GROUPS AND THEIR REPRESENTATIONS - FOURTH PILE, ALMOST IN ENGLISH

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22. Infinite Groups

Example 22.1. Many, in fact most well known and important groups are infinite:

(1) *The group of translations* $(\mathbb{R}^n, +)$
(2) *The group of invertible linear mappings* $GL(\mathbb{R}^n) \sim GL_n(\mathbb{R}) = \{$invertible $n \times n-$matrices$\}$ with matrix multiplication.
(3) *The group of volume and orientation preserving invertible linear mappings* $SL(\mathbb{R}^n) \sim SL_n(\mathbb{R}) = \{n \times n-$matrices with determinant $1\}$.
(4) *The group of length, angle and orientation preserving invertible linear mappings* also called *orhtogonaal linear mappings* $O(\mathbb{R}^n) \sim SO_n(\mathbb{R}) = \{n \times n-$matrices with orthogonal columns or, equivalently, rows and determinant $1\}$ $SO(\mathbb{R}^n) \sim SO_n(\mathbb{R})$
(5) The *Lorentz group* $SO_{3,1}(\mathbb{R})$, of invertible linear mappings preserving a given Minkowski product.
(6) *The group of complex invertible linear mappings* $GL(\mathbb{C}^n) \sim GL_n(\mathbb{R}) = \{$invertible complex $n \times n-$matrices$\}$.
(7) *The group of unitary* linear mappings $U(\mathbb{C}^n) \sim U_n(\mathbb{C}) = \{$complex $n \times n-$matrices preserving a Hermitian inner product$\}$=$\{$ $n \times n-$matrices with orthogonal columns or, equivalently, rows$\}$
(8) $SU(\mathbb{C}^n) \sim SU_n(\mathbb{C}) = \{$orientation preserving mappings in $U(\mathbb{C}^n)$
(9) A special case: $U(\mathbb{C}) = U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ $\sim\{$rotations in the plane$\}$.
(10) Products and many factor groups of the afore mentioned.

Remark 22.2. All these act by definition in sope set, most act linearly in a finite dimensional vector space, so they have *tautological representations.* It is a nontrivial task to find (all) other representations. We will find some. It is not quite evident how one could generalize the theory of finite groups, as the infinite groups are very large, all of the above are uncountable and most even lacking countable sets of generators. (Cf. **??**, where generators of $GL_n(\mathbb{R})$ are given). Fortunately, these groups carry other structure beside the group multiplication, all are topological spaces, and even better, smooth manifolds. Their group operations are smooth as well. Such groups are called *Lie groups*

23. Smooth manifolds and Lie groups

Remark 23.1. INTRODUCTION: ((((Jo aluksi voi hahmotella, mistä is kysymys. manifold is tavallisen smoothn pinnan yleistys, yleensä moniulotteinen — siitä nimi. manifold is siis joukko, usein jonkin korkeaulotteisen euklideisen avaruuden $\mathbb{R}^n$:n subset, jonka jokaisessa pisteessä is olemassa tangenttiavaruus, tangenttitason luonnollinen yleistys. Monistolla is sileitä käyriä and muita sileitä alimonistoja sekä ennen kaikkea monistolta is sileitä funktioita luvuille eli koordinaatteja sekä sileitä kuvauksia muille monistoille and itselleen. Monistolla voi siten harrastaa differentiaalilaskentaa ja, jos se is samalla mitta-avaruus, myös integraalilaskentaa.

Lie group is samalla group and smooth manifold, jossa laskutoimitus is differentioituva kuvaus, jolloin käy niin, että vasemmalta kertominen ryhmän alkiolla is diffeomorphism ryhmältä itselleen. Erityisesti jokaiseen pisteeseen piirretyt tangenttiavaruudet osoittautuvat täysin samanlaisiksi and voidaan siis samaistaa neutraalialkion kohdalle piirrettyyn tangenttiavaruuteen, joka is nimeltään ryhmän lie algebra. and "algebra" se onkin, sillä siinä is vektoriavaruuden rakenteen lisäksi myös eräänlainen kertolaskutoimitus, jota usein merkitään hakasulkein [ ].

Lien ryhmien esitysteorian pääidea is suunnilleen seuraava: Lien ryhmän esityksestä saadaan luonnollisella tavalla sen "Lien algebran esitys", joka is matemaattisesti paremmin hallittavissa oleva käsite, koska lie algebra is reaalinen vector space, joka kannattaa vielä täydentää kompleksiseksi vektoriavaruudeksi, koska täydellisessä kunnassa $\mathbb{C}$ is helpompi laskea kuin $\mathbb{R}$:ssä. Syntyvien ns. puoliyksinkertaisten Lien algebroiden esitykset is mahdollista luokitella and niistä saa rekonstruoitua alun perin etsityt reaaliset and edelleen Lien ryhmän esitykset. Tämä is ohjelmamme periaatteessa, pitkälti myös käytännössä.))))

Remark 23.2. The theory of representations of finite groups can also be generalized using another idea, *Haar measure*, which is a translation invariant measure is a group.

$$\mu(U) = \mu(gU) \quad \forall U \subset G,\ g \in G.$$

In particular, all compact topological groups carry a *finite Haar measure*, with $\mu(G) = 1$. This makes it easy to imitate the finite group theory, in particular the theory of characters, by replacing averages $\frac{1}{|G|}\sum_{g\in G}$ by the corresponding integrals $\int_G d\mu$. Unfortunately, only few interesting groups are compact, of the afore mentioned only $O(\mathbb{R}^n)$, $SO(\mathbb{R}^n)$, $U(\mathbb{C}^n)$ and $SU(\mathbb{C}^n)$. Infinite Haar measure is the other groups makes their theory much more complicated.

23.1. **Topology.**

Definition 23.3. The concepts of *Topological space* and its *topology* are considered well-known.

Remark 23.4. Well known examples of topological spaces and continuous mappings(((((Esimerkkejä topologioista:

1. *triviaali topologia* where tahansa joukossa,
2. *diskreetti topologia* where tahansa joukossa,
3. *kofiniittinen topologia* where tahansa joukossa,
4. *euklidinen topologia* $\mathbb{R}^n$:ssä,
5. *Zariskin topologia* $\mathbb{R}^n$:ssä (Zariskin topologiassa suljettuja ovat polynomien nollajoukot and niiden leikkauset),
6. *aliavaruustopologia* topologisen avaruuden osajoukossa,
7. *tulotopologia* topologisten avaruuksien tulojoukossa.

Remark 23.5. Topologinen avaruus is *topologisten avaruuksien kategorian objekti.* Kertamme joitakin tähän aihepiiriin liittyviä määritelmiä:

- Topologisten avaruuksien kategorian mielessä samoja eli *isomorfisia* ovat topologiset avaruudet, joiden välillä is bijection , joka kuvaa avoimet joukot avoimiksi joukoiksi, kuten myös sen käänteiskuvaus, siis topologian säilyttävä bijection .

- *Topologisten avaruuksien kategorian morphism* is *jatkuva* kuvaus, ts. kuvaus, jossa openten joukkojen alkukuvat ovat avoimia.

- *open kuvaus* is kuvaus, jossa openten joukkojen kuvat ovat avoimia. Huomataan, että jatkuva kuvaus ei yleensä ole open edes tavallisessa topologiassa $\mathbb{R} \to \mathbb{R}$, vastaesimerkkinä $x \to x^2$. Myöskään open kuvaus ei yleensä ole jatkuva eikä jatkuva open kuvaus ole yleensä bijection .

- *Homeomorphism* is kumpaankin suuntaan jatkuva bijection eli jatkuva and samalla open bijection . Tämä is sama asia kuin isomorphism!

- *Topologian kanta* eli *virittäjistö* is joukko avoimia joukkoja, "kantajoukkoja" jolla is se ominaisuus, että jokainen open joukko voidan lausua yhdisteenä kantajoukoista. Esimerkkinä avoimet pallot tavallisessa euklidisessa topologiassa.

- *Kahden topologisen avaruuden tulo* is niiden karteesinen tulo varustettuna topologialla, jonka kantana ovat alkuperäisten openten joukkojen tulot. Esimerkkinä tulotopologiasta olkoon $\mathbb{R}^2$:n euklidinen topologia avaruuksien $\mathbb{R}$ ja $\mathbb{R}$ topologioiden *tulotopologia*na.

- *Hausdorff*- avaruus is topologinen avaruus, joka toteuttaa *toisen numeroituvuusehdon* T_2, eli jossa kahdella eri pisteellä is aina erilliset avoimet ympäristöt. Esimerkiksi tavallinen euklidinen topologia and diskreetti topologia. Vastaesimerkkejä ovat äärettömän joukon triviaali topologia and kofiniittinen topologia sekä $\mathbb{R}^n$:n Zariskin topologia, kun $n \geq 2$.))))))

23.2. **Smooth mappings in Euclidean space and classical manifolds.**

Definition 23.6. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. Call $f : U \to V$ *smooth*, if at every $u \in U$ all partial derivatives of any order exist. From differential calculus we know that such functions have *derivatives* Notation: write $C^\infty(U, V)$.

Remark 23.7. Smoothness is a "local property" in the following sense: $f : U \to V$ obviously is smooth if and only if its every restriction to an open subset of U. It is sufficient to consider restrictions to some open cover.

Locality is expressed by saying that the smooth functions form a *function sheaf*. Similarly "sheaves" are formed by continuous functions, $C * 1$-functions, real analytic functions, holomorphic (component-wise complex analytic) functions, even rational functions etc. They all can be used to construct their own kinds of "manifolds".

Manifolds have two historical definitions, the older by Riemann. In the 1930:s *Hassler Whitney* proved their equivalence. In the classical definition, the manifold is a topological subspace of R^n inheriting also the differential structure from there. In the more abstract definition, the manifold is any topological space, equipped with an "atlas of charts" with smooth transition maps. Whitney's theorem allows to embed any abstract d-dimensional manifold in $\mathbb{R}^{2n+1}$.

We begin by extending the definition of smooth mapping from open sets to any subsets in $\mathbb{R}^n$.

Definition 23.8. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be arbitrary subsets. A mapping $f : U \to V$ is called *smooth*, if it is the restriction of some smooth map to the set U, i.e. there exist open $\tilde{U} \subset \mathbb{R}^m$ and $\tilde{V} \subset \mathbb{R}^n$ and a smooth mapping $\tilde{f} : \tilde{U} \to \tilde{V}$ such, that $U \subset \tilde{U}$, $V \subset \tilde{V}$ and $f(u) = \tilde{f}(u)$ for all $u \in U$. The set of all smooth functions $f : U \to V$ is a vector space $C^\infty(U, V)$.

Definition 23.9. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be arbitrary subsets. the mapping $f : U \to V$ is as *diffeomorphism*, if it is bijective and both $f : U \to V$ and $f^{-1} : V \to U$ are smooth.

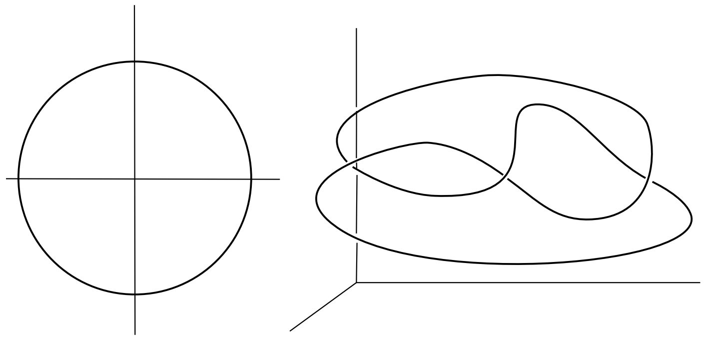
Remark 23.10. Diffeomorphisms are of course homeomorphisms, so diffeomorphic sets can be identified as topological spaces.

Remark 23.11. Diffeomorpism is an equivalence relation.

Remark 23.12. In the definitions above , m and n need not coincide. This is in particular true for diffeomorphisms, but it is well known that there exists no diffeomorphism between sets of different dimension. In the definitions above, m and n need not coincide since f and its inverse f^{-1} are only defined between the sets U and V whereas the extensions used $\tilde{f}$ and $(f^{-1})^{\sim}$ to test for smoothness need not be inverse to each other.

Example 23.13. Diffeomorfism:

(1) The unit circle in the plane $\mathbb{R}^2$ is diffeomorphic to *any circle* in the plane, in space or anywhere: $\mathbb{R}^3$. Also the circle is diffeomorphic to an ellipse or any loop, knotted or not.
(2) A square and a circle are not diffeomorphic.
(3) A square and a rectangle are diffeomorphic.
(4) A square and a quadrilateral are diffeomorphic.
(5) A sphere and a circle are not even homeomorphic.
(6) A sphere and a torus are not even homeomorphic..
(7) There exists a smooth bijection from the square to the circle but its inverse is not smooth.



Kuva 1: Ympyrä and solmu ovat diffeomorfiset

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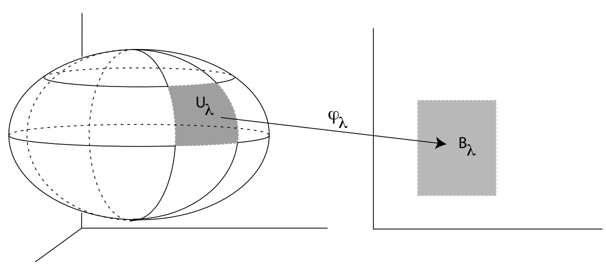
Definition 23.14. A subset $X \subset \mathbb{R}^m$ is a *(classical, smooth, $d-$dimensional) manifold*, same as *locally diffeomorphic to Euclidean space*$\mathbb{R}^n$, if X has an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$, s.th. every U_λ is diffeomorphic to an open $B_\lambda \subset \mathbb{R}^d$.

