R})-$valued continuous function

$$s \mapsto \tilde{\psi}(s)\tau_s\tilde{\varphi}$$

over the compact set $K_{\tilde{\psi}}$, otside of which it vanishes. Using theorems on vector valued integration (BOCHNER INTEGRAL) Wrt. a Borel measure will finish the proof.[34] Using this, 5.1 boils down to

$$(\varphi * \psi)^{\sim} = \int_{K_{\tilde{\psi}}} \tilde{\psi}(s)\,\tau_s\tilde{\varphi}\,ds \quad \in \mathcal{D}(\mathbb{R}). \tag{5.2}$$

This gives

$$\begin{aligned}(\Lambda * (\varphi * \psi))(0) &= \langle(\varphi * \psi)^{\sim}, \Lambda\rangle \\ &= \int_{K_{\tilde{\psi}}} \tilde{\psi}(s)\,\langle\tau_s\tilde{\varphi}, \Lambda\rangle\,ds \\ &= \int_{\mathbb{R}} \psi(-s)\,(\Lambda * \varphi)(s)\,ds \\ &= ((\Lambda * \varphi) * \psi)(0).\end{aligned}$$

This is (iii) at 0. The general case comes from translation by $\tau_{-x}\psi$. □

5.3. **Convolution -smoothing = Approximative identity.**

Definition 5.7. Let $\psi \in \mathcal{D}(\mathbb{R})$ be non-negative, $\operatorname{supp}\psi \subset [-1,1]$ and $\int_{\mathbb{R}} \psi = 1$. For all $n \in \mathbb{N}$ define *approximative identity* $\psi_n(x) = n\psi(nx)$, and notice $\psi_n \in \mathcal{D}(\mathbb{R})$ is non-negative, $\operatorname{supp}\psi \subset [-\frac{1}{n}, \frac{1}{n}]$ and $\int_{\mathbb{R}} \psi_n = 1$. Also $\|D^k\psi_n\|_\infty = n^{(k+1)}\|D^k\psi\|_\infty$ for all $k \in \mathbb{N}$.

Definition 5.8. The *Convolution -smoothing* of a Lebesgue-measurable function $f : \mathbb{R} \to \mathbb{R}$ is the convolution

$$f_n = (f * \psi_n) : \mathbb{R} \to \mathbb{R} : x \mapsto \int_{\mathbb{R}} f(x-t)\psi_n(t)\,dt.$$

Remark 5.9. The Approximative identity has the following properties: for all $\varphi \in \mathcal{D}(\mathbb{R})$, $\Lambda \in \mathcal{D}(\mathbb{R})^*$

1. $\varphi * \psi_n \to \psi \quad \in \mathcal{D}(\mathbb{R})$.
2. $\psi_n * \Lambda \to \Lambda \in \mathcal{D}(\mathbb{R})^*$ in the topology $\sigma(\mathcal{D}(\mathbb{R})^*, \mathcal{D}(\mathbb{R}))$. In particular smooth functions are weakly dense in $\mathcal{D}(\mathbb{R})^*$.

[34] Cf. Rudin, Functional Analysis Thm 3.27,

Proof. For each continuous function $f : \mathbb{R} \to \mathbb{C}$ we know $f * \psi_n \to f$ uniformly in compact sets. Applying this to function $f = D^k\varphi$ gives uniform convergence in compact sets $D^k(\varphi * \psi_n) = D^k\varphi * \psi_n \to D^k\varphi$. Since $\operatorname{supp}\varphi$ is compact and $\operatorname{supp}\psi \subset [-\frac{1}{n}, \frac{1}{n}]$, then the supports of all $\varphi * \psi_n$ are included in some same compact set this implying ia $\mathbb{R}$ for all k uniform convegence $D^k(\varphi * \psi_n) \to D^k\varphi$. This proves (1).

(2) follows now from 5.6 (iii), since

$$\begin{aligned} \langle\tilde{\varphi}, \Lambda\rangle &= (\Lambda * \varphi)(0) \\ &= \lim(\Lambda * (\psi_n * \varphi))(0) \\ &= \lim((\Lambda * \psi_n) * \varphi)(0) \\ &= \lim(\Lambda * \psi_n)(\tilde{\varphi}). \qquad \square \end{aligned}$$

5.4. **The convolution of two distributions.**

Definition 5.10. The *convolution* of two distributions Λ_1 and $\Lambda_2 \in \mathcal{D}(\mathbb{R})^*$ is the distribution $\Lambda_1 * \Lambda_2 = \delta_0 \circ (\Lambda_1 * \cdot) \circ (\Lambda_2 * \cdot)$:

$$\varphi \mapsto \langle \varphi\,,\, \Lambda_1 * \Lambda_2\rangle = (\Lambda_1 * (\Lambda_2 * \varphi))(0) \in \mathbb{C},$$

which turns out to be well defined at lest, when one of the original ones Λ_1 and Λ_2 has compact support. [35]

Remark 5.11. In order to find out when there exists a continuous linear mapping like

$$\mathcal{D}(\mathbb{R}) \xrightarrow{\Lambda_2 *\cdot} \mathcal{D}(\mathbb{R}) \xrightarrow{\Lambda_1 *\cdot} \mathcal{D}(\mathbb{R}) \xrightarrow{\delta_0} \mathbb{C}$$

we check,when the parts are well defined and continuous. We have to take care that that $(\Lambda_1 * \cdot)$ is defined and continuous in the image set $\{\Lambda_2 * \varphi \bigm| \varphi \in \mathcal{D}(\mathbb{R})\}$. The following theorems discus such that is.

Theorem 5.12. *Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$. The mapping $L = \Lambda * \cdot$ has the following properties:*

1. *L is continuous $\mathcal{D}(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$.*
2. L commutes with translations, *ie. for all $x \in \mathbb{R}$ we have $\tau_x L = L \circ \tau_x$*
3. *There are no other continuous linearmappings commuting with translation $\mathcal{D}(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$ except the ones mantioned above; $L = \Lambda *\cdot$, where $\Lambda\ in \mathcal{D}(\mathbb{R})^*$.*
4. $\Lambda_1 * \cdot = \Lambda_2 * \cdot \implies \Lambda_1 = \Lambda_2$*, so tje correspondence $\Lambda\ mapsto \Lambda * \cdot$ is bijective.*

Proof. (1) In **??** we proved that $\Lambda * \cdot \in \mathbb{C}^\infty(\mathbb{R})$. By**??** it is sufficinet to provethat the restriction of L to every subspace $\mathcal{D}_K(\mathbb{R})$ is continuous. Since both $\mathcal{D}_K(\mathbb{R})$ and $\mathbb{C}^\infty(\mathbb{R})$ are Fréchet-spaces, it is sufficient to prove that they have (sequentially) closed graphs [**??**]. Assume $\varphi_n \to \varphi \in \mathcal{D}_K(\mathbb{R})$ and $\Lambda * \varphi_n \to f \in \mathcal{D}_K(\mathbb{R})$. We have to prove that for all $x \in \mathbb{R}$ is $f(x) = (\Lambda * \varphi)(x)$. Let $x \in \mathbb{R}$. Since $\tau_x\tilde{\varphi}_n \to \tau_x\tilde{\varphi}$ in the space $\mathcal{D}(\mathbb{R})$, then

$$f(x) = \lim_{n\to\infty} (\Lambda * \varphi_n)(x) = \lim_{n\to\infty} \langle\tau_x\tilde{\varphi}_n, \Lambda\rangle = \langle\tau_x\tilde{\varphi}, \Lambda\rangle = (\Lambda * \varphi)(x).$$

(2) Implied by the previous and $L \circ \tau_x = \tau_x \circ L$.

[35] $\Lambda_2 * \varphi \in \mathcal{D}(\mathbb{R})$. But check also 5.14, and**??**.

(3) Let L be continuous and translation invariant. Define $\Lambda : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ by setting $\langle\varphi, \Lambda\ rangle = (L\tilde{\varphi})(0)$. Since reflection $\varphi \mapsto \tilde{\varphi}$ is a continuous linear mapping $\mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ and evaluation $\Delta_0 : \varphi \mapsto \varphi(0)$ is also continuous, Λ is a distribution. We assumed translation invariance $\tau_x L = L \circ \tau_x$, so

$$\begin{aligned}(L\varphi)(x) &= (\tau_{-x}L\varphi)(0) = (L\tau_{-x}\varphi)(0)\\ &= \langle(\tau_{-x}\varphi)^{\sim}, \Lambda\ rangle = \langle\tau_x\tilde{\varphi}\,, \Lambda\ rangle = (\Lambda * \varphi)(x).\end{aligned}$$

(4) It is sufficient to prove that $\Lambda * \cdot = 0 \implies \Lambda = 0$. Let $\Lambda * \cdot = 0$. For all $\varphi \in \mathcal{D}(\mathbb{R})$ is

$$\langle\Lambda, \tilde{\varphi}\rangle = (\Lambda * \varphi)(0) = 0.$$

□

Definition 5.13. Extend the definitions 5.4 by agreeing on that the convolution of a function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ and a compactly supported distribution $\Lambda \in \mathcal{D}(\mathbb{R})^*$ is the $C^\infty-$function

$$(\varphi * \Lambda)(x) = \langle\tau_x(\tilde{\varphi}), \Lambda\rangle.$$

Theorem 5.14. *The convolution of a function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ and compactly supported distribution $\Lambda \in \mathcal{D}(\mathbb{R})^*$ has the following properties for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$:*

(i) $\tau_x(\Lambda * \varphi) = (\tau_x\Lambda) * \varphi = \Lambda * (\tau_x\varphi)$
(ii) $\Lambda * \varphi \in \mathcal{C}^\infty(\mathbb{R})$
(iii) $D^n(\Lambda * \varphi) = (D^n\Lambda) * \varphi = \Lambda * (D^n\varphi)$

If $\psi \in \mathcal{D}(\mathbb{R})$, then

(iv) $\Lambda * \varphi \in \mathcal{D}(\mathbb{R})$ *and*
(v) $\Lambda * (\varphi * \psi) = (\Lambda * \varphi) * \psi = (\Lambda * \psi) * \varphi$.

Todistus. (i) – (iii) are proved like in 5.6. To prove (iv) notice that $\text{supp}(\tau_x\tilde{\psi}) = x - \text{supp}\,\psi$, so if $\text{supp}\,\Lambda \cap (x - \text{supp}\,\psi) = \emptyset$, then $(\Lambda * \psi)(x) = 0$. In other words $(\Lambda * \psi)(x) = 0$, when $x \notin \text{supp}\,\Lambda\ \cap \text{supp}\,\psi$ and so $\text{supp}(\Lambda * \psi) \subset \text{supp}\,\Lambda + \text{supp}\,\psi$ is compact.