The open cover sets U_λ , together with the corresponding diffeomorphisms are called *charts*, the sets B_λ *pages*, the chart maps $\varphi_\lambda : U_\lambda \to B_\lambda$ *local co-ordinate systems* and their inverses $\varphi_\lambda^{-1} : B_\lambda \to U_\lambda$ *local parametrisations.*



Kuva 2: Ellipsoidi is klassinen monisto

Remark 23.15. In the above definition one can take the $B_\lambda \subset \mathbb{R}^d$ to be open spheres, generalised rectangles or elements of any other basis of the Euclidean topology.

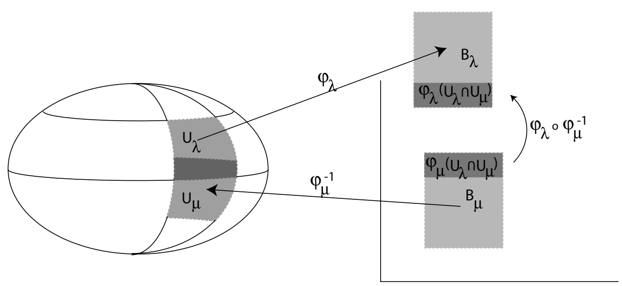
23.3. **Abstract manifolds.**

Definition 23.16. SA topological space X is an *abstract (smooth, d−dimensional) manifold* and $\{\varphi_\lambda\}_\Lambda$ is its *atlas*, if X has an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$, such that every U_λ is homeomorphic $\varphi : U_\lambda \to B_\lambda$ to some open $B_\lambda \subset \mathbb{R}^d$ such, that :

(1) X is Hausdorff
(2) the topology of X has a countable basis.
(3) If U_λ and U_μ intersect, then the *change of co-ordinates*

$$\varphi_\mu \circ \varphi_\lambda^{-1} : \varphi_\lambda(U_\lambda \cap U_\mu) \to \varphi_\mu(U_\lambda \cap U_\mu)$$

is a diffeomorphism.



Kuva 3: Ellipsoidi is abstract monisto

Remark 23.17 (Motivation). (((Määritelmän idea on, että pitää jotenkin pystyä määrittelemään, mitä voitaisiin tarkoittaa smoothllä funktiolla joukossa X. Topologia ei anna luontevaa tapaa. Kartasto antaa. is luonnollista pitää kuvausta $f : X \to \mathbb{R}$ smoothnä, jos jokainen

$$f \circ \varphi_\lambda^{-1} : B_\lambda \to \mathbb{R}$$

is smooth. Määritelmän kohta (3) takaa, että tämä ei riipu kartan valinnasta.

Remark 23.18. Määritelmässä olisi tietenkin riittänyt vaatia, että jokainen kartanvaihto is smooth funktio, ovathan niiden käänteiskuvaukset itsekin kartanvaihtoja.)))))))))

Remark 23.19. An open subset of an abstract manifold obviously is a manifold with restrictions of the original charts.

In particular an open $A \subset \mathbb{R}^n$ is an n-dim manifold and the obvious atlas consists of the embedding $U = A \to B = A \subset \mathbb{R}^n$.

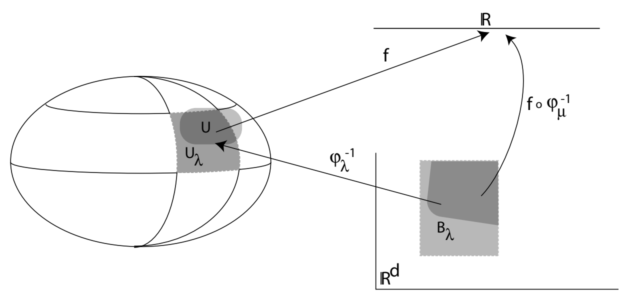
Remark 23.20. Any classical manifold is obviously an an abstract manifold. The inverse holds by Whitney's theorem. WE WILL USE BOTH DEFINITIONS.

Definition 23.21. Let X be an abstract manifold and U an open subset. A function $f : U \to \mathbb{R}$ is *smooth* in U,if it is *smooth is every chart*, i.e.. every composed map

$$f \circ \varphi_\lambda^{-1} : B_\lambda \cap \varphi_\lambda(U) \to \mathbb{R}$$

is smooth.

f is *smooth at the point* $x \in X$, if it is smooth in some neighbourhood of x.



Kuva 4: smooth funktio abstraktilla monistolla

Remark 23.22. To prove that a function $f : U \to \mathbb{R}$ is smooth at x one only has to check it for one chart containing x.

$f : U \to \mathbb{R}$ is smooth, if and only if f is smooth in every point $x \in U$. So smoothness is a local property, and smooth functions form a function sheaf.

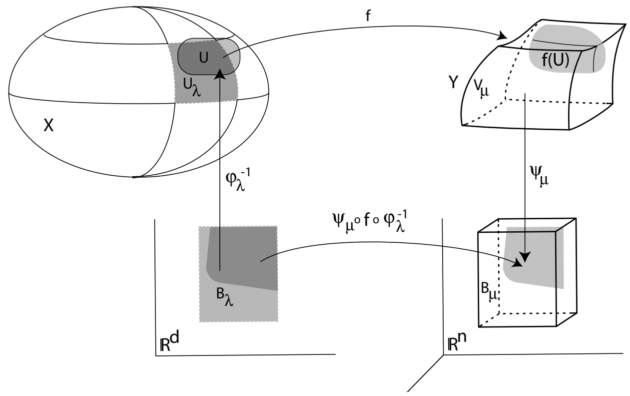
Definition 23.23. Let X and Y be two abstract manifolds. A mapping $f : X \to Y$ is *smooth*, if it is *smooth on each chart on both sides* i.e.. if every composed map

$$B_\lambda \xrightarrow{\varphi_\lambda^{-1}} U_\lambda \xrightarrow{f} f(U_\lambda) \cap V_\mu \xrightarrow{\psi_\mu} B_\mu \subset R^n$$

is smooth where it is defined , i.e.. in $\varphi_\lambda^{-1}(U_\lambda \cap f^{-1}(V_\mu))$, where the Y-chart is denoted $\{\psi_\mu : V_\mu \to B_\mu\}_{\mu \in M}$.

f is *smooth at* $x \in X$, if it is smooth in some neighbourhood of x.

Remark 23.24. smooth manifolds form a category ((((((muodostavat kategorian, jossa objekteina ovat monistot and morfismeina niiden väliset smootht kuvaukset. Isomorfismeja ovat monistojen väliset diffeomorphismt eli kumpaankin suuntaan smootht bijection t. Nämä ovat tietenkin homeomorfismeja, ovathan smootht kuvaukset selvästikin jatkuvia.))))))



Kuva 5: smooth kuvaus 2-ulotteiselta monistolta X 3-ulotteiselle monistolle Y

23.4. **Lie groups.**

Definition 23.25. *A topological group* is a group G, which is also a topological space and where both the group operation $\circ : G \times G \to G$ and taking inverses $G \to G : g \mapsto g^{-1}$ are continuous. Here the product set $G \times G$ carries the product topology.

Definition 23.26. A *Lie group*[1] is a group G, which is also a smooth manifold, and where both the group operation $\circ : G \times G \to G$ and taking inverses $G \to G : g \mapsto g^{-1}$ are continuous. Here the product set $G \times G$ carries the structure of a *product manifold* to be defined soon at**??**.

Example 23.27. All the infinite groups listed in the beginning are Lie groups, in fact classical ones i.e. embedded in Euclidean space. Let us check some of these statements:

$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \, xw - yz \neq 0 \right\}$ is open in $\mathbb{R}^4$ so $GL_2(\mathbb{R})$ is 4-dim manifold with chart = id.

[1]Sophus Lie 1842 - 1899, Norwegian.

$SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1 \right\}$ is a closed set in Euclidean space $\mathbb{R}^4$. Let us check directly by definition that it is a manifold. Choose:

$$U_x = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1, x \neq 0 \right\}$$
$$U_y = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1, y \neq 0 \right\}$$
$$U_z = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1, z \neq 0 \right\}$$
$$U_w = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1, w \neq 0 \right\}$$

And choose chart maps

$$\varphi_x : U_x \to B_x = \{(x, y, z) \in \mathbb{R}^3 \mid x \neq 0\} : \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto (x, y, z),$$
$$\varphi_y : U_y \to B_y = \{(x, y, w) \in \mathbb{R}^3 \mid y \neq 0\} : \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto (x, y, w),$$
$$\varphi_z : U_z \to B_z = \{(x, z, w) \in \mathbb{R}^3 \mid z \neq 0\} : \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto (x, z, w),$$
$$\varphi_w : U_w \to B_w = \{(y, z, w) \in \mathbb{R}^3 \mid w \neq 0\} : \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto (y, z, w),$$

such, that the deleted co-ordinate can always be calculated from the others by $xw - yz = 1$, which gives local parametrizations:

$$\varphi_x^{-1} : B_x \to U_x : (x, y, z) \mapsto \begin{bmatrix} x & y \\ z & \frac{yz+1}{x} \end{bmatrix},$$
$$\varphi_y^{-1} : B_y \to U_y : (x, y, w) \mapsto \begin{bmatrix} x & y \\ \frac{xw-1}{y} & w \end{bmatrix},$$
$$\varphi_z^{-1} : B_z \to U_z : (x, z, w) \mapsto \begin{bmatrix} x & \frac{xw-1}{z} \\ z & w \end{bmatrix},$$
$$\varphi_w^{-1} : B_w \to U_w : (y, z, w) \mapsto \begin{bmatrix} \frac{yz+1}{w} & y \\ z & w \end{bmatrix}.$$

All these are obviously smooth, so diffeomorfisms. By the classical def., $SL_2(\mathbb{R})$ is a smooth 3-dim manifold.

To learn it, let us calculate the coordinate transition functions also:

$$U_x \cap U_y = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| xw - yz = 1, x \neq 0, y \neq 0 \right\}$$

$$\varphi_x(U_x \cap U_y) = \{(x,y,z) \in \mathbb{R}^3 \mid x \neq 0, y \neq 0\} \subset B_x$$

$$\varphi_y \circ \varphi_x^{-1} : \varphi_y(U_x \cap U_y) \xrightarrow{\varphi_x^{-1}} U_x \cap U_y \xrightarrow{\varphi_y} \varphi_y(U_x \cap U_y) \subset B_y :$$
$$(x,y,z) \mapsto \begin{bmatrix} x & y \\ z & \frac{yz+1}{x} \end{bmatrix} \mapsto (x, y, \frac{yz+1}{x}).$$

This is a smooth bijection. The other 7 are similar.

In a similar fashion, $SL_2(\mathbb{C}) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbb{C}^{2\times 2} \middle| xw - yz = 1 \right\}$ is a complex manifold, since rational functions are holomorphic.

The other Lie groups will be considered later or in the exercises.

(Finite sets can be considered 0-dimensional manifolds!)

24. Tangent spaces and tangent mappings

The idea of a tangent space: Let $\psi(x) = \psi(x_1, \dots, x_d)$ be a point in a classical manifold $M \subset \mathbb{R}^n$. Here $\psi : B \to X$ is a local parametrisation. Since a local parametrisation is smooth, it has a derivative at $x = (x_1, \dots, x_d) \in B \subset \mathbb{R}^d$. This derivative is a linear mapping $L = d_x\psi : \mathbb{R}^d \to \mathbb{R}^n$ giving a close approximation to $y \mapsto \psi(y)$ in a neighbourhood of x. So its image set, the "plane" $\psi(x) + L(\mathbb{R}^d)$ is locally a good approximation to $\psi(B)$, which is a piece of the manifold M.

Instead of the "geometrically intuitive" tangent space, we will - for notational simplicity, consider the subspace parallel to it and call the subspace $L(\mathbb{R}^d)$ the *tangent space to M at $\psi(x)$*.

24.1. The derivative of a smooth function. The formal definition for a derivative can be found in any textbook.

The derivative is a linear mapping its matrix in any basis, often the standard basis of $\mathbb{R}^n$, is its *Jacobian matrix* consisting of its partial (same as directional!) derivatives

$$d_b f\,(h) = \lim_{t\to 0} \frac{f(b+th) - f(b)}{t}$$

in the well known way.

We will occasionally use some of the following terminology. A *marked* smooth manifold (M, x_0) is smooth manifold, with a distinguished point $x_0 \in M$. A *morphism of marked manifolds* $f : (M, x_0) \to (N, y_0)$ same as a smooth mapping between marked manifolds (M, x_0) and (N, y_0) is a smooth map $M \to N$ mapping $x_0 \mapsto y_0$.

The *tangent space* $T_x M$ of a smooth marked manifold (M, x_0) will be defined so that it will have the following properties:

To every morphism of marked manifolds

$$f : (M, x_0) \to (N, y_0)$$

there is a linear mapping

$$T_{x_0} f : (M, x_0) \to (N, y_0)$$

and this correspondence is a functor, i.e.

$$T_{x_0}(f \circ g) = T_{g(x_0)} f \circ T_{x_0} g,$$

whenever the composed morphism $f \circ g$ is defined at x_0. The *tangent mapping* same as *derivative* of $f : (M, x) \to (N, y)$ is this linear mapping $T_{x_0} f : (M, x_0) \to (N, y_0)$. [[[something is still missing: maybe we should say that the tangent mapping is the derivative whenever the derivative is already defined. note: this was no definition, just a prologue?]]]]