Claim (v) is proven by reducing it to the corresponding statement in theorem 5.6: Take an open, bounded set $W \supset \text{supp}\,\Lambda$. Choose a (compactly supported) function $\varphi_W \in \mathcal{D}(\mathbb{R})$, for which $\tilde{\varphi}_W = \tilde{\varphi}$ in $W + \text{supp}\,\psi$. Now $(\varphi * \psi)^{\sim} = (\varphi_W * \psi)^{\sim}$ in W, so

$$(\Lambda * (\varphi * \psi))(0) = (\Lambda * (\varphi_W * \psi))(0). \tag{5.3}$$

Since for all $-s \in \text{supp}\,\psi$ we have
$\tau_s\tilde{\varphi} = \tau_s\tilde{\varphi}_W$ in W, then $\Lambda * \varphi = \Lambda * \varphi_W$ in $-\text{supp}\,\psi$. So

$$((\Lambda * \varphi) * \psi)(0) = ((\Lambda * \varphi_W) * \psi)(0). \tag{5.4}$$

Since $\text{supp}(\Lambda * \psi) \subset (\text{supp}\,\Lambda + \text{supp}\,\psi)$,

$$((\Lambda * \psi) * \varphi)(0) = ((\Lambda * \psi) * \varphi_W)(0). \tag{5.5}$$

The claiim follows for $x = 0$since the right sides of the equations (5.3)–(5.5) arte the same by 5.6. The general case is again proved by applying the special case ti the function $\tau_{-x}\psi$. □

Remark 5.15. The definition 5.10 is OK so the convolution of the distributions Λ_1 and $\Lambda_2 \in \mathcal{D}(\mathbb{R})^*$, namely $\Lambda_1 * \Lambda_2$ exists, when at least one of them has compact support. By definition 5.10 for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle \varphi\,,\, \Lambda_1 * \Lambda_2 \rangle = (\Lambda_1 * (\Lambda_2 * \varphi))(0) \in \mathbb{C},$$

so $\Lambda_1 * \Lambda_2 \in \mathcal{D}(\mathbb{R})'$. Let us verify that $\Lambda_1 * \Lambda_2 \in \mathcal{D}(\mathbb{R})^*$. Assume that $\varphi_n \to 0$. By (1) in theorem 5.12 $(\Lambda_2 * \cdot)$ is continuous $\mathcal{D}(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$, so $\Lambda_2 * \varphi_n \to 0$ in the topology of $\mathcal{C}^\infty(\mathbb{R})$. If $\operatorname{supp} \Lambda$ is compact, then $\Lambda_2 * \varphi_n \to 0$ in the topology of $\mathcal{D}(\mathbb{R})$. Therefore $\langle \varphi\,,\, \Lambda_1 * \Lambda_2 \rangle = (\Lambda_1 * (\Lambda_2 * \varphi_n))(0) \to 0$.

Theorem 5.16. *Assume Λ_1, Λ_2 and $\Lambda_3 \in \mathcal{D}(\mathbb{R})^*$.*

1. *If at least one of the distributions Λ_1, Λ_2 is compactly supported, then $\Lambda_1 * \Lambda_2 = \Lambda_2 * \Lambda_1$.*
2. *If at least one of the distributions Λ_1, Λ_2 has compact supprot kantajista K_1, K_2, then $K_{\Lambda_1 * \Lambda_2} \subset K_1 + K_2$.*
3. *If at least two of the distributionsΛ_1, Λ_2, Λ_3 has compact support K_1, K_2, K_3, then $(\Lambda_1 * \Lambda_2) * \Lambda_3 = \Lambda_1 * (\Lambda_2 * \Lambda_3)$.*[36]
4. *For all $n \in \mathbb{N}$ is $D^n \Lambda_1 = (D^n \delta_0) * \Lambda_1$, in particular $\delta * \Lambda_1 = \Lambda_1$. has compact support K_1, K_2, then $D^n(\Lambda_1 * \Lambda_2) = (D^n \Lambda_1) * \Lambda_2 = \Lambda_1 * (D^n \Lambda_2)$.*

PROOF. (1) Let φ, $\psi \in \mathcal{D}(\mathbb{R})$. Since function's convolution is commutative then by 5.6

$$(\Lambda_1 * \Lambda_2) * (\varphi * \psi) = \Lambda_1 * (\Lambda_2 * (\varphi * \psi)) = \Lambda_1 * ((\Lambda_2 * \varphi) * \psi) = \Lambda_1 * (\psi * (\Lambda_2 * \varphi)).$$

If K_2 is compact, then apply theorem 5.6 uudelleen. If taas K_1 is compact, then apply theorem 5.14. In both casers we get

$$(\Lambda_1 * \Lambda_2) * (\varphi * \psi) = (\Lambda_1 * \psi) * (\Lambda_2 * \varphi).$$

Since $\varphi * \psi = \psi * \varphi$, a similarr calculation gives

$$(\Lambda_2 * \Lambda_1) * (\varphi * \psi) = (\Lambda_2 * \varphi) * (\Lambda_1 * \psi).$$

In both cases the right hand side is the convolution of 2 functions, hence commutatuive. So they are the same. Therefore

$$(\Lambda_1 * \Lambda_2) * (\varphi * \psi) = (\Lambda_2 * \Lambda_1) * (\varphi * \psi)$$

eli

$$((\Lambda_1 * \Lambda_2) * \varphi) * \psi = ((\Lambda_2 * \Lambda_1) * \varphi) * \psi.$$

By the uniqueness part of the proof of 5.12 $\Lambda_1 * \Lambda_2 = \Lambda_2 * \Lambda_1$.

(2) Let $\varphi \in \mathcal{D}(\mathbb{R})$. A littloe calculation show such that at

$$\langle \varphi, \Lambda_1 * \Lambda_2 \rangle = \langle (\Lambda_2 * \tilde{\varphi})^\sim, \Lambda_1 \rangle.$$

By (1) we may assume that K_2 is compact. By the proof of 5.6,

$$\operatorname{supp}(\Lambda_2 * \tilde{\varphi}) \subset K_2 - K_\varphi.$$

Therefore $(\Lambda_1 * \Lambda_2) = 0$, when $K_1 \cap K_\varphi - K_2 = \emptyset$, eli $K_1 + K_2 \cap K_\varphi = \emptyset$.

[36]WARNING! FAILS if not compact support!

(3) By (2) bith $(\Lambda_1 * \Lambda_2) * \Lambda_3$ and $u * (\Lambda_2 * \Lambda_3)$ are defined, when at lest 2 of K_1, K_2, K_3 are compact. If $\varphi \in \mathcal{D}(\mathbb{R})$, then by the definition of convolution of distributions

$$(\Lambda_1 * (\Lambda_2 * \Lambda_3)) * \varphi = \Lambda_1 * ((\Lambda_2 * \Lambda_3) * \varphi) = \Lambda_1 * (\Lambda_2 * (\Lambda_3 * \varphi)).$$

If K_1 is compact, then

$$((\Lambda_1 * \Lambda_2) * \Lambda_3) * \varphi = (\Lambda_1 * \Lambda_2) * (\Lambda_3 * \varphi) = \Lambda_1 * (\Lambda_2 * (\Lambda_3 * \varphi)),$$

since by 5.6 $\Lambda_3 * \varphi \in \mathcal{D}(\mathbb{R})$. Combining these observations gives (3) if K_3 is compact. If K_3 is not compact, then K_1 is compact, and the previous case combined with commutativity (1) gives:

$$\Lambda_1*(\Lambda_2*\Lambda_3) = \Lambda_1*(\Lambda_3*\Lambda_2) = (\Lambda_3*\Lambda_2)*\Lambda_1 = \Lambda_3*(\Lambda_2*\Lambda_1) = \Lambda_3*(\Lambda_1*\Lambda_2) = (\Lambda_1*\Lambda_2)*\Lambda_3.$$

(4) If $\varphi \in \mathcal{D}(\mathbb{R})$, then $\delta_0 * \varphi = \varphi$, since

$$(\delta_0 * \varphi)(x) = \delta_0(\tau_x\tilde{\varphi}) = (\tau_x\tilde{\varphi})(0) = \tilde{\varphi}(-x) = \varphi(x).$$

So by (3) and 5.6 we infer thata

$$(D^n\Lambda_1) * \varphi = \Lambda_1 * D^n\varphi = \Lambda_1 * D^n(\delta_0 * \varphi) = \Lambda_1 * (D^n\delta_0) * \varphi.$$

(5) By (4),(3) and (1):

$$D^n(\Lambda_1 * \Lambda_2) = (D^n\delta_0) * (\Lambda_1 * \Lambda_2) = ((D^n\delta_0) * \Lambda_1) * \Lambda_2 = (D^n\Lambda_1) * \Lambda_2$$

and

$$((D^n)\delta_0) * \Lambda_1 * \Lambda_2 = (\Lambda_1 * D^n\delta_0) * \Lambda_2 = \Lambda_1 * ((D^n\delta_0) * \Lambda_2) = \Lambda_1 * D^n\Lambda_2.$$

6. Fourier transforms of tempered distributions

6.1. The space $\mathcal{S}$ and tempered distributions.

Definition 6.1. *A tempered distribution* [37], is an element of the topological dual of the space $\mathcal{S}$ defined below.

Definition 6.2. *The space of rapidly decreasing functions* is

$$\mathcal{S} = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}}(1+x^2)^N|D^nf(x)| < \infty \quad \forall N, n \in \mathbb{N}\}.$$

So a smooth function f is rapidly decreasing, if and only if every derivative converges to 0 in $\pm\infty$, even if itis multiplied with any polynomial.

$\mathcal{S}$ has the locally convex topology of the seminorms

$$\sup_{x\in\mathbb{R}}(1+x^2)^N|D^nf(x)| \qquad (N, n \in \mathbb{N})$$

topology lla.

Theorem 6.3. *$\mathcal{S}$ is a Fréchet-space.*

Proof. Easy. □

Theorem 6.4. *$\mathcal{S}$ relates to $\mathcal{D}(\mathbb{R})$ like this:*

(1) *$\mathcal{D}(\mathbb{R})$ is dense in the space $\mathcal{S}$.*

---

[37] Engl. tempered distribution.