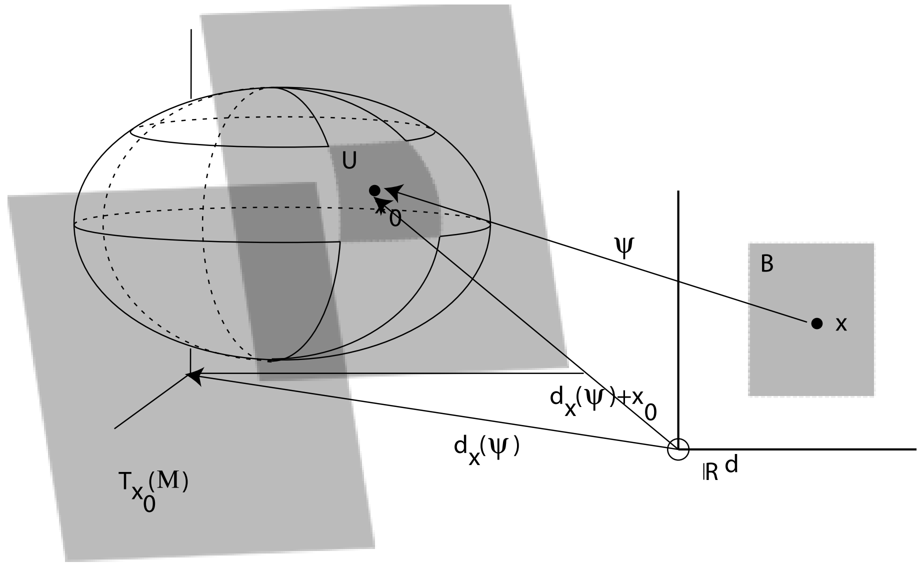
Definition 24.1. Consider a point $x_0 = \psi(x) = \psi(x_1, \ldots, x_d)$, in a d-dimensional manifold $M \subset \mathbb{R}^n$ where $\psi : B \to X$ is local parametrisation. Since a local parametrisation is smooth, it has a derivative at $x = (x_1, \ldots, x_d) \in B \subset \mathbb{R}^d$. The *tangent space* of the manifold M at $x_0 = \psi(x)$ is the image space $T_{x_0}(M) = d_x\psi(\mathbb{R}^d)$.

Definition 24.2. . Consider a smooth mapping between marked manifolds: $f : (M, x) \to (N, y)$ and a local paramterisations at the marked points $\psi : B \to U$, and assume $b \mapsto x$ and $\varphi : U' \to B'$, where $y \mapsto b' \in B'$ and : $M \subset \mathbb{R}^n, B \subset \mathbb{R}^d, N \subset \mathbb{R}^n, B' \subset \mathbb{R}^{d'}$.

The *tangent mapping* same as *derivative* of the mapping f is the derivative

$$d_b(\varphi \circ f \circ \psi) : \mathbb{R}^d \to \mathbb{R}^{d'}.$$

Remark 24.3. The definition above is not complete: One has to prove existence and uniqueness ei independence of the choice of charts. Let us do it. Before doing the general case, let us consider an example:



Kuva 7: Ellipsoidin tangenttitaso

Example 24.4. Consider the particular manifold $M = SL_2(\mathbb{R})$ at $x_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^4$. Use the local parametrization

$$\psi_x = \varphi_x^{-1} : B_x \to U_x : (x, y, z) \mapsto \begin{bmatrix} x & y \\ z & \frac{yz+1}{x} \end{bmatrix} = (x, y, z, \frac{yz+1}{x}) \in \mathbb{R}^4.$$

Calculate the partial derivatives at $(1, 0, 0) = \varphi_x \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and find the Jacobi matrix $\operatorname{Mat} d_{(1,0,0)} \psi_x$

$$\left. \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \\ \frac{\partial \frac{yz+1}{x}}{\partial x} & \frac{\partial \frac{yz+1}{x}}{\partial y} & \frac{\partial \frac{yz+1}{x}}{\partial z} \end{bmatrix} \right|_{(1,0,0)} = \left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{yz+1}{x^2} & \frac{z}{x} & \frac{y}{x} \end{bmatrix} \right|_{(1,0,0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

The image set of the linear mapping is the tangent space. So the tangent space is spanned by the columns:

$$\langle (1, 0, 0, -1), (0, 1, 0, 0), (0, 0, 1, 0) \rangle \subset \mathbb{R}^4 = \mathbb{R}^{2\times 2}$$

eli

$$T_I SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \, Tr \begin{bmatrix} z & y \\ z & w \end{bmatrix} = 0 \right\}.$$

Similarly $\subset \mathbb{R}^4$, for instance by the parametrization

$$\psi_w = \varphi_w^{-1} : B_w \to U_w : (y,z,w) \mapsto \begin{bmatrix} \frac{yz+1}{w} & y \\ z & w \end{bmatrix}$$

one finds at $(y,z,w) = (0,0,1)$ derivative

$$\left.\begin{bmatrix} \frac{z}{w} & \frac{y}{w} & -\frac{1+w}{w^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|_{(0,0,1)} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

from where one finds the tangent space to be

$$T_I SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} -w & y \\ z & w \end{bmatrix} \middle| \, y,z,w \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \, Tr \begin{bmatrix} z & y \\ z & w \end{bmatrix} = 0 \right\},$$

which is the same set that was found by the first parametrization. .

In fact it is intuitively clear why both parametrizations should give the same tangent space. The charts φ_i and their inverses, the local prametrizations $\varphi_i^{-1} = \psi_i$ form a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^3 \supset B_x \supset \varphi_x(U_x \cap U_w) & \xrightarrow{\psi_x} & U_x \cap U_w \\ \eta = \varphi_w \circ \psi_x \downarrow & \cdot & \downarrow I \\ \mathbb{R}^3 \supset B_w \supset \varphi_w(U_x \cap U_w) & \xrightarrow{\psi_w} & U_x \cap U_w \end{array}$$

The chart transition map $\eta = \varphi_w \circ \psi_x$ is diffeomorphism, and $\eta(1,0,0) = \varphi_w\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (0,0,1)$. The maps in the diagram are smooth. By taking derivatives we get the diagram

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{d_{(1,0,0)}\psi_x} & \mathbb{R}^4 \\ d_{(1,0,0)}\eta = d_{(1,0,0)}(\varphi_w \circ \psi_x) \downarrow & \cdot & \downarrow \text{identical mapping} \\ \mathbb{R}^3 & \xrightarrow{d_{(0,0,1)}\psi_w} & \mathbb{R}^4 \end{array}$$

Here one observes immediately $d_{(0,0,1)}\psi_w(\mathbb{R}^3) \subset d_{(1,0,0)}\psi_x(\mathbb{R}^3)$. "=" follows from symmetry or from the fact that the transition map η is a diffeomorphism, so its derivative is bijective.

Theorem 24.5. *The tangent space of a manifold at a given point does not depend on the choice of local coordinatization.*

Todistus. Given above!!! □

Remark 24.6. Remember, a Lie group is a group and a manifold, where the mappings $G \times G \to G : (x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth. Here the product set carries both the product group structure and the *product manifold structure*, the latter having products and product mappings of the original charts as charts.

Definition 24.7. A *Lie group map* also called a *homomorpism (of Lie groups)* is a smooth group homomorphism, similarly a *Lie group isomorphism* is a smooth isomorphism between Lie groups and whose inverse is also smooth.

Remark 24.8. Compositions of Lie group maps are Lie group maps, so Lie groups form a category.

Example 24.9. (1) $G = (\mathbb{R}, +)$ is Lie group, since $\mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto x + y$ and $\mathbb{R} \to \mathbb{R} : x \mapsto -x$ are smooth.

(2) The rotation group / circle S^1 is a Lie group, when it is identified with the circle as a manifold:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{(\cos 2\pi\alpha, \sin 2\pi\alpha) \mid \alpha \in \mathbb{R}\},$$

which leads to identifying its group multiplication with

$$(\cos 2\pi\alpha, \sin 2\pi\alpha) \circ (\cos 2\pi\beta, \sin 2\pi\beta) = (\cos 2\pi(\alpha + \beta), \sin 2\pi(\alpha + \beta))$$

and choosing for local parametrizations at $(\cos 2\pi\alpha, \sin 2\pi\alpha)$ the smooth mappings

$$\begin{aligned} \psi :]\alpha - \epsilon, \alpha + \epsilon[&\to S^1 \\ \theta &\mapsto (\cos 2\pi\theta, \sin 2\pi\theta). \end{aligned}$$

Check smoothness of group operations in S^1:

$$\begin{aligned}]\alpha - \epsilon, \alpha + \epsilon[\times]\beta - \epsilon, \beta + \epsilon[\xrightarrow{\psi \times \psi'} S^1 \times S^1 \xrightarrow{\circ} S^1 \xrightarrow{\varphi}]\alpha + \beta - 2\epsilon, \alpha + \beta + 2\epsilon[\\ (\theta, \theta') \overset{\psi \times \psi'}{\longmapsto} ((\cos 2\pi\theta, \sin 2\pi\theta), (\cos 2\pi\theta', \sin 2\pi\theta')) \overset{\circ}{\longmapsto} \\ \overset{\circ}{\longmapsto} (\cos 2\pi(\theta + \theta'), \sin 2\pi(\theta + \theta')) \overset{\varphi}{\longmapsto} \theta + \theta'. \end{aligned}$$

Remark 24.10. Notice: Above we also got an example of group map:

$$\iota : (\mathbb{R}, +) \to S^1 : \theta \mapsto (\cos 2\pi\theta, \sin 2\pi\theta)$$

is clearly a group homomorphism and we just proved it to be smooth also. It is not injective, hence no isomorphism.

Example 24.11. More Lie group maps:

(1) Since $SO(n) \subset SL(n) \subset GL(n)$, and all have usual matrix multiplication as the group operation and all carry the classical manifold structure inherited from $\mathbb{R}^{2n}$, so the inclusion mappings $SO(n) \to SL(n$ and $SL(n) \to GL(n)$ are Lie group maps.

(2) The circle group S^1 consists of rotations of the plane, also in the following sense:

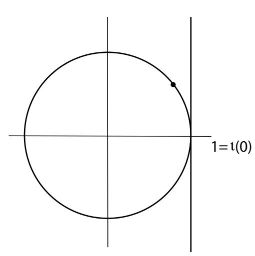
$$S^1 \to SO_{\mathbb{R}}(2) : (\cos 2\pi\theta, \sin 2\pi\theta) \mapsto \begin{bmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$$

is a Lie group isomorphism. Here the matrix group is a classical manifold in $\mathbb{R}^4$.

Also the group $U(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C}$ with complex number multiplication and the manifold structure coming from the identification $\mathbb{C} = \mathbb{R}^2$ is isomorphic to S^1. Details left as exercise.

Let us calculate the tangent space of $\iota : (\mathbb{R}, +) \to S^1$at the neutral element — using only the definition.

$$\begin{aligned} \iota : (\mathbb{R}, +) &\xrightarrow{\iota} S^1 : \\ \theta &\overset{\iota}{\mapsto} (\cos 2\pi\theta, \sin 2\pi\theta) \\ 0 &\overset{\iota}{\mapsto} (1, 0) \\ d_0\iota : \mathbb{R} = T_0\mathbb{R} &\xrightarrow{d_0\iota} T_{(1,0)}S^1 = \{0\} \times \mathbb{R} \subset \mathbb{R}^2 \\ t &\overset{d_0\iota}{\mapsto} \begin{bmatrix} \frac{\partial}{\partial\theta} \cos 2\pi\theta \\ \frac{\partial}{\partial\theta} \sin 2\pi\theta \end{bmatrix}_{\theta=0} [t] = \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} [t] = \begin{bmatrix} 0 \\ 2\pi t \end{bmatrix}. \end{aligned}$$



Kuva 8: Ympyrän S^1 tangenttiavaruus kohdassa $1 = \iota(0)$

25. Representations of Lie groups

Remark 25.1. In the following "vector spaces" will be finite dimensional, real — unless otherwise stated..

Definition 25.2. A *representation of a Lie group* G is a representation of the group G in a vector space V (a group homomorphism $G \to GL(V)$) , also a smooth mapping. Ie A Lie group representation is the same thing as a lie group mapping $\rho : G \to GL(V)$.

Example 25.3. (1) If course , the Lie groups which already by definition consist of linear mappings together with composition (matrix multiplication), like $GL(n), O(n), SL(n)$ etc. have a *tautological representation*, which is just the identical mapping of the group.

(2) The group of invertible numbers $(\mathbb{R}^*, \cdot)$ has many representations, already in dimension 2 i.e.. in $\mathbb{R}^2$:

(1) multiplication by the number itself: $\lambda \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \in GL(2)$
(2) multiplication by the square of number itself:: $\lambda \mapsto \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} \in GL(2)$
(3) Generalisations of the above: $\lambda \mapsto \begin{bmatrix} \lambda^m & 0 \\ 0 & \lambda^n \end{bmatrix} \in GL(2)$; $(m, n \in \mathbb{Z})$
(4) Action of $\lambda \in \mathbb{R}^*$ as by multiplication with $\|\lambda\|$. : $\lambda \mapsto \begin{bmatrix} |\lambda|^\alpha & 0 \\ 0 & |\lambda|^\beta \end{bmatrix}$

All these are reducible, in fact already "reduced". The group is abelian. The irreducible representations of the abelian group mentioned above are one dimensional. Exercise: are there any higher dimensional irreducible representations?

26. Linear and multilinear algebra

26.1. **Bilinear mappings.**

Definition 26.1. Let V, W and U be vector spaces and $\mathbb{F}$their coefficient field. A mapping $B : V \times W \to U$ is called *bilinear*, if both partial mappings

$$B(v, \cdot) : W \to U : w \mapsto B(v, w)$$

and

$$B(\cdot, w) : V \to U : v \mapsto B(v, w)$$

are linear. Examples:

Example 26.2. Almost all mappings called "products" in linear algebra are in fact bilinear:

(1) multiplication in $\mathbb{R}$ xy
(2) multiplication in $\mathbb{C}$ zz'
(3) multiplication of vectors by numbers λv
(4) inner product of vectors $(v|w)$
(5) matrix multiplication AB,
(6) in particular multiplication of vectors by matrices Ax
(7) evaluation of linear mappings at points Tv,
(8) in particular evaluation of linear forms $< v|u'\rangle$
(9) point-wise product of functions fg
(10) tensor product of vectors $x \otimes y$
(11) tensor product of linear mappings $T \otimes S$

26.2. **Tensor product of 2 vector spaces.**

Definition 26.3. A subset of a vector space $K \subset V$ is *free* (same as its elements are *linearly independent*, if the only way to express 0 as a linear combination of the vectors is the trivial combination $0 = \sum_{j=1}^{n} 0v_j,\ v_j \in K$, $n \in \mathbb{N}$.