(2) *Inclusion* $\mathcal{D}(\mathbb{R}) \to \mathcal{S}$ *is continuous*[38].
(3) $\mathcal{S}^* \subset \mathcal{D}(\mathbb{R})^*$ *so tempered distributions are distributions in the previous sense.*

Proof. (a) Let $f \in \mathcal{S}$. Choose $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(x) = 1$, when $|x| \leq 1$. Define $f_j(x) = f(x)\psi(\frac{x}{j})$. We have to prove that for all $N \in \mathbb{N}$ $(1+x^2)^N D^n(f_j - f) \to 0$ uniformly. We write $P(x) = (1+x^2)^N$

$$\begin{aligned} P(x)D^n(f_j(x) - f(x)) &= P(x)D^n\Big(f(x)\left(\psi(\tfrac{x}{j}) - 1\right)\Big) \\ &= P(x)\sum_{k=0}^{n} D^k f(x) \cdot D^{n-k}\big(\psi(\tfrac{x}{j})\big) - 1). \end{aligned}$$

When $j > m$, then $D^{n-k}\big(\psi(\frac{x}{j})\big) - 1) = 0$ in the interval $[-m, m]$. Since f is rapidly decreasing $p_{N,k}(f) = \sup_{x\in\mathbb{R}}(1+x^2)^N |D^n f(x)| < \infty$, for all $k = 0, \dots, n$. We are done!

(b) Since polynomials are continuous and therefore bounded in compact sets, $\mathcal{S}$ induces to its subspaces $\mathcal{D}_K(\mathbb{R})$ their original topology, and so does $\mathcal{D}(\mathbb{R})$. So the inclusions $\mathcal{D}_K(\mathbb{R}) \to \mathcal{S}$ are continuous, so by the properties of the direct inductive limit, **??** inclusion $\mathcal{D}(\mathbb{R}) \to \mathcal{S}$ is continuous.

(c) follows from (a) and (b). □

Example 6.5. The following are tempered distributions :

a) Consider
b) positive Borel-mesures μ, such that there exists $k \in \mathbb{N}$ such that:

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|^2)^k} < \infty$$

c) meassruable functions g, such that there exists $p \in [1, \infty[$ and $N > 0$ such that

$$\int_{\mathbb{R}} \left(\frac{g(x)}{(1+|x|^2)^N}\right)^p < \infty$$

d) polynomials.

Proof. Exercise (Ex set 11).

Lemma 6.6. *The derivative of a tempered distribution is a tempered distribution.*

Proof. The derivative of a rapidly decreasing function is rapidly decreasing, so the claim follows from the definition of distribution derivative. □

Remark 6.7. More properties of tempered distributions will follow later — partly proven using Fourier transforns.

The continuity of differentiation and of multiplication by certain functions can be proven now already:

Theorem 6.8. *The following linear mappings* $\mathcal{S} \to \mathcal{S}$ *are continuous :*

(1) *Differentiation.*
(2) *Multiplication by a polynomial.*

[38]The non-metrizable space $\mathcal{D}(\mathbb{R})$ cannot be a subspace of $\mathcal{S}$:n subspace.

(3) *Multiplication by a rapidly decreasing function.*

Todistus. Exercise (set 11). □

6.2. **The classical Fourier-transform.**

Definition 6.9. The *Fourier-transform* is defined in various function spaces by successive extension of the definition.

For simplicity of notation — later — we replace in $\mathbb{R}$ Lebesgue neasure dx, by $dm = \frac{1}{\sqrt{2\pi}}\,dx$.

Definition 6.10. The *Fourier-transform* of a $\mathcal{L}_1(\mathbb{R})-$function f, denoted $\mathcal{F}(f) = \hat{f}$ is defined by the classical formula:

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-itx}\,dx = \int_{\mathbb{R}} f(x)e^{-itx}\,dm.$$

Remark 6.11. f is complex valued (can be real), so is the transform $\hat{f}$.

Both $\hat{f}$ and the mapping $\mathcal{F} : f \mapsto \hat{f}$ are called *Fourier-transform* !

In short hand::

$$\hat{f}(t) = \int_{\mathbb{R}} f e_{-t}\,dm,$$

where the measure and also the exponential function are modified: by setting

$$e_t(x) = e^{ixt}.$$

We redefine also convolution by setting:

$$(f \hat{*} g)(x) = \frac{1}{\sqrt{2\pi}}(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dm(y).$$

Now the definition above becomes even shorter:

$$\hat{f}(t) = (f \hat{*} e_t)(0).$$

Remark 6.12. $e_t(x) = e^{itx}$ has the following properties:

a) $e_t(x+y) = e_t(x)e_t(y)$, ie. e_t is a group homomorphism $(\mathbb{R}, +) \to (\mathbb{C}^*, \cdot)$. The range $z \in \mathbb{C} \bigm| |z| = 1\}$. Such maps are called characters of the group[39]. $(\mathbb{R}, +)$ "*karakteeri*".

b) $e_t(x) = e_x(t)$

c) $De_t = ite_x$ so for each polynomial $P(z) = \sum \lambda_j z^j$ we have

$$\sum \lambda_j\, D^j\, e_t = P(it) \cdot e_t$$

If If we write $\sum \lambda_j\, D^j \; = P(D)$, then

$$P(D)\, e_t = P(it) \cdot e_t.$$

Theorem 6.13. *For $\mathcal{L}_1(\mathbb{R})-$functions the Fourier-transformation $\mathcal{F} : f \mapsto \hat{f}$ has the properties: For all $f,\, g \in \mathcal{L}_1(\mathbb{R})$ and $x,\, t \in \mathbb{R}$:*

a) *$\mathcal{F}$ is linear*

[39] This name is not needed in this course, nut gives the possibility ti generalize to groups. Haar measure!

b) $(\tau_x f)\hat{} = e_{-x}\hat{f}$
c) $(e_x f)\hat{} = \tau_{-x}\hat{f}$
d) $(f \hat{*} g)\hat{} = \hat{f}\,\hat{g}$
e) $(\frac{f}{\lambda})\hat{}(t) = \lambda\hat{f}(\lambda t)$, *when* $\lambda > 0$.

Proof. (a) is obvious.

(b) $$(\tau_x f)\hat{}(t) = \int_{\mathbb{R}} (\tau_x f)e_{-t}\,dm = \int_{\mathbb{R}} f\tau_{-x}e_{-t}\,dm = \int_{\mathbb{R}} fe_{-t}(x)e_{-t}\,dm = e_{-x}(t)\hat{f}(t).$$

(c) $$(e_x f)\hat{}(t) = \int_{\mathbb{R}} (e_x f)e_{-t}\,dm = \int_{\mathbb{R}} fe_{-(t-x)}\,dm = \tau_{-x}(t)\hat{f}(t).$$

(d) and (e) are exercises (Set 11). Hint: (d) Fubini, (e) linear change of variable. □

Definition 6.14. The *Fourier-transform* for $\mathcal{L}_2(\mathbb{R})-$functions f is defined by the following process:

(1) In $\mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$ the Fourier-transform is already defined.
(2) $\mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$ is dense in the space $\mathcal{L}_2(\mathbb{R})$.
(3) The Fourier-transform is a $\|\cdot\|_2$-isometry $\mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R})$ and can therefore be extended to an isometry in all of $L^2(\mathbb{R})$.
(4) The Fourier-transform $\mathcal{F}$ turns out to be surjective $\mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R})$, so it is a Hilbert-space isomorphism.**(Plancherel theorem.)** So $\mathcal{F}$ also preserves inner products in $\mathcal{L}_2(\mathbb{R})$ sisätulon.**(Parseval formula.)**
(5) In the space $\mathcal{L}_1(\mathbb{R}) \subset \mathcal{L}_2(\mathbb{R})$ The inverse of the Fourier-transform $\mathcal{F}^{-1}$ has the calssical foormula[40] **(Inversion formula.)**:

$$\mathcal{F}^{-1}g(x) = \frac{\mathbf{1}}{\sqrt{2\pi}}\int_{\mathbb{R}} g(t)e^{ixt}\,dt = \int_{\mathbb{R}} ge_x\,dm = (g \hat{*} e_x)(0).$$

Remark 6.15. We nwxt define Fourier-transformation for rapidly decreasing functions. As a side product we find proofs for the classical $\mathcal{L}_2(\mathbb{R})$–theorems above

6.3. **Fourier-transformations of rapidly decreasing functions.**

Theorem 6.16. a) *Por any function* $f \in \mathcal{S}$ *we have:*

$$(Df)\hat{}(t) = it\hat{f}(t) \quad \text{and}$$
$$(xf)\hat{}(t) = -D\hat{f}(t).$$

and so for every polynomial $P(z)$

$$(P(D)\,f)\hat{}(t) = P(it)\cdot\hat{f}(t) \quad \text{and}$$
$$(P\cdot f)\hat{}(t) = P(-D)\hat{f}(t).$$

b) *The Fourier-transform*

$$\mathcal{F} : f \mapsto \hat{f}(t) = \int_{\mathbb{R}} fe_{-t}\,dm$$

is a continuous linear mapping [41] $\mathcal{S} \to \mathcal{S}$.

[40] Pay attention to the sign of the exponent!
[41] Later in 6.19 it turns out to be aan isomorphism.

PROOF. (a) By 6.8 $Df \in \mathcal{S}$.

$$((Df)\hat{\ })(t) = ((Df) \hat{*} e_t)(0) = (f \hat{*} De_t)(0) = (f \hat{*} ite_t)(0) = it \cdot (f \hat{*} e_t)(0) = it\hat{f}(t),$$

so the first equation in (a) is proven. To prove the other calculate the derivative $D\hat{f}$ by its very original definition:

$$\begin{aligned} \frac{\hat{f}(t+\epsilon) - \hat{f}(t)}{\epsilon} &= \frac{1}{\epsilon}\left(\int_{\mathbb{R}} f(x)e^{-i(t+\epsilon)x}\,dm - \int_{\mathbb{R}} f(x)e^{-itx}\,dm\right) \\ &= i\int_{\mathbb{R}} xf(x)\frac{e^{-ix\epsilon}-1}{ix\epsilon}e^{-ixt}\,dm(x) \\ &\to i\int_{\mathbb{R}} xf(x)\ e^{-ixt}\,dm(x) = -(xf)\hat{\ }(t). \end{aligned}$$

Movin the limit out of the integral is legal by Lebesgue's dominated convergence theorem and the fact such that at $xf \in \mathcal{L}_1(\mathbb{R})$ and $e^{ixt}| = 1$. So the claim is true for the first derivatInduction gives the general case.

(b) Let $f \in \mathcal{S}$. We write $g(x) = (-1)^k x^k f(x)$. Then $g \in \mathcal{S}$. By (a) $\hat{g} = D^k\hat{f}$ and for all polynomials

$$P(x)D^k\hat{f}(x) = P(x)\hat{g}(x) = (P(-D)g)\hat{\ }(x)$$

is a bounded function, since $P(D)g \in \mathcal{L}_1(\mathbb{R})$. Therefore $\hat{f} \in \mathcal{S}$.

If $f_i \to f$ in the space $\mathcal{S}$, then a little calculationproves that $f_i \to f$ also in the space $\mathcal{L}_1(\mathbb{R})$. This implie such that at $\hat{f}_i(t) \to \hat{f}(t)$ for all $t \in \mathbb{R}$. By the closed graph theorem, pointwise convergence gives convergence in the space $\mathcal{S}$. (Exercise 9.1) $\square$.

The next theorems will lead to a proof of the inversion theorem. $\mathcal{S}$.

Theorem 6.17. *In $\mathcal{L}_1(\mathbb{R})-$ the Fourier-transform has the properties, for every $f \in \mathcal{L}_1(\mathbb{R})$*

$$\|\hat{f}\|_\infty \le \|f\|_1.$$

$\hat{f} : \mathbb{R} \to \mathbb{C}$ is a continuous function, vanishing in $\pm\infty$

PROOF. $\|\hat{f}\|_\infty \le \|f\|_1$ is implied by $|e_t(x)| = 1$ for all $x \in \mathbb{R}$.

Let $f \in \mathcal{L}_1(\mathbb{R})$. Since $\mathcal{S}$ is dense in the space $\mathcal{L}_1(\mathbb{R})$,we can find a sequence $(f_n)_{\mathbb{N}} \subset \mathcal{S}$, for which $\|f_n - f\|_1 \to 0$. Since $\mathcal{F} : \mathcal{S} \to \S$ every $\hat{f}_n \in \mathcal{S}$, and so it is a continuous function and häviää at $\pm\infty$. Since we have proven $\|\hat{f}\|_\infty \le \|f\|_1$ for all $f \in \mathcal{L}_1(\mathbb{R})$, also $\|f_n - f\|_\infty \le \|f_n - f\|_\infty \to 0$, so f is a continuous function and vanishes at $\pm\infty$. $\square$

Lemma 6.18. *The "Gauss function" $\phi(x) = e^{-x^2/2}$ is rapidly decreasing and has the properties*

a) $\hat{\phi} = \phi$.

b) $\phi(0) = \int_{\mathbb{R}} \hat{\phi}\,dm$.

Todistus. (a) ϕ is a solution of the linear differential equation $y' + xy = 0$ So is $\hat{\phi}$. Therefore $\phi/\hat{\phi}$ is a constant . Since both functions have value 1 at 0, they must coincide: $\hat{\phi} = \phi$.

(b) In combination with (a) the classical definition of Fourier-transform gives:

$$\phi(0) = \hat{\phi}(0) = \int_{\mathbb{R}} \phi\, dm = \int_{\mathbb{R}} \hat{\phi}\, dm. \qquad \square$$

Theorem 6.19. (Inversion formula) *The Fourier-trnsformation for rapidly decreasing function has the following properties:*

a)

$$g(x) = \int_{\mathbb{R}} \hat{g} e_x\, dm.$$

b) *The Fourier-transform*

$$\mathcal{F} : f \mapsto \hat{f}(t) = \int_{\mathbb{R}} f e_t\, dm$$

is a linear homeomorphism $\mathcal{S} \to \mathcal{S}$.

c) *The inversion formula holds for* $\mathcal{L}_1(\mathbb{R})$*–functions in the sense that if both* f *and* $\hat{f}$ *are integrable, then for almost all* $x \in \mathbb{R}$ *i*

$$f(x) = \int_{\mathbb{R}} \hat{f} e_x\, dm.$$

Todistus. If both f and g are integrablem then Fubini applies to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(y) e_{-ixy} dm^2,$$

giving

$$\int_{\mathbb{R}} \hat{f} g\, dm = \int_{\mathbb{R}} f \hat{g}\, dm. \tag{6.1}$$

(a) Since $g(x) = \tau_{-x} g(0)$, we only have to prove the inversion formula at $x = 0$, ie. that

$$g(0) = \int_{\mathbb{R}} \hat{g}\, dm.$$

In (6.1) choose $f(x) = \phi(\frac{x}{\lambda})$, where $\lambda > 0$ and ϕ is the function in lemma 6.18 $\phi(x) = e^{-x^2/2}$. This gives

$$\int_{\mathbb{R}} g(t) \lambda \hat{\phi}(\lambda t)\, dm(t) = \int_{\mathbb{R}} \phi\left(\tfrac{y}{\lambda}\right) \hat{g}(y)\, dm(y).$$

A change of variables on the left:

$$\int_{\mathbb{R}} g\left(\tfrac{t}{\lambda}\right) \hat{\phi}(t)\, dm(t) = \int_{\mathbb{R}} \phi\left(\tfrac{y}{\lambda}\right) \hat{g}(y)\, dm(y).$$

When $\lambda \to \infty$, then $g\left(\frac{t}{\lambda}\right) \to 0$ and $\phi\left(\frac{y}{\lambda}\right) \to \phi(0)$ and Lebesgue dominated convergence gives

$$g(0) \int_{\mathbb{R}} \hat{\phi}\, dm = \phi(0) \int_{\mathbb{R}} \hat{g}\, dm,$$

which is what we want, since $\phi(x) = e^{-x^2/2}$.

(b) By (a) the Fourier-transform $\mathcal{F} : g \mapsto \hat{g}$ is injective $\mathcal{S} \to \mathcal{S}$. By the inversion formula $(\mathcal{F} \circ \mathcal{F}) g\,(x) = g(-x)$, so $\mathcal{F}^4$ is the identical mapping and therefore $\mathcal{F}(\mathcal{S}) \supset$

$\mathcal{F}(\mathcal{F}^3(\mathcal{S})) = \mathcal{F}^4(\mathcal{S}) = \mathcal{S}$, so $\mathcal{F}$ is surjektive, hence a bijection. Continuity of $\mathcal{F}$ was proven at **??**, so the inverse $\mathcal{F}^3$ is continuous.

(c) Let

$$f_0(x) = \int_{\mathbb{R}} \hat{f} e_x \, dm.$$

Inserting this into the equation 6.1 and using Fubinbi once more gives

$$\int_{\mathbb{R}} f_0 \hat{g} \, dm = \int_{\mathbb{R}} f \hat{g} \, dm.$$

This is valid for all $g \in \mathcal{S}$ so for all $\hat{g} \in \mathcal{S}$, in particular for all $\hat{g} \in \mathcal{D}(\mathbb{R})$, same as $\int_{\mathbb{R}} (f_0 - f)\varphi \, dm = 0$ to all $\varphi \in \mathcal{D}(\mathbb{R})$. Therefore $f_0 = f$ ae. □

Corollary 6.20. *For rapidly decreasing f and g:*

a) $(fg)\hat{} = \hat{f} \hat{*} \hat{g}$ *and*
b) $f \hat{*} g \in \mathcal{S}$.

Todistus. (a) By 6.13 $\mathcal{F}(f \hat{*} g) = \mathcal{F}f \cdot \mathcal{F}g$. Replace f:n and g by $\mathcal{F}f$ and $\mathcal{F}g$:

$$(\mathcal{F}(\hat{f} \hat{*} \hat{g}))(x) = (\mathcal{F}^2 f)(x) \cdot (\mathcal{F}^2 g)(x) = f(-x) \cdot g(-x) = (fg)(-x) = \mathcal{F}^2(fg)(x).$$

To $(\mathcal{F}(\hat{f} \hat{*} \hat{g})) = \mathcal{F}^2(fg)$ apply $\mathcal{F}^{-1}$ and you have proven (a). Since obviously $fg \in \mathcal{S}$, $\mathcal{S} \ni (fg)\hat{} = \hat{f}\hat{*}\hat{g}$, so (b) is true, since $\mathcal{F}$ is a surjektion $\mathcal{S} \to \mathcal{S}$. □

Theorem 6.21. (Plancherel and Parseval) *The Fourier-transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ has a unique extension to an isometrical isomorphism $\mathcal{F} : \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R})$.*

Proof. Obviously $\mathcal{S} \subset \mathcal{L}_2(\mathbb{R})$. We prove, that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ preservs inner products in Hilbert space $\mathcal{L}_2(\mathbb{R})$. Let f and $g \in \mathcal{S}$. By the inversion formula ($\bar{z}$ is the complex conjugate of z).

$$\begin{aligned}
(g, f) = \int_{\mathbb{R}} \bar{g} f \, dm &= \int_{\mathbb{R}} \bar{g}(x) \big(\int_{\mathbb{R}} \hat{f}(t) e^{itx} \, dm(t) \big) \, dm(x) \\
&= \int_{\mathbb{R}} \hat{f}(t) \big(\int_{\mathbb{R}} \bar{g}(x) e^{itx} \, dm(x) \big) \, dm(t) \\
&= \int_{\mathbb{R}} \hat{f}(t) \big(\int_{\mathbb{R}} \overline{g(x) e^{-itx}} \, dm(x) \big) \, dm(t) = (\hat{g}, \hat{f}) \\
&= \int_{\mathbb{R}} \hat{f}(t) \overline{\hat{g}(t)} \, dm(t) = (\hat{g}, \hat{f}).
\end{aligned}$$

Like in the space $\mathcal{L}_1(\mathbb{R})$ $\mathcal{S}$ is also dense in the space $\mathcal{L}_2(\mathbb{R})$. The Fourier-transform is a bijection $\mathcal{S} \to \mathcal{S}$, soit is an isometric isomorphism from a dense subset of the metric space $\mathcal{L}_2(\mathbb{R})$ to a dense subset of the metric space $\mathcal{L}_2(\mathbb{R})$. Its extension to completions is obviously isometric, so it preserves inner products also, and w¨we have proved Parseval's formula in the space $\mathcal{L}_2(\mathbb{R})$. □

6.4. **Fourier-transforms of tempered distributions.**

Definition 6.22. The Fourier-transform of a tempered distribution $\Lambda \in \mathcal{S}^*$ is defined by

$$\langle \varphi, \hat{\Lambda} \rangle = \langle \hat{\varphi}, \Lambda \rangle,$$

in other words, it is the transpose of the Fourier-transform for rapidly decreasing test functions.

Example 6.23. We have proved in 6.8 that calculus with tempered distributions goes almost like calculus with functions, for example differentiation and multiplication by polynomials are possible and are continuous linear mappings $\mathcal{S}^* \to \mathcal{S}^*$. The same is true for tempered distributions: (Some proofs come a little later)

Theorem 6.24. *For all $\Lambda \in \mathcal{S}^*, \phi, \psi \in \mathcal{S}, f \in \mathcal{L}_1(\mathbb{R})$)*

a) $\Lambda_{\hat{f}} = (\Lambda_f)\hat{}$ *for all $f \in \mathcal{L}_1(R)$, in particular:*
b) $\hat{1} = \delta_0$, $\hat{\delta}_0 = 1$
c) *The Fourier-transform $\mathcal{F} : \Lambda \mapsto \hat{\Lambda}$ is a linear homeomorphism $\mathcal{S}^* \to \mathcal{S}^*$.*
d) *$\mathcal{F}^4$ is the identical mapping, ie. $\mathcal{F}^{-1} = \mathcal{F} \circ \mathcal{F} \circ \mathcal{F}$.*
e) $(D\Lambda)\hat{} = it\hat{\Lambda}$ *and* $(x\Lambda)\hat{} = -D\hat{\Lambda}$
f) $(\hat{\Lambda})\hat{} = \tilde{\Lambda}$, *where* $\langle \varphi, \tilde{\Lambda} \rangle = \langle \tilde{\varphi}, \Lambda \rangle$.**(inversion formula)**

Todistus. (a) $(\Lambda_f)\hat{}(\phi) = \Lambda_f(\hat{\phi}) = \int_{\mathbb{R}} f\hat{\phi} = \int_{\mathbb{R}} \hat{f}\phi = (\Lambda_{\hat{f}})(\phi)$.