A subset of a vector space $K \subset V$ *spans the space* V, if every vector $v \in V$ is the linear combination $v = \sum_{j=1}^{n} \lambda_j v_j$; $v_j \in K$, $n \in \mathbb{N}$ of some vectors in K.

A (Hamel) *basis* of a vector space V is a free set $K \subset V$,spanning V.

Remark 26.4. A set $K \subset V$is a basis of V if and only if every vector $v \in V$ can be expressed as a linear combination $v = \sum_{j=1}^{n} \lambda_j v_j$, where $\lambda_j \in \mathbb{F}$, $v_j \in K$, $n \in \mathbb{N}$, in exactly one way.

Liner combinations $v = \sum_{j=1}^{n} \lambda_j v_j$, where $\lambda_j \in \mathbb{F}$, $v_j \in K$, $n \in \mathbb{N}$, are often denoted in short hand: $v = \sum_K \lambda_k k$, where — of course — only finite many λ_k are distinct of 0.

All bases of the same vector space have the same cardinality. A vector space is d-dimensional when it has a basis with d elements. The axiom of choice implies that every vector space has a basis — generally infinite, of course.

A linear mapping is uniquely defined by the images of the basis elements, and the se images can be any vectors. In particular, in the finite dimensional case, representing linear mappings by matrices is based on this fact. Similarly, a bilinear mapping $B : V \times W \to U$ is completely determined by the values $B(k,l)$, where k goes through a basis of K of V and a basis L of W. This is so, since $B\left(\sum_K \lambda_k k, \sum_L \mu_l l\right) = \sum_{K\times L} \lambda_k \mu_l kl$.

Definition 26.5. The *The free vector space spanned by set* K is a vector space $\mathbb{F}^{(K)}$ whose basis is K.

Remark 26.6. The free vector space spanned by K is unique up to isomorphism. Also, it exists: Every set K spans some vector space, since a vector space with basis K can be constructed as follows: $\mathbb{F}^K = \{f \mid f \text{ is a function } K \to \mathbb{F}\}$. Obviously $\mathbb{F}^K$ is a vector space. Define $\mathbb{F}^{(K)} = \mathbb{F}^{(K)} = \{f \in \mathbb{F}^K \mid f(k) \neq 0 \text{ for only finitely many } k \in K\}$. Finally, identify K with the subset $K \subset \mathbb{F}^{(K)}$ by identifying the element $l \in K$ with the mapping $k \mapsto \delta_{lk}$, where δ_{kl}=1, if $k = l$ and 0 else. In this notation, every $f \in V$ is a finite sum[2] $f = \sum_{k\in K} \lambda_k k$, where $\lambda_k = f(k)$.

All in all every vector space is a free vector space and every set is the basis of some vector space.[3]

Remark 26.7. Next we define the *tensor product* of two vector spaces V and W as consisting of a space $V \otimes W$ and a bilinear mapping $V \times W \xrightarrow{\otimes} V \otimes W$. There exist several commonly used definitions for the tensor product, preferences depending on intended use. The following will be either included in the definition or proven as theorems:

(1) The tensor product of two vector spaces V and W is the most general i.e. *universal* bilinear mapping in $V \times V$ in the following sense: Each bilinear $B : V \times W \to U$ can be factored as $B = L \circ \otimes$, where

[2]only finitely many terms $\neq 0$!

[3]If the coefficient field of the vector space is replaced by just a ring(!), we arrive at the concept of a module. A free module is defined much like a free vector space. It turns out that not every module is free.

$L : V \otimes W \to U$ is a bilinear mapping depending uniquely (!) on B.

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \underset{B}{\searrow} & \downarrow L \\ & & U \end{array}$$

We will have to prove to existence and uniqueness of such a bilinear mapping $\otimes$.

(2) The tensor product of two vector spaces $V = \mathbb{F}^{(K)}$ and $W = \mathbb{F}^{(L)}$ is the free vector space $V \otimes W = \mathbb{F}^{(K)} \otimes \mathbb{F}^{(L)} = \mathbb{F}^{(K\times L)}$ equipped with the bilinear mapping

$$V \times W \xrightarrow{\otimes} V \otimes W,$$

mapping the pair of basis vectors (k, l) is mapped to the basis vector $(k, l) \in V \otimes W = \mathbb{F}^{(K\times L)}$. In particular $\dim(V \otimes W) = \dim V \cdot \dim W$. The basis vectors in the tensor product are usually denoted by $\otimes(k, l) = k \otimes l$. Therefore, the elements of the tensor product $V \otimes W$ generally are finite sums of the form

$$\sum_{(k,l)\in K\times L} \lambda_{k,l}\, k \otimes l$$

and the bilinear mapping $\otimes : V \times W \to V \otimes W$ is

$$(\sum_{k\in K} \lambda_k\, k) \otimes (\sum_{l\in L} \mu_l\, l) = \sum_{(k,l)\in K\times L} \lambda_k \mu_l\ k \otimes l.$$

Bilinearity is expressed by:

$$\begin{aligned} (v + v') \otimes w &= v \otimes w + v' \otimes w \\ \lambda \cdot (v \otimes w) &= (\lambda \cdot v) \otimes w \\ v \otimes (w + w') &= v \otimes w + v \otimes w' \\ \lambda \cdot (v \otimes w) &= v \otimes (\lambda \cdot w) \end{aligned}$$

for all $v, v', \in V,\ w, w'\ \in W$, $\lambda \in \mathbb{F}$.

These properties might not make it evident that the tensor product $\otimes : V \times W \to V \otimes W$ is unique up to isomorphism, in particular independent of the choice of a basis. [4]

The following definition of tensor product includes an explicit construction without referring to any basis. This definition also generalises to outer and symmetric products of spaces

[4]Also, definitions should generally be " coordinate free" to avoid such questions as well as the questions of existence of basis, which was solved above.

Definition 26.8. let V and W be vector spaces with coefficients in a field $\mathbb{F}$. look at the vector space $\langle V \times W\rangle$, with basis $V \times W$. Here the vectors are finite linear combinations of the basis vectors $(v, w) \in \langle V \times W\rangle$. The idea is to identify, in $\langle V \times W\rangle$, for instance $((\lambda v), w)$, to $\lambda(v, w)$ and $(v + v', w)$ to $(v, w) + (v', w)$. this is done by introducing a suitable factor space: Let $R \subset V \times W$ be the smallest subspace containing the following vectors: $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{F}$:

(1) $(v + v', w) - (v, w) - (v', w)$
(2) $\lambda(v, w) - (\lambda v, w)$
(3) $(v, w + w') - (v, w) - (v, w')$
(4) $\lambda(v, w) - (v, \lambda w)$

Define: $V \otimes W = \frac{\langle V\times W\rangle}{R}$. Finally, equip it with a bilinear mapping:

$$\begin{aligned} \otimes : V \times W \to \ & \langle V \times W\rangle \to V \otimes W \\ (v, w) \mapsto \quad & (v, w) \quad \mapsto v \otimes w = (v, w) + R. \end{aligned}$$

At least, $\otimes$ is well defined as the composition of two well defined mappings. Check bilinearity: $\otimes$ for all $v, v' \in V$, $w \in W$.

$$\begin{aligned} (v + v', w) &\mapsto (v + v') \otimes w = (v + v', w) + R \\ (v, w) &\mapsto v \otimes w = (v, w) + R \\ (v', w) &\mapsto v' \otimes w = (v', w) + R. \end{aligned}$$

Since, by (1) in the definition of R , we have $(v+v', w)-(v, w)-(v', w) \in R$, we can conclude

$$(v + v', w) + R = ((v, w) + R) + ((v', w) + R),$$

i.e..

$$(v + v') \otimes w = v \otimes w + v' \otimes w.$$

The other statements in the definition of bilinearity follow from (2),(3) and (4) in a similar way.

Remark 26.9. Now we have defined the tensor product. The next task is to prove that it has the properties (1) and (2) in the alternative definitions.

(1) Universlity of $\otimes$: Consider any bilinear $B : V \times W \to P$. Try to prove the existence of exactly one a linear mapping $L : V \times W \to P$

such, that the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \overset{B}{\searrow} & \downarrow L \\ & & P \end{array}$$

commutes, i.e.. $B = L \circ \otimes$. Uniqueness of L is obvious, since the only candidate for L maps simple tensors like this:

$$L : v \otimes w \mapsto B(v, w)$$

and L is determined by the images of the generating vectors of $V \otimes W$. What remains is to construct such a linear mapping L. The existence of such an L is non-trivial, since the simple tensors are not linearly independent, so their images cannot be chosen arbitrarily. We construct L like this: First define a linear mapping $\tilde{L} : \langle V \times W \rangle \to P$ by fixing suitable the images of the basis vectors of $\langle V \times W \rangle$: Take $(v, w) \mapsto B(v, w)$. Prove that $\tilde{L}$ can be factored through the factor space i.e. there exists a linear mapping $L : V \otimes W \to P$, s th. $\tilde{L} = L \circ \phi$, same as

$$\begin{array}{ccc} \langle V \times W \rangle & \xrightarrow{\phi} & V \otimes W \\ & \overset{\tilde{L}}{\searrow} & \downarrow L \\ & & P \end{array}$$

where ϕ is the canonical surjection

$$\begin{aligned} \langle V \times W \rangle \to \frac{\langle V \times W \rangle}{R} &= V \otimes W \\ (v, w) \mapsto (v, w) + R &= v \otimes w. \end{aligned}$$

such an L exists since $\operatorname{Ker} \phi = R \subset \operatorname{Ker} \tilde{L}$. This in turn follows from the observation that if $a + R = b + R \in \frac{\langle V \times W \rangle}{R} = V \otimes W$, then $a - b \in R \subset \operatorname{Ker} \tilde{L} \subset \langle V \times W \rangle$, so $\tilde{L}a = \tilde{L}b$. Therefore, the mapping

$$V \otimes W \to P : a + R \mapsto \tilde{L}a$$

is well defined and of course linear.

(2) The images of the basis vectors of $\langle V \times W \rangle$ where $v \in V$ and $w \in W$, are not linearly independent in $v \otimes w$ as is seen for example by looking at

$$(v + v') \otimes w - v \otimes w - v' \otimes w = 0.$$

But they span $V \otimes W = \frac{\langle V \times W \rangle}{R}$, whose elements, the *tensors* are finite sums of these *simple tensors*. If K is a basis for V and L for W, then the elements of V are finite sums $\sum_{k \in K} \lambda_k \, k$ and the elements of W are sums $\sum_{l\in L} \mu_l\, l$ so the elements of the tensor product space are finite sums

$$(\sum_{k\in K} \lambda_k\, k) \otimes (\sum_{l\in L} \mu_l\, l) = \sum_{(k,l)\in K\times L} \lambda_k\mu_l\ k\otimes l = \sum_{(k,l)\in K\times L} \rho_{k,l}\ k\otimes l.$$

We have just proved that the tensor products of the original basis vectors span the tensor product space. $V\otimes W$. Now we prove their linear independence. Let

$$\sum_{(k,l)\in K\times L} \lambda_{k,l}\ k\otimes l = 0 \in V\otimes W.$$

By universality of the bilinear mapping $\otimes$ every bilinear mapping $B : V\times W \to \mathbb{F}$ corresponds to a unique linear mapping $L : V\times W \to P$ making the diagram

$$\begin{tikzcd} V\times W \arrow[r, "\otimes"] \arrow[dr, "B"'] & V\otimes W \arrow[d, "L"] \\ & \mathbb{F} \end{tikzcd}$$

commutative, i.e.. such that $B = L\circ\otimes$. In particular, for every bilinear mapping $B : V\times W \to \mathbb{F}$ we have $\sum_{(k,l)\in K\times L} \lambda_{k,l}\ B(k,l) = 0$. But the images of the pairs of basis vectors cab e chosen freely when setting up a bilinear mapping. Therefore $\sum_{(k,l)\in K\times L} \lambda_{k,l}\ B_{k,l} = 0$ for all choices $B_{k,l}\in\mathbb{F}$, so every $\lambda_{k,l}$ is 0. $\square$

Example 26.10. Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $\{e_1,\dots,e_n\}$, $\{f_1,\dots,f_m\}$ their bases. Now $V\otimes W = \mathbb{R}^{nm}$, which as a vector space is the same as the space of all $n\times m$-matrices $M_{n\times m} = \mathbb{R}^{n\times m}$. In particular, a basis for $V\otimes W$ consists of all the tensor products of the original basis vectors

$$e_i\otimes f_j = \begin{array}{c} \begin{bmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ \dots & 1 & \dots \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix} \\ (i) \end{array} \begin{array}{c} \\ \\ (j) \\ \\ \\ \\ \end{array} = [\delta_{\alpha i}\delta_{\beta j}]_{\alpha,\beta}.$$

In these bases tensor products of vectors Näissä kannoissa saadaan siis yleisten vektorien tensorituloksi eli yksinkertaiseksi tensoriksi $(a_1,\dots,a_n)\otimes$

$(b_1, \dots b_m)$ [5]

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_m \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \dots & a_nb_m \end{bmatrix}, \text{ joka is } \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}.$$

The product matrix has linearly dependent columns, so it has rank 1 (ir 0). It is obvious that every rank one matrix can be constructed like this, so it represents a simple tensor.

26.3. **Symmetric and alternating bilinear mappings.**

Definition 26.11. Let V and P be vector spaces.

(1) A bilinear mapping $B : V \times V \to P$ is called *symmetric*, if

$$B(v, w) = B(w, v) \quad \forall v, w \in V.$$

(2) A bilinear mapping $B : V \times V \to P$ is called *alternating*, if

$$B(v, v) = 0 \quad \forall v \in V.$$

Remark 26.12. If the coefficient field is $\mathbb{F}$ is $\mathbb{C}$ or any subfield of $\mathbb{C}$ [6], then $B : V \times V \to P$ is alternating, if and only if it is *anticommutative*:

$$B(v, w) = -B(w, v) \quad \forall v, w \in V,$$

. To prove this, just notice that if B is alternating, then $0 = B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) = 0 + B(v, w) + B(w, v) + 0$ and if we assume that B is anticommutative, then $0 = B(x, x) + B(x, x) = 2B(x, x)$, so $B(x, x) = 0$.

Example 26.13. The fundamental example of asymmetric bilinear mapping is the product of polynomials: let $V = \mathbb{F}[x] = \{f \mid f \text{ be a one variable } \mathbb{F}\text{- polynomial}\}$. The usual multiplication of polynomials $V \times V \to V$ is biklinear. The same holds in th space of several polynomials of several variables ie. in the space $V = \mathbb{F}[x_1, \dots, x_d]$.

More generally, in any commutative $\mathbb{F}$-algebra A internal multiplication $A \times A \to A$ is $\mathbb{F}$-bilinear and symmetric; in fact a *commutative* $\mathbb{F}$-*algebra* is by definition a $\mathbb{F}$-vector space together with a symmetric bilinear mapping $B : A \times A \to A$. [7]

[5] Ei ole Kroneckerin tulo?!

[6] It is sufficient to assume that $2 \neq 0$ in $\mathbb{F}$.

[7] More genreally, if a bilinear mapping $B : A \times A \to A$ is associative, then $(A, +, B)$ is simultaneously a vector space and a ring, ie an *associative algebra*. An assosiatiive algebra is not always commutative and a *general algebra* same as just

Example 26.14. In $\mathbb{R}^3$ the "cross product" of two vectors is an example of an alternating bilinear mapping. A more fundamental example is given by the determinant of a 2×2-matrix:

Let $V = \mathbb{R}^2$. The mapping associating to a pair of vectors $v = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $w = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$ the determinant $B(v, w) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is an alternating bilinear mapping $V \times V \to \mathbb{R}$.

26.4. **Multilinear mappings.** It is natural to generalise the determinant example to $\mathbb{R}^d$. The determinant is linear in the column vectors and has the value 0, whenever two columns coincide. This gives the motivation to consider alternating multilinear mappings of more than 2 variables.

Definition 26.15. Let $V_1, \dots, V_n$ and P be vector spaces and $\mathbb{F}$ their common coefficient field.

(1) The mapping $M : V_1 \times \cdots \times V_n \to P$ is *multilinear*, in this case $n-$*linear*, if its all partial mappings
$$\begin{aligned} V_1 \to P : v_1 &\mapsto M(v_1, \dots, v_n), \\ V_2 \to P : v_2 &\mapsto M(v_1, \dots, v_n), \\ &\cdots \\ V_n \to P : v_n &\mapsto M(v_1, \dots, v_n), \end{aligned}$$
are linear.

(2) An n-linear mapping $M : V^n \to P$ is *symmetric*, if $M(v_1, \dots, v_n)$ does not depend on the order of the variables $v_1, \dots, v_n$. This means that for all permutations $\sigma \in S_n$ we have
$$M(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = M(v_1, \dots, v_n)$$

(3) An n-linear mapping $M : V^n \to P$ is *alternating*, if
$$M(v_1, \dots, v_n) = 0 \text{ whenever } v_i = v_j \text{ for some } i \neq j.$$

Remark 26.16. When $\mathbb{F}$ is a subfield of $\mathbb{C}$, then a multilinear mapping is alternating if and only if $M(v_1, \dots, v_n) = 0$ whenever $v_1, \dots, v_n$ are linearly dependent. This is equivalent to saying that the mapping changes sign whenever two variables are interchanged. The classical example

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a bilinear mapping, is not always associative. In particular, lien algebras will not be associative, nor commutative.

of an alternating multilinear mapping is — as was already mentioned– the determinant considered as a function of its column (or rows)

26.5. **Symmetric and outer powers.**

Definition 26.17. Let $V_1 \times \cdots \times V_n$ be ($\mathbb{F}$-) vector spaces. An n-linear map $\otimes : V_1 \times \cdots \times V_n \to W$ is called the *tensor product of the spaces* $V_1, \ldots, V_n$ and denoted $W = V_1 \otimes \cdots \otimes V_n$, if $\otimes$ is a *universal n-linear mapping* in the following sense:

Every $n-$linear mappings $M : V_1 \otimes \cdots \otimes V_n \to U$ can be factored into $B = L \circ \otimes$, where $L : V_1 \otimes \cdots \otimes V_n \to U$ is a linear mapping uniquely determined by the $n-$linear mapping M.

$$\begin{array}{ccc} V_1 \times \cdots \times V_n & \stackrel{\otimes}{\longrightarrow} & V_1 \otimes \cdots \otimes V_n \\ & \stackrel{M}{\searrow} & \downarrow L \\ & & U \end{array}$$

The construction of the tensor product of two spaces can readily be generalised to the case of several spaces and proves its existence and uniqueness. The same result can be obtained by paying attention to that $V_1 \otimes \cdots \otimes V_n$ can be defined as $(\ldots(V_1 \otimes V_2) \otimes \cdots \otimes V_n)$. A basis for the tensor product is given by $K_1 \times \cdots \times K_n$, where K_j is a basis of the original space V_j. In particular, the d:th *tensor power* $\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n = (\mathbb{R}^n)^{\otimes d}$ has a basis consisting of all $e_{i_1} \otimes \cdots \otimes e_{i_d}$, where $\{e_1, \ldots, e_n\}$ is a basis of the original space $\mathbb{R}^n$.

By the same principle that we used the concept of bilinear mappings for constructing the tensor product, we can define and construct the symmetric and alternating (same as outer) powers of a vector space using symmetric and alternating multilinear mappings:

Definition 26.18. Let V be a ($\mathbb{F}$-) vector space A symmetric n-linear mapping $\cdot : V \times \cdots \times V = V^n \to W$ is the n:th *symmetric power* $W = S^nV$, if it is the *universal symmetric n-linear mapping* in the following sense:

Every symmetric $n-$linear mapping $M : V^n \to U$ is of the form $M = L \circ \cdot$, where $L : S^nV \to U$ is a linear mapping depending uniquely on the symmetric $n-$linear mapping M.

$$\begin{array}{ccc} V^n & \stackrel{\cdot}{\longrightarrow} & S^nV \\ & \stackrel{M}{\searrow} & \downarrow L \\ & & U \end{array}$$

The existence of a symmetric power is proved by construction. This can be done using bases and/or factor spaces. We skip the construction since the following construction of the outer power is almost identical. We just remark that a basis for the symmetric power is given by all products of the original basis vectors $e_{i_1} \cdot \dots \cdot e_{i_d}$, where $i_1 \leq \dots \leq i_n$, and the original index set I is ordered.

Example 26.19. Let V={degree 1 homogenous polynomials of n variables} $= \langle x_1, \dots, x_n \rangle$. In this case $S^d V = \{$ degree d homogeneous polynomials of n variables$\}$, for example

$$S^2 V = \langle x_1^2, x_1x_2, x_1x_2, \dots x_1x_n, x_2^2, x_2x_3, \dots, \dots x_n^2 \rangle.$$

In this example the symmetric product is the same thing as the "formal product" of polynomials.

Definition 26.20. Let V be a ($\mathbb{F}$-) vector space. An alternating n-linear mapping $\wedge : V \times \dots \times V = V^n \to W$ is called the n:th *outer power* of the space V and denoted $W = \Lambda^n V$, if it is the *universal alternating n-linear mapping*:

Each alternating $n-$linear mapping $M : V^n \to U$ is of the form $M = L \circ \wedge$, where $L : \Lambda^n V \to U$ is a linear mapping depending only on the alternating $n-$linear mapping M.

$$\begin{array}{ccc} V^n & \xrightarrow{\wedge} & \Lambda^n V \\ & \overset{M}{\searrow} & \downarrow L \\ & & U \end{array}$$

The existence of an outer power is proved by constructing it. For simplicity, we do it for $n = 2$ only. Like above in the construction of the tensor product, we begin by considering the free vector space $\langle V^2 \rangle$, which has all of V^2 as basis. Next define the subspace R, spanned by the "relations" mentioned above in the tensor product construction, and one new. This means: $R \subset V^2$ is the smallest vector subspace containing the following vectors: for all $v, v'w, w' \in V$ and $\lambda \in \mathbb{F}$:

(1) $(v + v', w) - (v, w) - (v', w)$
(2) $\lambda(v, w) - (\lambda v, w)$
(3) $(v, w + w') - (v, w) - (v, w')$
(4) $\lambda(v, w) - (v, \lambda w)$
(5) $(v, w) + (v, w)$

Now the factor space $\Lambda^2 V = \frac{\langle V^2 \rangle}{R}$ has the universal property defining the outer power, when equipped with the bilinear mapping

$$\begin{aligned} \wedge : V \times V \ &\rightarrow \langle V^2 \rangle \rightarrow \Lambda^2 V \\ (v, w) &\mapsto (v, w) \mapsto v \wedge w = (v, w) + R. \end{aligned}$$

follows from condition (5)). We sum up the properties of $\wedge$: bilinearity

$$\begin{aligned} \lambda \cdot (x \wedge y) &= (\lambda \cdot x) \wedge y = x \wedge (\lambda \cdot y) \\ (x + y) \wedge z &= x \wedge z + y \wedge z \\ x \wedge (z + w) &= x \wedge z + x \wedge w \end{aligned}$$

and alternation:

$$v \wedge w = -w \wedge v.$$

An outer power satisfying the universal property does exist.

The images of the pairs $e_i \wedge e_j$ of original basis vectors of V have images $e_i \wedge e_j$ spanning all of $\Lambda^2 V = \frac{\langle V \times W \rangle}{R}$, so the elements of this space are the finite sums of these generators. If the basis K of V is an ordered, then already the products , $e_k \wedge e_l$ with $k < l$, span $\Lambda^2 V$, since $e_k \wedge e_l = 0$, for $k = l$ and $e_k \wedge e_l = -e_l \wedge e_k$, where $k > l$. Linear independence of the remaining basis vector candidates is proven essentially in the same way as we did for the basis of the tensor product.

the basis becomes $\{e_k \wedge e_l \mid k < l\} = \{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, \ldots e_1 \wedge e_d, e_2 \wedge e_3, e_2 \wedge e_4, \ldots e_2 \wedge e_d, e_3 \wedge e_4, \ldots, \ldots e_{d-1} \wedge e_d, \}$, so $\dim(\Lambda^2 V) = \frac{1}{2} n(n-1)$.

Example 26.21. Take $V = \mathbb{R}^n$ with basis $\{e_1, \ldots, e_n\}$. Now$\Lambda^2 V$ can be identifies a as a vector space with the space of all anti-symmetric $n \times n$-matrices.

Remark 26.22. Similar considerations can be carried through for higher powers $\Lambda^n V$. In particular, if V has its basis K ordered, then the basis vectors for $\Lambda^n V$ can be taken to be the outer products of the original basis vectors, $e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_d}$, with$k_1 < k_2 < \cdots < k_d$.

Example 26.23. By what was just proved, $\Lambda^d \mathbb{R}^d$ is one dimensional with one basis vector being for instance $e_1 \wedge e_2 \wedge \cdots \wedge e_d$. Remember that the determinant of a $n \times n$-matrix is an alternating multilinear mapping from the column vectors to the ground field $\mathbb{R}$.

$$\begin{array}{ccc} (\mathbb{R}^n)^n & \xrightarrow{\wedge} & \Lambda^n \mathbb{R}^n \sim \mathbb{R} \\ & \overset{\det}{\searrow} & \downarrow L = \lambda \cdot \\ & & \mathbb{R} \end{array}$$

We notice that the determinant is, up to a multiplicative constant, the only alternating n-linear mapping from n-dimensional space to the numbers. The same holds for any ground field $\mathbb{F}$.

27. Tensor products of representations

Example 27.1. In example **??** we mentioned the space S^dV of homogeneous polynomials of degree d and n variables. A special case is

$$S^2V = \langle x_1^2, x_1x_2, x_1x_2, \dots x_1x_n, x_2^2, x_2x_3, \dots, \dots x_n^2\rangle.$$

A *linear change of co-ordinates* is a linear mapping

$$S^dV \to S^dV : P \mapsto P \circ L,$$

where $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear.[8] If $\rho : G \to GL_n(\mathbb{R})$ is a representation, then also $g \mapsto (P \mapsto P \circ \rho(g))$ is a representation of G. This is a special case of the following general construction.

27.1. **Induced representations.** [9]

By definition, a representation of a Lie group G in a vector space V is a smooth group homomorphism $G \to GL(V)$.

Definition 27.2. Let ρ and $\rho' : G \to GL(V)$ be representations of a group G in finite dimensional spaces V and W. Then the following are representations

(1) $\rho \otimes \rho' : g \mapsto (\rho \otimes \rho')_g : V \otimes W \to V \otimes W : u \otimes w \mapsto \rho_g u \otimes \rho_g v.$
(2) $\rho \cdot \rho' : g \mapsto (\rho \cdot \rho')_g : S^2V \to S^2V : u \cdot w \mapsto \rho_g u \cdot \rho'_g v.$
(3) $\rho \wedge \rho' : g \mapsto (\rho \wedge \rho')_g : S^2V \to S^2V : u \wedge w \mapsto \rho_g u \wedge \rho'_g v.$

Here linear mappings are determined by giving the images of the basis vectors.

Example 27.3. The tautological representation of the group $G = GL_2(\mathbb{R})$ acts in R^2. It induces a representation in the outer product $\lambda^2(\mathbb{R}^2)$. The induced representation is one dimensional with basis vector $e_1 \wedge e_2$. Let us find an explicit expression for it: An element of the group

[8] Strictly speaking, polynomials are not functions but can often be interpreted as such.

[9] Seems, in Serre's book, induced representations are something different.

$G = GL_2(\mathbb{R})$is a matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It maps the basis vector of the outer product like this:

$$
\begin{aligned}
e_1 \wedge e_2 \mapsto ge_1 \wedge ge_2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \wedge \begin{bmatrix} b \\ d \end{bmatrix} \\
&= (ae_1 + ce_2) \wedge (ce_1 \wedge de_2) \\
&= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\
&= (ad - bc)e_1 \wedge e_2 \\
&= \det g \, e_1 \wedge e_2.
\end{aligned}
$$

All in all, the second outer power of any linear mapping in $\mathbb{R}^2$, in particular a representation matrix, is multiplication by its determinant in one dimensional space.