(b) $\langle \phi, 1 \rangle = \int_{\mathbb{R}} \phi \, dm$, so $\langle \phi, \hat{1} \rangle = \langle \hat{\phi}, 1 \rangle = \int_{\mathbb{R}} \hat{\phi} \, dm = \phi(0) = \langle \phi, \delta_0 \rangle$. Toiset yhtälöt todistetaan correspondingsti.

(c) Follows from the fact that $\mathcal{F}$ is the transpose of a linear homeomorphism (Cf. **??**)

(d), (e) and (f) Follow from the definition and that rapidly decreasing functions enjoy the same property. □

Definition 6.25. The *convolution* of a tempered distribution and a rapidly decreasing function is defined by

$$(\varphi * \Lambda)(x) = \langle \tau_x(\tilde{\varphi}), \Lambda \rangle,$$

which extends definition 5.4, which was for compactly supported functions $\Lambda = \Lambda_f$.

The next lemma is used for differentiation of convolutions of tempered distributions .The proof uses Fourier transforms.At last some use!

Lemma 6.26. *Differentiation is the limit of the classical defining fraction also in the topology of $\mathcal{S}$ ie. for all $\varphi \in \mathcal{S}$ and $x \in \mathbb{R}$:*

$$\lim_{\epsilon \to 0} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} - D\varphi(x) = 0$$

in the topology of $\mathcal{S}$.

Proof. Since the Fourier-transform is a homeomorphism in the topology of $\mathcal{S}$, we may instead prove that

$$\lim_{\epsilon \to 0} \mathcal{F}\left(\frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon} - D\varphi(x) \right) = 0$$

same as

$$\lim_{\epsilon\to 0}\left(\frac{\int_{\mathbb{R}}(\varphi(x+\epsilon)-\varphi(x))e^{-itx}\,dm(x)}{\epsilon}-\mathcal{F}(D\varphi)(t)\right)=0$$

same as

$$\lim_{\epsilon\to 0}\left(\frac{e^{i\epsilon t}-1}{\epsilon}\int_{\mathbb{R}}\varphi(x)e^{-itx}\,dm(x)-it\hat{\varphi}(t)\right)=0$$

same as

$$\lim_{\epsilon\to 0}\left(\frac{e^{i\epsilon t}-1}{\epsilon}\hat{\varphi}(t)-it\hat{\varphi}(t)\right)=0$$

same as

$$\frac{e^{i\epsilon t}-1}{\epsilon}\,\hat{\varphi}(t)\to 0.$$

We know that $\hat{\varphi}\in\mathcal{S}$. For any number $k\in\mathbb{N}$ and polynomial - in particular for $P(t)=(1+t^2)^N$, we have

$$P(t)D^k\left(\frac{e^{i\epsilon t}-1}{\epsilon}\,\hat{\varphi}(t)\right)=P(t)\cdot\left(\sum_{j=0}^{k}\binom{k}{j}D^j\left(\frac{e^{i\epsilon t}-1}{\epsilon}\right)\,D^{k-j}\hat{\varphi}(t)\right).$$

Calculating the derivative gives

$$D^j\left(\frac{e^{i\epsilon t}-1}{\epsilon}-it\right)=\begin{cases}\frac{e^{i\epsilon t}-1}{\epsilon}-it, & \text{when } j=0\\ \frac{i\epsilon e^{i\epsilon t}}{\epsilon}-i=ie^{i\epsilon t}-i, & \text{when } j=1\\ i^k\epsilon^{j-1}e^{i\epsilon t}, & \text{when } j\geq 2\end{cases}$$

so

$$\sup_{t\in\mathbb{R}}|D^j\left(\frac{e^{i\epsilon t}-1}{\epsilon}-it\right)|\to 0.$$

Since $\varphi\in\mathcal{S}$, then convergence in the topology of $\mathcal{S}$ is proven. □

Theorem 6.27. *Convolutions and Fourier-transforms of tempered distributions have the following properties: (For all $\Lambda\in\mathcal{S}^*,\varphi,\psi\in\mathcal{S},f\in\mathcal{L}_1(\mathbb{R})$)*

a) $\Lambda\,\hat{*}\,\varphi\in\mathcal{C}^\infty(\mathbb{R})$ *and* $D^n(\Lambda\,\hat{*}\,\varphi)=(D^n\Lambda)\,\hat{*}\,\varphi=\Lambda\,\hat{*}\,(D^n\varphi)$.

b) $\Lambda\,\hat{*}\,\varphi$ *is a tempered distribution, in fact a polynomially increasing function .*

c) $(\Lambda\,\hat{*}\,\varphi)\hat{}=\hat{\varphi}\Lambda$.

d) $(\Lambda\,\hat{*}\,\varphi)\,\hat{*}\,\psi=\Lambda\,\hat{*}\,(\varphi\,\hat{*}\,\psi)$.

e) $\hat{\Lambda}\,\hat{*}\,\hat{\varphi}=(\varphi\Lambda)\hat{}$.

Todistus. (a) Proven just like the corresponding theorem for Schwartzin distributions 5.6, except that **lemma 6.26** is used in the critical stage.[42]

(b)

No too difficult (?)

THE END (SOME APPLICATIONS STILL COME)

[42]Do it! XXX

Solutions or hints to Exercises
Topological vector spaces 2010

Exercise set 8.

8.5. *Let $P_n = \{f : \mathbb{K} \to \mathbb{K} \bigm| p$ is a polynomial of degree at most $n-1$ $\}$ with its natural structure as Hausdorff- topological vector space. Let $P = \bigcup_{n\in\mathbb{N}} P_n = \{f : \mathbb{K} \to \mathbb{K} \bigm| p$ is a polynomial$\}$ with the structure of direct inductive limit: $P = \varinjlim P_n$. Is P metrizable in this topology?τ? Is P a Montel space?*

Solution. The Hausdorff-tvs-topology is unique. We can define it by the norm $\| \sum_{n\in\mathbb{N}} a_n x^n \| = \sum_{n\in\mathbb{N}} |a_n|$.

Since every P_n on finite dimensiona, it is a Banach space, so P is a $\mathcal{LB}$- space and therefore a bornological Montel space. (Remenber: TA barreled space where in which every closed, bounded set is compact, is a *Montel space*, in particular any finite dimensional (!) normed (or tvs) space is a bornological Montel space (Heine-Borel)). Therefore also P is a bornological Montel space, ksince these properties are inherited by the direct inductive limit space. **??**. As a countable union of its subspaces, P is of Baire 1 category. But it is complete, so it cannot be metrizable — by Baire's theorem.

8.6. *(continuation to (a)) Introduce another topology on P, call it τ_0, whose defining seminorms are $q_k(\sum_{i\in\mathbb{N}} \lambda_i x^i) = |\lambda_k|$. I such that is topology metrizable ? Or Montel? Is one of the topologies τ and τ_0 finer than the other?*

Solution. τ_0 is obviously metrizable, since the seminorm family is countable. (+Hausd). Since the inclusions $P_i \to P$ are continuous also in τ_0, we conclude that τ is finer than τ_0. Since the topologies are not identical, τ is strictly finer than τ_0. □

8.7. *Prove that the function spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$ are Montel.*

Solution. Since $\mathcal{D}(\Omega)$ is a direct indudctive limit of the spaces $\mathcal{D}_K(\Omega)$, it is sufficient to prove that the spaces $D_K(\Omega)$ are Montel. As Fréchet-space such that ey are barreled, so it is sufficient to prove that they satisfy the Heine-Borel-property, ie. every closed, bounded set is compact. Let $H \subset D_K(\Omega) \subset \mathcal{C}^\infty(\Omega)$ be closed and bounded. Let $f \in H$ and x_1 and $x_2 \in K$.

$$|f(x_1) - f(x_2)| \le \|Df\|_\infty |x_1 - x_2|.$$

Similarly:

$$|D^k f(x_1) - D^k f(x_2)| \le \|D^{k+1} f\|_\infty |x_1 - x_2|.$$

By Ascoli's theorem, since $\mathbb{R}$ on complete metrizable and K compact metrizable and $\mathcal{H} \subset \mathcal{C}(K, X) = \{f : K \to X \bigm| f \text{ is continuous}\}$, the following are equivalent:

(1) $\mathcal{H}$ is relatively compact ie. $\overline{\mathcal{H}}$ is compact **in the sup-norm**.
(2) (a) $\mathcal{H}$ is equicontinuous.
  (b) $\mathcal{H}(x) \subset \mathbb{R}$ is relatiively compact (same as bounded, since we are in $\mathbb{R}$:ssä) for all $x \in K$.

Obviously our equation implies (a) and (b), so the closure of the (sup-(!))closed set H wrt. **sup-norm** $\overline{\mathcal{H}}$ is compact in the **sup-norm**. The same construction and ideas work in the other equation for higher derivatives. It may be difficult to prove directly that H is compact in its locally convex topology, given by the sup-norms of derivatives. But since this topology is metrizable, it is sufficient to prove that H is totally bounded

The topology can be defined by the incresing sequence of norms $|||\cdot|||_n = \|\cdot\|_\infty + \|\cdot\|_1 + \|\cdot\|_2 + \cdots + |||\cdot\|_n$. Let $\epsilon > 0$. By what we already knoe, H is included in jsome compact, siis in particular precompact set separately for each norm $\|f\|_n = \|D^n f\|_\infty$, so for each $n \in \mathbb{N}$ there exists a finite covering of H by subsets $H_{n,i} \subset H,\ i = 1, 2, \dots m_n$ such that each $H_{n,i}$ has diameter (in the $\|\cdot\|_n$-sense) iat most ϵ:

$$\|D^n f - D^n g\|_n < \epsilon \text{ for all } f, g \in H_{n,i}.$$

Now we can construct a finite covering of H by the sets $H_{n,i} \subset H,\ i = 1, 2, \dots m_n$ such that the diameter of each $H_{n,i}$ in $|||\cdot|||_n$-sense is at most ϵ: The intersections $H_{0,i_0} \cap H_{1,i_1} \cap \cdots \cap H_{n,i_n}$, form such a covering when the indices i_j run through a ll possible values, ie each $i_j \in \{1, \dots, m_j\}$. (!) □

(There was a shorter proof inthe exercises?)