This can be generalised. The tautological action of $GL_n(\mathbb{R}^n)$ in $\mathbb{R}^n$ induces a one dimensional representation $\Lambda_n(\mathbb{R}^n)$, which is multiplication by the determinant of the original matrix. We leave it as a n exercise to find out what the lower outer powers of the tautological action of $GL_n(\mathbb{R}^n)$ in $\mathbb{R}^n$ induces into $\Lambda_m(\mathbb{R}^n)$, for $1 < m < n$.

Example 27.4. The tautological action of the group $G = GL_2(\mathbb{R})$ in R^2 induces a representation in the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$. . Let us decompose it into a direct product of irreducible representations.

The first observation is that something is needed, since the representation of $G = GL_2(\mathbb{R})$

$$g : \mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2 : g(v \otimes w) = gv \otimes gw$$

is not irreducible but has at least one proper sub-representation, namely

$$W_s = \left\langle \{v \otimes w + w \otimes v \;\middle|\; v, w \in \mathbb{R}^2\} \right\rangle.$$

This motivates to check whether possibly also

$$W_\wedge = \left\langle \{v \otimes w - w \otimes v \;\middle|\; v, w \in \mathbb{R}^2\} \right\rangle$$

is a sub-representation, and it turns out to be. let us check these two representations for reducibility without using any theory, just by calculating them explicitly. The first step is to calculate the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2 \to$ explicitly:

The elements of $GL_2(\mathbb{R})$ are all invertible 2×2-matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The space $\mathbb{R}^2 \otimes \mathbb{R}^2 \to$ is spanned by the simple tensors

$$v \otimes w = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}.$$

In particular the standard basis vectors will be

$$e_1 \otimes e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_1 \otimes e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_2 \otimes e_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_2 \otimes e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The generating vectors of the subspace $W_\wedge$ are of the form

$$\begin{aligned} v \otimes w - w \otimes v &= \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix} - \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & a_1b_2 - a_2b_1 \\ a_2b_1 - a_1b_2 & 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \end{aligned}$$

where $\lambda = a_1b_2 - a_2b_1$. In other words

$$W_\wedge = \lambda \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mid \lambda \in \mathbb{R} \right\} = \left\{ \lambda \left(e_1 \otimes e_2 - e_2 \otimes e_1 \right) \bigm| \lambda \in \mathbb{R} \right\}.$$

The action of the group in this one-dimensional space is determined by its action on the basis vector, which, by definition, is the following:

$$\begin{aligned} &\begin{bmatrix} a & b \\ c & d \end{bmatrix} (e_1 \otimes e_2 - e_2 \otimes e_1) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) (e_1 \otimes e_2 - e_2 \otimes e_1) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} e_1 \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} e_2 - \begin{bmatrix} a & b \\ c & d \end{bmatrix} e_2 \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} e_1 \\ &= \begin{bmatrix} a \\ c \end{bmatrix} \otimes \begin{bmatrix} b \\ d \end{bmatrix} - \begin{bmatrix} b \\ d \end{bmatrix} \otimes \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} ab & ad \\ cb & cd \end{bmatrix} - \begin{bmatrix} ba & cd \\ ad & cd \end{bmatrix} = \begin{bmatrix} 0 & ad - bc \\ cb - ad & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (e_1 \otimes e_2 - e_2 \otimes e_1). \end{aligned}$$

So we have proved by direct calculation, that the sub-representation $W_\wedge$ of the representation $\mathbb{R}^2 \otimes \mathbb{R}^2$ is multiplication in one-dimensional space by the determinant of the group element. So this is isomorphic as a representation to $\Lambda^2\mathbb{R}^2$.

In the subspace W_s we can choose a basis

$$\{e_1 \otimes\, e_1,\ e_2 \otimes e_2,\ e_1 \otimes e_2 + e_2 \otimes e_1\} \subset W_s,$$

since these three are linearly independent and the space W_s is at most 3-dimensional, since is a nontrivial subspace of a 4-dimensional space.

The action of $GL_2(\mathbb{R})$ in $S^2\mathbb{R}$ can nicely be interpreted like we did in example **??** in the beginning of this section, i.e. as linear changes of co-ordinates in $\langle x^2, y^2, xy\rangle$.

We leave it as an exercise to show that the mapping $W_s \to S^2\mathbb{R}$

$$\begin{aligned} e_1 \otimes\ e_1 &\mapsto x^2 \\ e_2 \otimes e_2 &\mapsto y^2 \\ e_1 \otimes e_2 + e_2 \otimes e_1 &\mapsto 2xy \end{aligned}$$

is an isomorphism of representations.

Finally, we check that $S^2\mathbb{R}$ is irreducible. It is sufficient to find a vector — same as a polynomial— whose orbit will span the whole 3-dimensional space $S^2\mathbb{R}$, which is the case if there are 3 linearly independent polynomials in the orbit. This is easily found by trial (and error). Remember, that an element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $G = GL_2(\mathbb{R})$ acts in the space of polynomials by

$$\begin{aligned} x &\mapsto ax + by \\ y &\mapsto cx + dy. \end{aligned}$$

In particular, the basis vectors of $S^2\mathbb{R}$ are mapped like this:

$$\begin{aligned} x^2 &\mapsto (ax+by)^2 = a^2x^2 + 2abxy + b^2y^2 \\ y^2 &\mapsto (cx+dy)^2 = c^2x^2 + 2cdxy + d^2y^2, \\ xy &\mapsto (ax+by)(cx+dy)^2 = acx^2 + (ad+bc)xy + bdy^2, \end{aligned}$$

so by choosing $a = b = c = -d = 1$ we get

$$xy \mapsto acx^2 + (ad+bc)xy + bdy^2 = x^2 - y^2,$$

and by choosing $a = 1, = b = c = d = 0$ we get

$$x^2 - y^2 \mapsto x^2,$$

and by $a = b = c = 0, d = 1$ we get

$$x^2 - y^2 \mapsto -y^2.$$

The linearly independent polynomials xy, x^2 and $-y^2$ are in the same orbit!

Now we have reduced the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$ to a direct sum which up to isomorphism is

$$\mathbb{R}^2 \otimes R^2 = \Lambda^2\mathbb{R} \oplus S^2\mathbb{R}.$$

This result can be generalised to higher tensor powers, which we will do later in **??**.

28. Lie algebras — introduction

28.1. An abstract definition.

Definition 28.1. A *lie algebra* is a vector space V together with an alternating bilinear mapping, called the *lie bracket*

$$V \times V \to V : (X, Y) \mapsto [X, Y],$$

satisfying the *Jacobi identity*: for all $X, Y, Z \in V$:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The coefficient field of the vector space $\mathbb{F}$ can be any field, but in this text it is most often $\mathbb{R}$, sometimes C.

A *lie algebra homomorphism* is a liner mapping $f : V \to V'$, where V and $V'$are lie algebras and f preserves brackets, in other words $[f(X), f(Y)] = [X, Y]$ for all $X, Y \in V$. A *lien algebra isomorphism* is also a bijection, and its inverse will be a homomorphism.

A *lie sub-algebra* is a vector subspace of a lie algebra which is stable under the bracket operation, i.e. ,it contains the brackets of its elements, so it is a lie algebra in its own right.

Remark 28.2. 1) Lie algebras form a category, in particular composed mappings of homomorphisms are homomorphisms.

2) A lie algebra is generally not associative. The Jacobi identity can be seen as a surrogate property.

3) The motivation to consider lie algebras in the context of Lie groups is the following: To each Lie group one can — in a natural way — associate a lie algebra, and essentially all lie algebras arise in this way. The same is true with respect to representations.

Example 28.3. The most trivial example of a lie algebra is the is *trivial lie algebra*, i.e. any vector space with the zero bracket: $[X, Y] = 0$

Example 28.4. The fundamental example of a lie algebra is the space of all (invertible or not!) $n \times n$-matrices

$$gl_n = gl_n(\mathbb{R}) = \mathbb{R}^{n\times n}$$

equipped with the *standard bracket*

$$[X, Y] = XY - YX.$$

It is easy to check that the standard bracket is an alternating bilinear mapping. The Jacobi identity is trivial as well: :

$$\begin{aligned}
&[X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] \\
&= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) \\
&\quad - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z \\
&= X(YZ) - (XZ)Y - (YZ)X + (ZY)X + Y(ZX) - Y(XZ) \\
&\quad - (ZX)Y + (XZ)Y + Z(XY) - Z(YX) - (XY)Z + (YX)Z \\
&= XYZ - XZY - YZX + ZYX + YZX - YXZ \\
&\quad - ZXY + XZY + ZXY - ZYX - XYZ + YXZ \\
&= 0.
\end{aligned}$$

It will turn out that all lie algebras of finite dimensional lie groups are sub-algebras of this.

Example 28.5. The former ca be generalised. starting with any associative algebra, $\mathcal{U}$ one can equip it with the *usual bracket*

$$[X, Y] = XY - YX.$$

Example 28.6. The following example of a lie algebra is also an example of the lie algebra of a Lie group, and also an example of a sub-lie-algebra of $gl_n(\mathbb{R})$. Recall **??** that the tangent space of the Lie group $SL_n(\mathbb{R})$ at the neutral element is s

$$T_e(SL_n(\mathbb{R})) = sl_n(\mathbb{R}) = \{X \in gl_n(\mathbb{R}) \mid Tr(x) = 0\}.$$

This is a lie sub-algebra, which can be checked by noticing that is is a vector subspace and $[X, Y] \in sl_n(\mathbb{R})$ whenever $X, Y \in sl_n(\mathbb{R})$, something that must be proved by a little calculation — left to the reader.

Example 28.7. The *Heisenberg lie algebra* is a three dimensional vector space V, where the brackets of the bais vectors are jossa kantavektorien Lien sulkeet ovat

$$\begin{aligned}
[X, Y] &= Z \\
[Y, Z] &= 0 \\
[Z, X] &= 0.
\end{aligned}$$

It is rather easy to notice that this is isomorphic to the lie sub-algebra of $sl_3(\mathbb{R})$ consisting of proper upper triangular matrices. The isomorphism is given by

$$X \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$Y \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$Z \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

29. The lie algebra of a Lie group

29.1. **Introduction.** The lie algebra $\mathcal{G}$ of a Lie group G is as a vector space identical to the tangent space of the smooth manifold G at the identity element $e \in G$. smoothn moniston G tangenttiavaruus ryhmän G neutraalialkion $e \in G$ kohdalla. Also, it is equipped with a Lie bracket, the definition of which depends on the group operaion in G. Before giving any definition of the bracket, consider a basic example of a Lie group, say the group of invertible $n \times n$-matrices $GL_n(\mathbb{R})$. Its neutral element is the unity matrix I. Since $GL_n(\mathbb{R})$ is an open subset of the space of all $n \times n$-matrices $M_{n\times n}(\mathbb{R}^n) = \mathbb{R}^{n\times n} \sim \mathbb{R}^{n^2}$ open subset, its tangent space at all points, in particular at I, is all of $\mathbb{R}^{n^2}$ same as $M_{n\times n}$. This space was equipped with a lie bracket in example **??**. We defined $[X, Y] = XY - YX$. It turns out that this is in this case just the lie bracket coming from the general definition of the lie algebra of a Lie group. The general definition and the proof of this fact will fill the next few pages. Let us sketch the construction:

Remark 29.1. **Phase 1.** Let γ be the action of an n-dimensional Lie group by conjugationin the group G itself, i.e.

$$\begin{aligned} \gamma_g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

Each mapping γ_g is a diffeomorphism and $\gamma_g(e) = e$, so its derivative at e is a bijective linear mapping

$$Ad_g = d_e\gamma_g : T_eG \to T_eG.$$

In this way we have constructed a representation of the Lie group G in its own tangent space:

$$Ad : G \to GL(T_eG) \sim GL_n(\mathbb{R})$$
$$g \mapsto (Ad_g : T_eG \to T_eG)$$

Phase 2. We notice that also the mapping Ad is smooth. So it has a derivative at $e \in G$. Define

$$ad = d_e(Ad) : T_eG \to T_I(GL(T_eG))$$
$$X \mapsto (ad_X : T_eG \to T_eG).$$

The derivative ad of the mapping Ad is a linear mapping from the tangent space T_eG of G to the tangent space of the vector space $GL(T_eG)$ at I, which of course is the linear space itself

$$T_I(GL(T_eG)) \sim M_{n\times n} \sim \{\text{lineaarikuvaukset } T_eG \to T_eG\}.$$

Phase 3. Finally, define the *lie bracket in T_eG* by

$$[X, Y] = ad_X Y.$$

It turns out that this mapping is a lie bracket in the abstract sense: bilinear, alternating and satisfying the Jacobi identity. We call the tangent space T_eG with this bracket the *lie algebra $\mathcal{G}$ of the Lie group G.*

Remark 29.2. It will take a while until we have proven all these statements. Let us begin by proving the existence and bilinearity of the lie bracket in the tangent plane.

Phase1. Conjugation in a Lie group G is composed of two diffeomorphisms,

$$\gamma_g : h \mapsto gh \mapsto g^{-1},$$

therefore itself a diffeomorphism, so its derivative

$$Ad_g = d_e\gamma_g : T_eG \to T_eG$$

at e exists and is a linear bijection. By the chain rule $Ad_{gg'} = d_e\gamma_{gg'} = d_e(\gamma_g \circ \gamma_{g'}) = d_{\gamma'(e)}\gamma_g \circ d_e\gamma_{g'} = d_e\gamma_g \circ d_e\gamma_{g'} = Ad_g Ad_{g'}$, so we get a representation

$$Ad : G \to GL(T_eG) \sim GL_n(\mathbb{R}).$$

To be a Lie group representation, Ad has to be smooth. This may be difficult to verify without using co-ordiantes, but a minute's thought will reveal that in local co-ordinates all components of the conjugation mapping $G\times G \to G : (g, h) \mapsto ghg^{-1}$ are smooth real valued functions. Its derivative, which is nothing but Ad_g has a Jacobi matrix whose all entries are partial derivatives of these components, hence smooth as well. So the mapping $g \mapsto \mathrm{Mat}\, Ad_g$ is smooth i.e. Ad is smooth.

Phase 2. Ad_g has a derivative at $e \in G$, called ad. This derivative is, by definition, a linear mapping, i.e.

$$T_eG \to T_I(GL(T_eG)) \sim M_{n\times n} \sim \{\text{linear mappings} T_eG \to T_eG\}.$$

Vaihe 3. a) Bilinearity: Since ad_X is a linear mapping $T_eG \to T_eG$, the mapping $Y \mapsto [X, Y] = ad_X Y$ is tof course linear in the variable Y. Linearity in X depends on the fact that the mapping $ad : X \mapsto ad_X$ is the derivative of Ad, so it is linear.

We will only later (in **??**) prove the rest, namely that our bracket is:

b) alternating and

c) satisfies the Jacobi identity

Instead of giving a proof, we inspect some examples::

Example 29.3. The Lie group S^1 can be identified with the unit circle $U(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C} \sim \mathbb{R}^2$, where multiplication of complex numbers is the group operation and 1 is the neutral element. Conjugation by the number $g \in U(1)$ is nothing but $z \mapsto gzg^{-1} = zgg^{-1} = g$, in other words every γ_g is the identical mapping. The derivative of the identical mapping is of course the identical mapping of the tangent space, so Ad is a constant mapping $S^1 \to GL(T_1S^1)$. The derivative of a constant is zero, so $ad_X = 0$ for all $X \in T_1S^1$ and consequently $[X, Y] = ad_X Y = 0$ for all X and Y.

The same argument shows that the lie algebra of every abelian Lie group is trivial.

Example 29.4. Next we prove that the lie algebra of $G = GL_n(\mathbb{R})$is what we dafined already in **??** namely $gl_n(\mathbb{R})$. We alrady know that the tangent space of the manifold $GL_n(\mathbb{R})$ at the neutral element I is the correct set $T_I(GL_n(\mathbb{R})) = \mathbb{R}^{n\times n}$. What remains to be proven is that the lie bracket defined by conjugation and the derivatives Ad and ad will be the same thing as the standard bracket in $gl_n(\mathbb{R})$.

Conjugation by a matrix g is the mapping $\gamma_g : h \mapsto gh \mapsto ghg^{-1}$, which is the composition of two linear mappings, hence linear, strictly speaking the restriction of a linear mapping $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ to the open set $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$. Its derivative at I is its continuation as a linear mapping $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$:

$$Ad_g Y = gYg^{-1}.$$

To calculate the derivative of the mapping

$$Ad : G \to GL(T(\mathbb{R}^{n^2})) : g \mapsto (Y \overset{Ad_g}{\mapsto} gYg^{-1})$$

with respect to g: we notice that it is the composition of the restriction of the bilinear mapping

$$B : G \times G \to GL(T(\mathbb{R}^{n^2})) : (g, \tilde{g}) \mapsto (Y \overset{B_{g,\tilde{g}}}{\mapsto} gY\tilde{g})$$

to an open set and the matrix inversion mapping $G \to G \times G : g \mapsto (g, g^{-1})$, so the derivative is given by the chain rule, once we find the derivative of these two mappings.

The derivative of any bilinear mapping is well known[10] $B : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$:

$$d_{(A,B)}B(X,Y) = B(A,Y) + B(X,B).$$

At the identity element we get

$$d_{(I,I)}B(X,Y) = B(I,Y) + B(X,I).$$

Applying the formula for the derivative of a bilinear mapping to $AA^{-1} = I$ one can find the derivative of inversion $k : G \to G : A \mapsto A^{-1}$

$$d_g k(X) = -g^{-1}Xg^{-1}.$$

At $e = I \in G = GL_n(\mathbb{R})$ this becomes

$$d_I k(X) = -X.$$

Combining these results by the chain rule gives what we were searching for:

$$\begin{aligned} ad : T_I G &\overset{d_I(Id,k)}{\to} T_I G \times T_I G \overset{d_{(I,I)}B}{\to} T_I(GL(T(\mathbb{R}^{n^2}))) \\ X &\overset{d_I(Id,k)}{\to} (X, -X) \overset{d_{(I,I)}B}{\to} B(I,-X) + B(X,I) \\ &= (Y \mapsto (IY(-X) + XYI)) \\ &= (Y \mapsto (XY - YX)), \end{aligned}$$

which is the standard bracket. □

[10]On Purmosen diff lask 1 monisteessa harjoitustehtävänä! Hyvä kaava. Tämän erikoistapauksena saadaan mm. tunnettu tulon derivoimiskaava.

Theorem 29.5. *Let G be a Lie group. The linear mapping $ad : T_eG \to gl(T_eG)$ same as $\mathcal{G} \to gl(\mathcal{G})$ preserves the lie bracket, in other words $X, Y \in \mathcal{G}$:*

$$ad_{[X,Y]} = [ad_X, ad_Y],$$

where on the left we have the bracket in the lie algebra $\mathcal{G}$ and on the right the bracket in the lie algebra of the group $GL(T_eG)$, which was just proven to be $gl(T_eG)$ with the standard bracket.

Remark 29.6. We will not give a final proof yet. We will only prove that the statement of the preceding theorem is equivalent to another statement, namely that the bracket in general$\mathcal{G}$ does satisfy the axioms of a lie bracket, alternating and Jacobi.

Todistus. Let us believe in the Jacobi identity for the bracket in $\mathcal{G}$

$$ad_{[X,Y]} = [ad_X, ad_Y],$$

where on the left we have the bracket in the lie algebra $\mathcal{G}$ and on the right the bracket in the lie algebra of the group $GL(T_eG)$, which was just proven to be $gl(T_eG)$ with the standard bracket.

We will not give a final proof yet. We will only prove that the statement of the preceding theorem is equivalent to another statement, namely that the bracket in general $\mathcal{G}$ does satisfy We have to prove

$$\begin{aligned} & ad_{[X,Y]}(Z) = [ad_X, ad_Y](Z) \\ \text{eli} \quad & ad_{[X,Y]}(Z) = (ad_X \circ ad_Y)(Z) - (ad_Y \circ ad_X)(Z) \end{aligned}$$

$X, Y, Z \in \mathcal{G} = T_eG$.

We begin by writing the Jacobi identity in $\mathcal{G}$:

$$\begin{aligned} [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= 0 \\ [X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z] &= 0 \\ ad_X([Y, Z]) - ad_Y([X, Z]) - ad_{[X,Y]}(Z) &= 0 \\ ad_X(ad_Y(Z)) - ad_Y(ad_X(Z)) - ad_{[X,Y]}(Z) &= 0 \\ ad_X \circ ad_Y - ad_Y \circ ad_X &= ad_{[X,Y]} \end{aligned}$$

□

The result can be generalised for any lie group homomorphism, in particular for representations:

Theorem 29.7. *Let $f : G \to H$ be aLie group homomorphism. Its derivative $df_e : T_eG \to gl(T_eG)$ same as $\mathcal{G} \to \mathcal{H}$ is a* lie algebra homomorphism *which means a linear mapping preserving the bracket: in other words for all $X, Y \in \mathcal{G}$:*

$$df_e[X,Y] = [df_eX, df_eY],$$

where on the left there is the Lie bracket in $\mathcal{G}$ and on the right the bracket in $\mathcal{H}$.

Sketch of proof. Conjugation in a group commutes with a homomorphism, of course, so there is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \gamma_g \downarrow & \cdot & \downarrow \gamma_{f(g)} \\ G & \xrightarrow{f} & H \end{array}$$

i.e. $f(ghg^{-1}) = f(g)f(h)f(g)^{-1}$ for all $g, h \in H$. Calculating derivatives with the chain rule gives a commutative diagram for all $g \in G$

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{d_ef} & \mathcal{H} \\ Ad_g \downarrow & \cdot & \downarrow Ad_{f(g)} \\ \mathcal{G} & \xrightarrow{d_ef} & \mathcal{H} \end{array}$$

so there are mappings

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ Ad \downarrow & \cdot & \downarrow Ad \\ GL(\mathcal{G}) & & GL(\mathcal{H}) \end{array}$$

such that for all $X \in Ad(G) \subset GL(\mathcal{G})$ we have

$$df_e(Ad_g(X)) = Ad_{f(g)}d_ef(X).$$

The derivative of this with respect to g gives

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{d_ef} & \mathcal{H} \\ ad \downarrow & & \downarrow ad \\ gl(\mathcal{G}) & & gl(\mathcal{H}), \end{array}$$

where again in the respective image set $ad(\mathcal{G}) \subset gl(\mathcal{G})$ there is commutativity of the diagram, i.e. for every $Y \in \mathcal{G}$

$$\begin{aligned} d_ef(ad_X)(Y) &= ad_{d_ef(X)}d_ef(Y), \\ \text{eli} \quad d_ef([X,Y]) &= [d_ef(X), d_ef(Y)]. \end{aligned}$$

30. The exponential mapping — a preliminary definition

Definition 30.1. A *representation of a real lie algebra* is a lie algebra homomorphism to $GL_n(\mathbb{R})$. Similarly we define a complex and a general lie algebra representation.

30.1. **The idea of the exponential mapping.** We have found a natural way to attach a lie algebra $\mathcal{G}$ to every Lie group G. As a vector space $\mathcal{G} = T_eG$. By the previous theorem, this is a functorial correspondence in the sense that it also respects homomorphisms, in particular representations. So a representation $\rho : G \to GL_n(\mathbb{R})$ of a Lie group G gives rise to a lie algebra representation $d_e\rho : \mathcal{G} \to gl_n(\mathbb{R})$. Our next task is to find a way to search for a way to produce an inverse of this correspondence: given a representation of the lie algebra, try to find a representation of the original group. The exponential mapping is a tool for this purpose.

Remark 30.2. The exponential mapping will be a smooth mapping with the following properties:

(1) $0 \mapsto e \in G$
(2) $d_0 \exp = Id_{\mathcal{G}}$.

Some more condition. (The two properties above do not characterise any map, as is seen from the following example.

Example 30.3. Take $G = (\mathbb{R}^*, \cdot)$, the group of invertible real numbers with identity element 1. Its tangent space at 1 is $\mathcal{G} = T_1(\mathbb{R}^*) \sim \mathbb{R}$, where the lie bracket is trivial $[X, Y] = 0$, since G is commutative (**??**). In this example, the usual exponential function will have the properties (1) and (2): $x \mapsto e^x$ is smooth $\mathcal{G} \to G$ same as $\mathbb{R} \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and

(1) $0 \mapsto e^0 = 1 \in G$
(2) $d_0(x \mapsto e^x) = Id_{\mathbb{R}}$, since the Jacobi matrix of the derivative is the 1×1- matrix $[\frac{\partial e^x}{\partial x}]_{x=0}$ and $e^0 = 1$.

This example shows why properties (1) and (2) are insufficient to characterise the exponential mapping: Any function $f : \mathbb{R} \to \mathbb{R}^*$,which coincides with *exp* in any neighbourhood of 0 will satisfy (1) and (2) since it has the same derivtive $\left[\frac{\partial f}{\partial x}\right]_{x=0}$ and the same value $f(0) = 1$.

It should be no surprise that the definition is still incomplete; after all we have not used the group operation at all. Also, in this example it happens tha exp happens to be a group homomorphism from the additive group of the vector space to the group $\mathcal{G}$ but this is not true for a general exponential map of a Lie group. It is just a consequence of this example happening to be one dimensional.

Next we define the exponential mapping for all "classical" Lie groups/algebras.

Definition 30.4. The *classical exponential mapping* is the mapping

$$\begin{aligned} \exp : M_{n\times n} &\longrightarrow M_{n\times n} \\ A &\mapsto \sum_{p=0}^{\infty} \frac{1}{p!} A^p = I + A + \frac{1}{2}AA + \frac{1}{6}AAA + \cdots \end{aligned}$$

It is easy to verify the absolute convergence of this series in the usual Euclidean topology of $M_{n\times n} = \mathbb{R}^{n\times n}$. One can do it by using the well known matrix norm $\|A\|_M = \{\sup \|Ax\| \bigm| \|x\| \le 1\}$ or by estimating the Euclidean norm using the inequalities

$$\|A\| \le n^2 \max_{i,j} |A_{ij}|$$

and

$$[A^p]_{ij} \le (n \cdot \max_{i,j} |A_{ij}|)^p.$$

Example 30.5. Let $G = SO_2(\mathbb{R})$ be the greoup of rotations of the plane $\mathbb{R}^2$, consisting of the toation matrices

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta \in \mathbb{R}.$$

A local parametrization for the smooth manifold $SO_2(\mathbb{R})$ is $\varphi : \mathbb{R} \to M_{2\times 2} \sim \mathbb{R}^4 : \theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. In particular $0 \overset{\varphi}{\mapsto} \mathbf{1}$, so the tangent plane of $SO_2(\mathbb{R})$ at the neutral element $\mathbf{1}$ can be calculated from the derivative of the local parametrization at 0. It is the linear mapping

$$d_0\varphi : \mathbb{R} \to \mathbb{R}^4 \sim M_{2\times 2} : \lambda \mapsto \lambda \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix},$$

so the tangent space $so_2(\mathbb{R})$ is

$$T_0(SO_2(\mathbb{R})) = d_0\varphi(\mathbb{R}) = \left\{ \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \bigm| \lambda \in \mathbb{R} \right\} \sim \mathbb{R}.$$

Because $SO_2(\mathbb{R})$ is commutative, the lie bracket is zero. The classical exponential mapping $\exp : so_2(\mathbb{R}) \to M_{2\times 2}$ maps the matrix $A =$

$\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \in so_2(\mathbb{R})$ to

$$\begin{aligned}
& \mathbf{1} + \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}^2 + \tfrac{1}{3!}\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}^3 + \tfrac{1}{4!}\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}^4 + \dots \\
= & \mathbf{1} + \lambda\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \tfrac{1}{2}\lambda^2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 + \tfrac{1}{3!}\lambda^3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 + \tfrac{1}{4!}\lambda^4\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 + \dots \\
= & \mathbf{1} + \lambda\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \tfrac{1}{2}\lambda^2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \tfrac{1}{3!}\lambda^3\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \tfrac{1}{4!}\lambda^4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \dots \\
= & \mathbf{1} + \lambda\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \tfrac{1}{2}\lambda^2\mathbf{1} - \tfrac{1}{3!}\lambda^3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \tfrac{1}{4!}\lambda^4\mathbf{1} + \dots \\
= & (1 - \frac{\lambda^3}{3!} + \dots)\mathbf{1} + (\lambda - \frac{\lambda^2}{2!} + \dots)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
= & \cos\lambda\mathbf{1} + \sin\lambda\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix} \in SO_2(\mathbb{R}).
\end{aligned}$$

we observe that the classical exponential mapping at least in this one case really maps the lie algebra $so_2(\mathbb{R})$ of the group $SO_2(\mathbb{R})$ to the group $SO_2(\mathbb{R})$ itself. Obviously, it also is smooth and maps the zero element $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ to the neutral element, i.e. the unit matrix. Also, its derivative at zero is

$$d_0\mathrm{exp} = \left(\begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}\right) = Id_{so_2(\mathbb{R})},$$

so it satisfies the preliminary conditions for the exponential mapping.