8.8. *Prove that the derivative of a $\mathbb{C}^\infty$–function f coincides with its distribution-derivative.(interpreted correctly).*

Solution. Easy.

8.9. *Prove that in the distribution derivaative sense*

$$\frac{d}{dx} \log |x| = v.p.\frac{1}{x}.$$

Notation v.p. (value principale) means Cauchy principal value for integral. Define

$$v.p. \int_\infty^\infty f = \lim_{\epsilon \to 0+} \left(\int_\infty^{-\epsilon} f + \int_\epsilon^\infty f \right)$$

As a distribution $v.p.f$ neans the linear form

$$\mathcal{D} \to \mathbb{R} : \varphi \mapsto \langle \varphi,\, v.p.f \rangle = v.p \int_\infty^\infty f\varphi.$$

Solution. By the definition of distribution derivative, $\langle D\log|x|,\varphi\rangle = -\langle \log|x|, D\varphi\rangle$. Since $\log|x|$ is locally integrable in the set $\mathbb{R}$[43], so the right hand side can be interpreted as a distribution by $-\int_{-\infty}^{\infty}\log|x|\; D\varphi\,dx$ and we arrive at:

$$\begin{aligned}
\langle D\log|x|,\varphi\rangle &= -\int_{-\infty}^{\infty}\log|x|\; D\varphi\,dx \\
&= \lim_{\epsilon\to 0}\big(-\int_{-\infty}^{-\epsilon}\log|x|\; D\varphi\,dx - \int_{\epsilon}^{\infty}\log|x|\; D\varphi\,dx\big) \\
&= \lim_{\epsilon\to 0+}\left(\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx - \Big|_{-\infty}^{-\epsilon}\log|x|\;\varphi(x) + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}\,dx - \Big|_{\epsilon}^{\infty}\log|x|\;\varphi(x)\right) \\
&= v.p.\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx - \lim_{\epsilon\to 0+}\left(\Big|_{-\infty}^{-\epsilon}\log|x|\;\varphi(x) - \Big|_{\epsilon}^{\infty}\log|x|\;\varphi(x)\right) \\
&= v.p.\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx - \lim_{\epsilon\to 0+}\big(\log(\epsilon)\,(\varphi(-\epsilon)-\varphi(\epsilon))\big) = v.p.\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx.
\end{aligned}$$

8.10. *Let E be a Fréchet-spaces and $F = \lim\limits_{\to} F_n$ a $\mathcal{LF}$–space. Let $T : E \to F$ be a continuous linear mapping. Prove that there exists a number $k \in \mathbb{N}$ such that $T(E) \subset F_k$.*

Solution. (This was also improved in the exx but I have not yet written the better proof. Here is the longer one:)

Let $H_n = \{(x,Tx) \in E \times F \;\big|\; Ty \in F_n\}$. They are closed subspacdes in the products of 2 Fréchet-spaces so they are Fréchet-spaces. Let $\pi_n : H_n \to F : (x,y) \mapsto x$ be the projections. Since $\bigcup_n F_n = F$, we have $\bigcup_n H_n = T$ so by Baire some closed set H_n has an interior point. The mapping $\pi_2 : Gr(T) \to F : (x,y) \mapsto y$ is a continuous linear mapping betwween Fréchet–spaces, so it is open . It maps int points to int points in the complete set $H_n \subset T\;\pi_2(H_n)$, but this is a subspoace, so it must be all of F. But $\pi_2(H_n) \subset F_n$.

[43] $\mathbb{R}_+$, $\int \log x = -x + x\log x$ is even continuous, since $\lim_{x\to 0}(-x + x\log x) = 0$.

Exercise set 9.

9.11. *Assume $f \in \mathcal{C}^\infty(\mathbb{R})$ and $\Lambda \in \mathcal{D}(\mathbb{R})$. Prove or disprove:*

$$D(f\Lambda) = Df\,\Lambda + f\,D\Lambda.$$

Solution. The derivative of the istribution $\Lambda \in D^*$. is $D\Lambda := -\Lambda \circ D$, ie.

$$\langle \varphi, D\Lambda \rangle := -\langle D\varphi, \Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

So we calculate:

$$\begin{aligned}
\langle \varphi, D(f\Lambda) \rangle &= -\langle D\varphi, f\Lambda \rangle \\
&= -\langle fD\varphi, \Lambda \rangle \\
&= -\langle D(f\varphi) - (Df)\varphi, \Lambda \rangle \\
&= -\langle D(f\varphi), \Lambda \rangle + \langle \varphi(Df), \Lambda \rangle \\
&= \langle f\varphi, D\Lambda \rangle + \langle \varphi, (Df)\Lambda \rangle \\
&= \langle \varphi, fD\Lambda \rangle + \langle \varphi, (Df)\Lambda \rangle = \langle \varphi, Df\,\Lambda + f\,D\Lambda \rangle.
\end{aligned}$$

9.12. *The space $\mathcal{D}(\mathbb{R})^*$ has the weak topology $\sigma(\mathcal{D}(\mathbb{R})^*, \mathcal{D}(\mathbb{R}))$. Assume that $(f_n)_\mathbb{N} \to f$ is convergent sequence in the Fréchet space $\mathcal{C}^\infty(\mathbb{R})$ (Seminorms sup-norms of derivatives in compact sets) and the sequence $(\Lambda_n)_\mathbb{N} \to \Lambda$ is convergent in the distribution space $\mathcal{D}(\mathbb{R})^*$.Prove that $(f_n\Lambda_n)_\mathbb{N} \to f\Lambda$ is a convergent sequence in the space $\mathcal{D}(\mathbb{R})^*$.*

Solution. Since $\Lambda_n \to \Lambda$, for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle \varphi, \Lambda_n \rangle \to 0.$$

Since $f_n \to f$, for all $k \in \mathbb{N}$ and compact $K \subset \mathbb{R}$

$$\sup_K |D^k f| \to 0.$$

The product of a distribution and a function is defined like in exercise (1).We have to prove that for all $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle \varphi, f_n\Lambda_n \rangle = \langle f_n\varphi, \Lambda_n \rangle \to 0.$$

Since $f_n \mapsto f_n\varphi$ is continuous $\mathcal{C}^\infty(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$, we have $f_n\varphi \to f\varphi \in \mathcal{D}(\mathbb{R})$.

The bilinear mapping

$$\mathcal{C}^\infty(\mathbb{R}) \times \mathcal{D}(\mathbb{R})^* \to \mathbb{C} : (f, \Lambda) \mapsto \langle \varphi, f\Lambda \rangle = \langle f\varphi, \Lambda \rangle$$

is continuous for each variablke separately. The next lemma prove such that at this implies sequential continuitu in the product topology – and this is continuity, since the space is metrizable.

Lemma 9.28. *(For exercise 9.2 Let $B : E \times F \to \mathbb{R}$ be a bilinear mapping; E and F Fréchet-spaces. (Sufficient: E is Fréchet.) The following are equivalent.*

a) *B is (sequentially) continuous (in the product topology) (at the origin).*
b) *B is (separately) continuous (at the origin).*

Proof. Obviously a) $\implies$ b) Assume b), and consider a sequence $(x_n, y_n) \to 0 \in E \times F$ in other words $x_n \to 0 \in E$ and $y_n \to 0 \in F$. We prove, that $B(x_n, y_n) \to 0 \in \mathbb{R}$.

Let $U \in \mathcal{U}_{\mathbb{R}}$.

For fixed $x \in E$, the sequence $B(x, y_n) \subset \mathbb{R}$ is convergent, so bounded. Every mapping $x \mapsto B(x, y_n)$ is continuous, so $\{x \mapsto B(x, y_n) \bigm| n \in \mathbb{N}\}$ is a pointwise bounded family of continuous linear mappings. By the principle of uniform boundedness such a family is family is equicontinuous, so there exists neighbourhood of the origin $W \in \mathcal{U}_E$ such that $B(x, y_n) \in U$ for every $n \in \mathbb{N}$ and $x \in W$. Choose a $n_U \in \mathbb{N}$ such that $n \geq n_U \implies x_n \in W$. Now $B(x_n, y_n) \in U$. □

9.13. *Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$ be a distribution. Assume*

$$W = \bigcup \{\omega \overset{open}{\subset} \Omega \bigm| \langle f, \Lambda\rangle = 0, \ \textit{When} \ \operatorname{supp} f \subset \omega\}.$$

Prove that $\langle f, \Lambda\rangle = 0$, when $\operatorname{supp} f \subset W$. (Hint: Partitions of unity.)

Solution. $\{\omega \overset{\text{open}}{\subset} \Omega \bigm| \langle f, \Lambda\rangle = 0, \ \text{When} \ \operatorname{supp} f \subset \omega\} = (\omega_i)_{i \in I}$ is a family of open sets and $\omega_i \subset \mathbb{R}$ and $W = \bigcup_{i \in I} \omega_i$. By 9.6. (Partitions of unity) there exists a sequence of functions $(\psi_n)_{n \in \mathbb{N}}$ such that

a) $\forall n \in \mathbb{N} \, \exists i \in I$ such that $\operatorname{supp} \psi_n \subset \omega_i$.
b) $\sum_{n \in \mathbb{N}} \psi_n = 1$ in the set W.
c) To each compact $K \subset W$ there exists an open $A \supset K$ and a number $m \in \mathbb{N}$ such that $\sum_{n=1}^{m} \psi_n = 1$ in the set A.

Let $f \in \mathcal{D}(\mathbb{R})$ and $K = \operatorname{supp} f \subset W$. Since K is compact, c) can be applied. We may assume that $A \subset W$. (Intersect with W if needed.)

$$f = f \cdot 1 = \sum_{n=1}^{m} \psi_n f$$

in the set A, and outside A $f = \sum_{n=1}^{m} \psi_n f$, namely 0. Therefore

$$\langle f, \Lambda\rangle = \sum_{n=1}^{m} \langle \psi_n f, \Lambda\rangle.$$

But every $\langle \psi_n f, \Lambda\rangle$ is 0, since $\operatorname{supp} \psi_n f \subset \operatorname{supp} \psi_n \subset \omega_i$ for some $i \in I$. □

9.14. *Let $\Lambda \in \mathcal{D}(\mathbb{R})^*$ be a distribution. Prove that*

a) *If $\varphi \in \mathcal{D}(\Omega)$ and $\operatorname{supp} \varphi \cap \operatorname{supp} \Lambda = \emptyset$, then $\langle \varphi, \Lambda\rangle = 0$.*
b) *If $\psi \in \mathcal{C}^\infty(\Omega)$ and $\psi(x) = 1$ in some open set $A \supset \operatorname{supp} \Lambda$, then $\Lambda\psi = \Lambda$.*

Solution. a) By definition, $\operatorname{supp} \Lambda = \mathbb{R} \setminus W$, where

$$W = \bigcup \{\omega \overset{\text{open}}{\subset} \Omega \bigm| \langle f, \Lambda\rangle = 0, \ \text{when} \ \operatorname{supp} f \subset \omega\}.$$

So if $\operatorname{supp} \varphi \cap \operatorname{supp} \Lambda = \emptyset$, then $\operatorname{supp} f \subset W$, so by the previous exercise $\langle f, \Lambda\rangle = 0$
b) Let $\psi \in \mathcal{C}^\infty(\Omega)$ and $\psi(x) = 1$ in some open $A \supset \operatorname{supp} \Lambda$. We prove that $\Lambda\psi - \Lambda = 0$. Since $\psi(x) = 1$ in $A \supset \operatorname{supp} \Lambda$, then for all $\varphi \in \mathcal{D}(\Omega)$ and $x \in A$ is

$$\varphi(x) - (\psi\varphi)(x) = \varphi(x) - (1\varphi)(x) = 0.$$

Therefore $\operatorname{supp}(\varphi - \psi\varphi) \cap \operatorname{supp}\Lambda = \emptyset$. So by a)

$$\langle \varphi - \psi\varphi, \Lambda \rangle = 0 \text{ eli}$$
$$\langle \varphi, \Lambda \rangle = \langle \psi\varphi, \Lambda \rangle = \langle \varphi, \psi\Lambda \rangle \text{ for all } \varphi \in \mathcal{D}(\Omega).$$

□

9.15. *Let $\varphi \in \mathcal{D}(\mathbb{R})$ and let $K_B(0,r)$ be a 0–centered compact interval. Assume that $D^k\varphi(0) = 0$ for all $k = 0, 1, \ldots, N$ and $\|D^N\varphi\big|_K\|_\infty \le \eta$. Prove that for all $k \le N$ and $x \in K$*

$$|D^k\varphi(x)| \le \eta |x|^{N-k}.$$

Solution. $K = [-r, r]$, ie. $x \in K \iff |x| \le r$. Assumption: $D^k\varphi(0) = 0$ for all $k = 0, 1, \ldots, N$ and $\|D^N\varphi\big|_K\|_\infty \le \eta$.

Induction:
I) $(n = 0)$: $\|D^N\varphi\big|_K\|_\infty \le \eta$ gives for all $x \in K$: $|D^{N-0}\varphi(x)| \le \eta|x|^{N-(N-0)}$.
II) Induction assumption: for all $x \in K$: $|D^{N-k}\varphi(x)| \le \eta|x|^{N-(N-k)}$.
Induction claim: for all $x \in K$: $|D^{N-k-1}\varphi(x)| \le \eta|x|^{N-(N-k-1)}$. (kunnes $k = N$)
Step: when $0 < x < r$, then — since $D^k\varphi(0) = 0$ for all $k = 0, 1, \ldots, N$ —

$$\begin{aligned}
|D^{N-k-1}\varphi(x)| &= |\int_0^x D^{N-k}\varphi(t)\,dt| \\
&\le \int_0^x |D^{N-k}\varphi(t)|\,dt \\
&\le \int_0^x \eta|t|^{N-(N-k)}\,dt \\
&= \eta\int_0^x t^k\,dt = \eta\tfrac{1}{k}x^{k+1}
\end{aligned}$$

Similarly for negative.

9.16. **(Partitions of unity!)** *Let $(\omega_i)_{i\in I}$ be a family of open sets and $\omega_i \subset \mathbb{R}$ and $\Omega = \bigcup_{i\in I}\omega_i$. There exists a sequence (!) of functions $(\psi_n)_{n\in\mathbb{N}}$ such that*

a) $\forall n \in \mathbb{N}\, \exists i \in I$ *such that* $\operatorname{supp}\psi_n \subset \omega_i$.
b) $\sum_{n\in\mathbb{N}} \psi_n = 1$ *in* Ω.
c) *To each compact $K \subset \Omega$ there exists an open $A \supset K$ and a number $m \in \mathbb{N}$ such that $\sum_{n=1}^m \psi_n = 1$ in A.*

Solution. Classical real analysis construction. Very nice! Look in books or search the the net.

Exercise set 10.

10.17. *Let $(\Lambda_n)_{\mathbb{N}} \subset \mathcal{D}(\Omega)^*$ be a sequence of distributions such that for all $\varphi \in \mathcal{D}(\Omega)$ there exists the limit*

$$\langle \varphi, \Lambda \rangle = \lim_{n\to\infty} \langle \varphi, \Lambda_n \rangle \in \mathbb{C}.$$

Prove using Banach and Steinhaus uniform boudedness that $\Lambda \in \mathcal{D}(\Omega)^*$ and $D^k\Lambda_n \to D^k\Lambda$ in the standard topology of $\mathcal{D}(\Omega)^*$, which you should remember. It is ....

Solution. ...the weak topology $w^* = \sigma(\mathcal{D}(\Omega)^*, \mathcal{D}(\Omega))$.
(a) Λ is obviously linear. Tarkastetaan its continuity: Let $K \subset \Omega$ be compact. Consider the family of mappings $(\Lambda_n\big|_K)_{\mathbb{N}} \subset \mathcal{D}_K(\Omega)^*$. It is **pointwise bounded**, since atg each point $\varphi \in \mathcal{D}_K(\Omega)$ the set $\{\langle \varphi, \Lambda_n \rangle \bigm| n \in \mathbb{N}\}$ is a convergent sequence in $\mathbb{R}$, hence bounded. Since $\mathcal{D}_K(\Omega)$ is a Fréchet-space, the family is by Banach and Steinhaus equicontinuous: To each neighbourhood $B(0,r) \in \mathcal{U}_{\mathbb{C}}$ there exists a neighbourhood $U_r \in \mathcal{U}_{\mathcal{D}_K(\Omega)}$ such that for all $n \in \mathbb{N}$

$$\langle U_r, \Lambda_n \rangle \subset B(0,r)$$

so for all $\varphi \in U_r$:

$$|\langle \varphi, \Lambda \rangle| = \lim_{n\to\infty} |\langle \varphi, \Lambda_n \rangle| \le r$$

The restrictions of Λ to the spaces $\mathcal{D}_K(\Omega)$ are all continuous, so $\Lambda \in \mathcal{D}(\Omega)^*$.
(b) By the assumption $\langle \varphi, \Lambda \rangle = \lim_{n\to\infty} \langle \varphi, \Lambda_n \rangle \in \mathbb{C}$ at least $\Lambda_n \to \Lambda$ in the standard topolgy of $\mathcal{D}(\Omega)^*$.

For all $\varphi \in \mathcal{D}(\Omega)$, by the definitionof distribution derivative

$$\langle \varphi, D^k\Lambda_n \rangle = (-1)^k \langle D^k\varphi, \Lambda_n \rangle \to (-1)^k \langle D^k\varphi, \Lambda \rangle = \langle \varphi, D^k\Lambda \rangle,$$

same as $D^k\Lambda_n \to D^k\Lambda$ in the standard topolgy of $\mathcal{D}(\Omega)^*$. (You could also use continuity of differentiation which is proved similarly)

10.18. *Is* $\quad \langle \varphi, \Lambda \rangle = 0 \implies \varphi\Lambda = 0, \quad$ *valid for $\varphi \in \mathcal{D}(\Omega)$ and $\Lambda \in \mathcal{D}(\Omega)^*$?*

Solution. $\varphi\Lambda = 0$ means by the definition of the product$\varphi\Lambda$, that $\langle \psi\varphi, \Lambda \rangle = \langle \psi, \varphi\Lambda \rangle = 0$ for all $\psi \in \mathcal{D}(\Omega)$.

Slect (Use approximative unity!) functions $\psi_n \in \mathcal{D}(\Omega)$ such that $\psi_n\varphi \to \varphi$ in the space $\mathcal{D}(\Omega)$. Then $0 = \langle \psi_n\varphi, \Lambda \rangle \to \langle \varphi, \Lambda \rangle$, so $\varphi, \Lambda \rangle = 0$.

The other directionis — of course — false. A counterexample: $\Lambda = \Lambda_g$ with $g = 1$ on $[-1,1]$ and 0 elsewhere, and $\varphi = \psi$ is x on $[-1,1]$ elsewhere 0.

10.19. *Express Dirac's delta eksplicitely as a higher — as low as possible – derivative of a continuous function.*

Solution. easy. cf. later exx.

10.20. (Convolution of functions). The convolution of 2 functions $u, v : \mathbb{R} \to \mathbb{C}$ is defined as

$$(u * v)(x) = \int_{\mathbb{R}} u(t)v(x-t)\,dt,$$

when the right hand side Lebesgue–integral exists. Prove that if we denote by $\tau_x(v)$ the function v shifted by x: $\tau_x(v)(t) = v(t-x)$, and we denote by $\tilde{v}$:lla the function v reflected $\tilde{v}(t) = v(-t)$, then

$$(u * v)(x) = \langle \tau_x(\tilde{v}), \Lambda_u \rangle.$$

Solution. EASY! □.

10.21. Convolution has many nice properties like

(1) In general $u * v$ is as"smooth" as the more smooth one of u and v.
(2) $u * v$ is bilinear.
(3) $u * v = v * u$.
(4) $(u * v) * w == v * (u * w)$.
(5) yms.
(6) ...
(7) Fourier–transform turn convolutions to products: $\mathcal{F}(uv) = \mathcal{F}u * \mathcal{F}v$.

Find more nice results – or prove some of these.

Solution. Nice animations can be found in the net — already in Wikipedia and MathWorld.

10.22. *In probability theory, discuss the distribution(!) of the sum of 2 "continuous" independent random variables. How about "discrete variables? Or one "discrete" and one "continuous"?*

10.23. (The convolution of a distribution and a function). *Assume $\psi \in \mathcal{D}(\mathbb{R})$ and that Λ is a compactly supported distribution and $x \in \mathbb{R}$. Prove that if* $\operatorname{supp}\Lambda \cap (x - \operatorname{supp}\psi) = \emptyset$, *then* $(\Lambda * \psi)(x) = 0$.

Solution. Easy □

10.24. *Consider Heaviside step function* $H(x) = \begin{cases} 0, & forx \le 0 \\ 1, & for\ x > 0. \end{cases}$

Its derivative in the classical sense is ae. 0, but its distribution derivative is δ_0. Prove that for all $\varphi \in \mathcal{D}(\mathbb{R})$:

a) $(H * \varphi)(x) = \int_{-\infty}^{x} \varphi(t)\,dt$
b) $D\delta_0 * H = \delta_0$.
c) $\Lambda_1 * H = 0.(UUPPSmaybewrong)$ *(Here 1 is the constant function 1 and $1 = \Lambda_1$ the corresponding distribution.)*
d) *There exist distributions Λ_a, Λ_b and $\Lambda_c \in \mathcal{D}(\mathbb{R})^*$, such that*

$$(\Lambda_a * \Lambda_b) * \Lambda_c \neq \Lambda_a * (\Lambda_b * \Lambda_c).$$

Solution. Not written yet. Easy? Check for errors.

Exercise set 11.

11.25. *a) Is e^x a temperede distribution? b) How about $e^x\cos(e^x)$? c) Are You in coflict with Hahn and Banach?)*

Solution. At least n $\int_{\mathbb{R}} \varphi(x)e^x\,dx$ is not defined for all rapidly decreasing functions φ. Counterexample $\varphi(x) = e^{-|x|/2}$. But i such that is the full answer to the question? If $\Lambda = e^x$ were a tempered distribution, then it would be a Schwartzin distribution [**??**] and $\langle \varphi, \Lambda\rangle = \int_{\mathbb{R}} \varphi(x)e^x\,dx$ at least for all $\varphi \in \mathcal{D}(\mathbb{R})$. I such that ere an extension of this to $\mathcal{S}$? Since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}$, we only have to check continuituy in the topology of $\mathcal{S}$-. We will prive discontinuity of $\mathcal{D}(\mathbb{R}) \to \mathbb{K} : \langle \varphi, \Lambda\rangle = \int_{\mathbb{R}} \varphi(x)e^x\,dx$ in the topology induced by $\mathcal{S}$. Choose some $\psi_n \in \mathcal{D}(\mathbb{R})$, such that $0 \le \psi \le 1$, $\operatorname{supp}\psi \in [-2n, 2n]$ and $\psi = 1$ on $[-n, n]$. Now $\psi_n(x)e^{-|x|/2} \to e^{-|x|/2}$ in $\mathcal{S}$, in particular $(\psi_n(x)e^{-|x|/2})_{\mathbb{N}}$ is a Cauchy- sequence in the subspace topology. But $(\langle\psi_n(x)e^{-|x|/2}, \Lambda\rangle)_{\mathbb{N}}$ is not convergent:

$$\langle \psi_n(x)e^{-|x|/2}, \Lambda\rangle = \int_{\mathbb{R}} \psi_n(x)e^{-|x|/2} \cdot e^x\,dx \to \infty,$$

jso Λ is not convergent in that topology.

b) Easy: yes!

c) Not at all. Different topologies! □

11.26. *Why are the following tempered distributions:*

a) *Consider*

b) *positive Borel-mesures μ, such that there exists $k \in \mathbb{N}$ such that:*

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|^2)^k} < \infty$$

c) *meassruable functions g, such that there exists $p \in [1,\infty[$ and $N > 0$ such that*

$$\int_{\mathbb{R}} \left(\frac{g(x)}{(1+|x|^2)^N}\right)^p < \infty$$

d) *polynomials.*

Solution. (a) Let $K = \operatorname{supp}\Lambda$ be compact. Choose $\psi \in \mathcal{D}(\mathbb{R})$ with value 1 in an open set saa arvon 1 avoimessa $U \supset K$. let

$$\langle f, \tilde\Lambda\rangle = \langle f\psi, \Lambda\rangle.$$

If $f_i \to 0$ in the topology of $\mathcal{S}$, then $f_i\psi \to 0$ in the topology of $\mathcal{D}(\mathbb{R})$, so $\tilde\Lambda \in \mathcal{S}^*$. Nut for all $\varphi \in \mathcal{D}(\mathbb{R})$ we know $\langle f, \tilde\Lambda\rangle = \langle f, \Lambda\rangle$. □

(b) Let μ be a Borel-measure and $k \in \mathbb{N}$ such that $\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|^2)^k} < \infty$. Prove that $f \mapsto \int_{\mathbb{R}} f\,d\mu$ is continuous in the topology of $\mathcal{S}$. consider a sequence $f_j \to 0$ in the topology of $\mathcal{S}$, so on particular $\|(1+|x|^2)^k f_j(x)\|_\infty \to 0$. Now

$$|\langle f_j, \mu\rangle| = |\int_{\mathbb{R}} f_j(x)\,d\mu| = |\int_{\mathbb{R}} \frac{(1+|x|^2)^k}{(1+|x|^2)^k} f_j(x)\,d\mu| \le \underbrace{\|(1+|x|^2)^k f_j(x)\|_\infty}_{\to 0} \underbrace{\int_{\mathbb{R}} \frac{d\mu(x)}{(1+|x|^2)^k}}_{<\infty}.$$

(c) Here $\langle\varphi,\Lambda_g\rangle = \int_{\mathbb{R}} \varphi g\,dx$. The case $p=1$ is a aspecial case of (b). If $p\in]1,\infty[$, then remeber *Hölder's inequality*: "If $p>1$ ja $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and $f\in L^p$ ja $g\in L^q$, then

$$|\int_{\mathbb{R}} fg\,d\mu| \le \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}} |g|^q\,d\mu\right)^{\frac{1}{q}}."$$

Divide the integrand to factors, use Hölder and remebr tha aim:, $|\langle\varphi,\Lambda_g\rangle| \le C\|D^k\varphi(x)(1+|x|^2)^N\|_\infty$ for some N,k ja C.

$$\begin{aligned}
|\langle\varphi,\Lambda_g\rangle| &= |\int_{\mathbb{R}} \varphi g\,dx| \\
&= |\int_{\mathbb{R}} \varphi(x)(1+|x|^2)^N \frac{g(x)}{(1+|x|^2)^N}\,dx| \\
&\le \left(\int_{\mathbb{R}} \left|\varphi(x)(1+|x|^2)^N\right|^q\,dx\right)^{\frac{1}{q}} \cdot \underbrace{\left(\int_{\mathbb{R}} \left(\frac{|g(x)|}{(1+|x|^2)^N}\right)^p\,dx\right)^{\frac{1}{p}}}_{\text{=C=vakio}} \\
&\le C\left(\int_{\mathbb{R}} \left|\varphi(x)(1+|x|^2)^N\right|^q\,dx\right)^{\frac{1}{q}} \\
&= C\left(\int_{\mathbb{R}} \left|\varphi(x)(1+|x|^2)^M\cdot(1+|x|^2)^{N-M}\right|^q\,dx\right)^{\frac{1}{q}} \\
&\le C\|\varphi(x)(1+|x|^2)^M\|_\infty \cdot \underbrace{\left(\int_{\mathbb{R}} \left|(1+|x|^2)\right|^{(N-M)q}\,dx\right)^{\frac{1}{q}}}_{\text{constant}<\infty\text{ kun } M \text{ large enough}}.
\end{aligned}$$

(d) A special case of (c). □

11.27. *Prove that the derivative of a tempered distribution is a tempered distribution.*

Solution.

$$\langle f, D\Lambda\rangle = -\langle Df,\Lambda\rangle$$

and a) of the **next** exercise. □

11.28. *The following linear mappings are continuous $\mathcal{S}\to\mathcal{S}$:*

(1) *Differentiation*
(2) *Multiplication by a polynomial.*
(3) *Multiplication by a rapidly dectreasing function.*

Hint: A corollary of the closed graph theorem: A linear mapping $T:E\to F$ between Fréchet-spaces is continuous if and only if for all $(x_n)_{\mathbb{N}}\subset E$

$$(x_n, Tx_n)\to(0,y) \implies y=0.$$

Solution. Let $\varphi\in\mathcal{S}$.

(1) Differentiation: Claim $\|D^k(D\varphi)(x)(1+|x|^2)^N\|_\infty \le C\|D^{k'}\varphi(x)(1+|x|^2)^{N'}\|_\infty$ for some N,k ja C. Yes! Choose $C=1, k'=k+1$ ja $N'=N$.

(2) Almost by definition: $\varphi \in \mathcal{S} \iff \|P(x)D^k\varphi(x)\|_\infty < \infty$ for all polynomials P and $k \in \mathbb{N}$, so $\varphi \in \mathcal{S} \implies D\varphi \in \mathcal{S}$. Use the hint: Let $(\varphi_n)_\mathbb{N} \subset \mathcal{S}$ such that $\varphi_n \to 0$ and $P \cdot \varphi_n \to \psi \in \mathcal{S}$. Claim: $\psi = 0$.

The assumption mean such that at$\|Q_1(x) \cdot D^k\varphi_n(x)\|_\infty \to 0$ for all polynomials Q_1 and numbers $k \in \mathbb{N}$ and $\|Q_2 \cdot D^k(P \cdot (\varphi_n - \psi))\|_\infty \to 0$ for all polynomials Q_2 and numbers $k \in \mathbb{N}$. The differentiation formula for products (Leibnitz) gives

$$\|Q_2 \cdot \sum_{j=0}^{k}(D^jP)(D^{k-j}(\varphi_n - \psi))\|_\infty \to 0$$

eli

$$\|\sum_{j=0}^{k}((Q_2 \cdot D^jP) \cdot D^{k-j}\varphi_n - Q_2 \cdot D^j\psi))\|_\infty \to 0.$$

since $Q_2 \cdot D^jP$ is a polynomial, every$\|(Q_2 \cdot D^jP) \cdot D^{k-j}\varphi_n\|_\infty \to 0$, so ....(fill in!)— $\psi = 0$.

(3) Similar.

11.29. The Fourier- transformation of $\mathcal{L}_1(\mathbb{R})-$functions has the following properties: for all $f,\, g \in \mathcal{L}_1(\mathbb{R})$ ja $x,\, t \in \mathbb{R}$:

a) $\mathcal{F}$ is linear
b) $(\tau_x f)\hat{} = e_{-x}\hat{f}$
c) $(e_x f)\hat{} = \tau_{-x}\hat{f}$
d) $(f \hat{*} g)\hat{} = \hat{f}\,\hat{g}$
e) $(\frac{f}{\lambda})\hat{}(t) = \lambda\hat{f}(\lambda t)$, kun $\lambda > 0$.

Todistus. (a) is obvious.

(b) $(\tau_x f)\hat{}(t) = \int_\mathbb{R}(\tau_x f)e_{-t}\,dm = \int_\mathbb{R} f\tau_{-x}e_{-t}\,dm = \int_\mathbb{R} fe_{-t}(x)e_{-t}\,dm = e_{-x}(t)\hat{f}(t).$

(c) $(e_x f)\hat{}(t) = \int_\mathbb{R}(e_x f)e_{-t}\,dm = \int_\mathbb{R} fe_{-(t-x)}\,dm = \tau_{-x}(t)\hat{f}(t).$

Prove (d) and (e).

Solution. (d) Fubini . (e) A linear change of variables. □

11.30. Solve one exercise from Rudin's book "Functional Analysis" Chapter 7.

11.31. Solve one more exercise from Rudin's book "Functional Analysis" Chapter 7 (or 8).