Like in the previous example, the Lie group is again one-dimensional, and the exponential mapping does again map sums to products. We repeat the warning: this is not true for exponential maps in general.

Definition 30.6. (Move to a better place!) In addition to the conditions in **??** the exponential mapping should be *natural* in the following sense: (3) If $f : G \to H$ is a Lie group homomorphism, then the diagram

$$\begin{array}{ccc} G & \stackrel{f}{\to} & H \\ \exp\uparrow & \cdot & \uparrow\exp \\ \mathcal{G} & \stackrel{d_e f}{\to} & \mathcal{H} \end{array}$$

is commutative

31. Vector fields on manifolds

31.1. **Introduction.** Every homomorphism of a Lie group $\rho : G \to H$, in particular every representation, has a derivative at the neutral element, and this derivative $d_0\rho : \mathcal{G} \to \mathcal{G}$ is a homomorphism of lie algebras. In fact every lie algebra homomorphism is the derivative of some Lie group homomorphism:

If G is a "concrete" Lie group, same as aa sub-Lie group of $GL_n(\mathbb{R})$, then the classical exponential mapping is a mapping $\mathcal{G} \to G$, and if $T : \mathcal{G} \to \mathcal{G}$ is a homomorphism of lie algebras, then there exists a Lie group homomorphism $\rho : G \to H$ such, that $T = d_0\rho$. Since Euclidean space $\mathcal{G}$ is connected, its continuous image $\exp(\mathcal{G}) \subset G$ is connected as well. Therefore the image of the exponential mapping is contained in the connected component of the Lie group which contains the neutral element. It turns out that if G happens to be simply connected, then the classical exponential mapping will define a 1-1-correspondence between representations of the Lie group G and its lie algebra $\mathcal{G}$.

The following Lie groups are simply connected:

(1) $\{M \in GL_n(\mathbb{R}) \mid M$ is upper diagonal with all diagonal elements equal to 1.$\}$.
(2) closed Lie subgroups of the above
(3) the neutral element's connected component in a ny Lie group.
(4) the *universal covering group* of any Lie group. By this we mean the following: Tällä tarkoitetaan seuraavaa: Every manifold M has a unique *universal covering space* whgich is a manifold P together with a smooth surjection, the *covering map* $\pi : P \to M$ such, that every $x \in M$ has a neighbourhood U, whose preimage $\pi_1(U)$ consists of a collection of distinct sets each diffeomorphic to U:

$$U = \bigcup_{i \in I} U_i \qquad \pi \big|_{U_i} : U_i \to U \text{ is a diffeomorphism.}$$

The universal covering space of a Lie group can be given the structure of a Lie group such that π becomes a homomorphism.[11].

It is obvious that a Lie group and its universal covering group have the same lie algebra.

[11] OK?

For example $(\mathbb{R}, +)$ is a simply connected n Lie group and $\pi : \mathbb{R} \to U = SL_2(\mathbb{R}) : \lambda \mapsto \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix}$ is a universal covering map and a homomorphism. So both of these groups have the same lie algebra, namely $\mathbb{R}$ with the zero bracket.

The following Lie groups are not simply connected:

(1) finite groups — except the trivial group
(2) the rotation group $U = SL_2(\mathbb{R})$,
(3) $SL_n(\mathbb{R})$,
(4) $SL_n(\mathbb{C})$,
(5) $O_n(\mathbb{R})$ is not even connected (det is 1 in one component, -1 in the other.)
(6) For $n \geq 3$ the connected group $SO_n(\mathbb{R})$ is not simply connected. Its universal covering group is called the *spin group* and the covering map is almost injective in the sense that every point has precisely 2 preimages. We say that the covering group has 2 *sheets*.

We will now prove that every Lie group has an exponential mapping $\mathcal{G} \to G$ which maps representations to representations. The construction of such an exponential mapping is based on the concept of a vector field on manifolds and on some theory of differential equations.

31.2. **The classical definition of a vector field.** We consider smooth manifolds which need not necessarily be Lie groups.

Definition 31.1. Let $M \subset \mathbb{R}^n$ be a classical d-udimensional smooth manifold and $x_0 = \psi(x) = \psi(x_1, \ldots, x_d) \in M$, where $\psi : B \to X$ is a local parametrisation; $B \subset \mathbb{R}^d$. The tangent space of the manifold M at a point $x_0 = \psi(x)$ is the image space $T_{x_0}(M) = d_x\psi(\mathbb{R}^d)$.

The *tangent bundle* of the manifold M is the disjoint union of all its tangent spaces, formally:

$$TM = \{(x, v) \mid x \in M,\, v \in T_x M\}.$$

For the tangent bundle we define the *projection mapping*

$$\pi : TM \to M : (x, v) \mapsto x,$$

so for all $x \in M$ we have $\pi^{-1}(x) = \{x\} \times T_x M \sim T_x M$. These preimages, same as the individual tangent spaces, are called the *fibres* of the tangent bundle TM .

The tangent bundle is also equipped with the structure of a smooth manifold. This is done by taking the *local trivialisations*

$$\begin{aligned}\varphi_T : \pi^{-1}(U) &\to \varphi(U) \times \mathbb{R}^d \\ (x, v) &\mapsto (\varphi(x), d_x\varphi v),\end{aligned}$$

as charts where the $\varphi : U \to B \subset \mathbb{R}^d$ are the charts of the original manifold M.

PICTURE TO BE DRAWN LATER

Kuva 999: tangenttikimpun lokaali trivialisointi

Example 31.2. The tangent bundle of the circle $M = S_1$ is the cylinder $S_1 \times \mathbb{R}$ with the structure of the product manifold and the projection map $\pi : S_1 \times \mathbb{R} \to S_1 : (x, u) \mapsto x$. A tangent bundle isomorphic to $M \times \mathbb{R}^d$ is called a *trivial tangent bundle.* so the tangent bundle of the circle is trivial but we will soon notice that the tangent bundle of the two dimensional sphere is not. On the other hand, the tangent bundle of every Lie group is trivial.

Definition 31.3. A *vector field* on a manifold M is a smooth mapping $X : M \to TM$, such that $\pi \circ X = Id_M$ or equivalently $\forall x \in M$: $X(x) \in \{x\} \times T_xM$.

Proposition 31.4. ***"You cannot comb the hair on a sphere — theorem"***[12] *Every vector field on the 2-dimensional sphere $S_2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ has a zero point.*

Todistus. The proof would lead us too much astray. Consult the literature. $\square$

Corollary 31.5. *The tangent bundle TS_2 of the sphere S_2 has no* global frame *by which we mean such a pair of vector fields X_1, X_2, that $X_1(x), X_2(x)$ would be a basis for the tangent space T_xS_2 at each $x \in S_2$.*

[12]There are nice videos on Youtube, where this is tried!

Corollary 31.6. *The tangent bundle TS_2 of the sphere S_2 has no* global trivialisation *by which we mean a diffeomorphism $f : TS_2 \to S_2 \times \mathbb{R}^2$, with $\pi \circ f = f \circ \pi$.*

Todistus. It is left as an exercise to prove that the tangent bundle of a manifold has a global trivialisation if and only if it has a global frame. □

For example the tangent bundle of the circle S_1 i.e. the cylinder is gobally trivial but the manifold S_1 has no atlas consisting of one chart only.

(CORRECT THE FINNISH TEXT!)

31.3. **Left invariant vector fields on a Lie group.**

Definition 31.7. A vector field X on a Lie group ryhmän G is *left invariant*, if for all $g, h \in G$

$$d_x(m_g)X(h) = X(gh),$$

where $m : G \to G : h \mapsto gh$ is *multiplication from the left* the element g and $d_x(m_g) : T_hG \to T_{gh}G$ is its derivative.

Proposition 31.8. *Let $v \in T_eG = \mathcal{G}$. Now*

$$X_v; g \mapsto (g, d_x(m_g)v)$$

is a left invariant vector field, where $m : G \to G : h \mapsto gh$ is multiplication from the left by the element g and $d_x(m_g) : T_hG \to T_{gh}G$ is its derivative.

Todistus. The mapping X_v is a vector field, since it is a composition of smooth mappings, hence a smooth mapping $X : M \to TM$ and evidently $X(x) \in \{x\} \times T_xM$ for all $x \in M$. Left invariance follows from the chain rule:

$$\begin{array}{cccccc} m_{gh}: & G & \xrightarrow{m_h} & G & \xrightarrow{m_g} & G \\ & | & & | & & | \\ d_e m_{gh}: & T_eG & \xrightarrow{d_e m_h} & T_hG & \xrightarrow{d_e m_g} & T_{gh}G \end{array}.$$

□

If $v \neq 0$, the vector field X_v has no zeros, in particular it — of course — cannot be the zero vector field, so $v \mapsto X_v$ is a linear injection, in fact a bijection $T_eG \to$ {left invariant vector fields of G}. By this bijection we can carry over the lie algebra structure of T_eG to {left invariant vector fields of G :}.

By this construction we have been able to identify a single tangent vector $X \in T_eG$ with a vector field: The correspondence is

$$X \mapsto (g \mapsto d_e m_g X).$$

Having carried over the vector space structure and the lie bracket from $X \in T_eG$ we have made the set of left invariant vector fields into a lie algebra. It is worth while to notice that the vector space operations coincide with the "natural pointwise" operations on general vector fields: if $X(x) = (x, v)$ and $Y(x) = (x, w)$, then $(\alpha X + \beta Y)(x) = (x, \alpha v + \beta w)$. This is so, because the derivative of left multiplication m_g is a linear isomorphism $T_eG \to T_gG$. By this isomorphism, we can – of course — also transfer the basis of T_eG to T_eG to T_mG or to {left invariant vector fields of G :}, so every Lie group has a global frame.

Remark 31.9. We transferred the lie bracket by the linear isomorphism $T_eG \to T_mG$, but in fact it is possible, and common, to define a lie bracket of vector fields on any manifold. It turns out that this is not in conflict with our definition: the lie algebra {left invariant vector fields of G :} is a sub lie algebra of {all vector fields on G} with the more general lie bracket.

Let us also notice that the mapping

$$g \mapsto (X \mapsto d_e m_g X)$$

is a representation of G in the space of its own left invariant vector fields which we now have completely identified with the lie algebra $\mathcal{G}$.

This can be generalised: If a Lie group G acts as diffeomorphisms on any manifold M, then G acts linearly in the space VK_M of all vector fields over M which is a representation of G. The representation is defined by the following construction: Let $X : M \to TM : x \mapsto (x, X(x))$ be a vector field and take $g \in G$. The vector field gX is the mapping

$$y \mapsto (y, d_x m_g X(x)),$$

where we have denoted the action of $g \in G$ in M by g and $x = g^{-1}y$.

31.4. **The general definition of the exponential mapping.** Since not all Lie groups are matrix groups, and some are matrix groups only by cumbersome identifications, there is need to re-define the exponential mapping in an abstract setting. exp should be a mapping from the lie algebra $\mathcal{G}$ of a general Lie group G to the Lie group G itself and have the following properties:

(1) $0 \mapsto e$
(2) $d_0 \exp = Id_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} = T_0(\mathcal{G})$
(3) for all $X \in \mathcal{G}$ the restriction $\exp\big|_{\mathbb{R}X}$ is a group homomorphism $\mathbb{R}X \sim \mathbb{R} \to G$, in other words for all $\lambda, \mu \in \mathbb{R}$ and $X \in \mathcal{G}$

$$\exp((\lambda + \mu)X) = \exp(\lambda X) \cdot \exp(\mu X).$$

Condition (3) is often expressed by saying that the restriction $\exp\big|_{\mathbb{R}X}$ is a *one parameter subgroup* of$G$.

Proposition 31.10. *there exists a unique mapping satisfying the conditions (1)-(3).*

Todistus. we prove the proposition by giving a construction in two steps:

(Step 1) Interprete the elements X of the lie algebra $\mathcal{G}$ as left invariant vector spaces on G. This was done above.

(Step 2) From the theory of differential equations, adopt the fact that any everywhere non-zero vector field has a *flow* consisting of its *integral curves*. Roughly this means that it is possible to start at any point p on the manifold and then "follow the vectors" interpreted as "velocities" for some time. If it is possible to follow a given vector field everywhere for a a given time, say time 1, this will create a smooth map from the manifold to itself, and the procedure can be repeated generating a flow that "goes on forever". Let us do this in detail:

Define an *integral curve* of a given vector field X in a neighbourhood U of a given point $p \in M$ to be a smooth curve $\gamma_X :]-\epsilon, \epsilon[\to M$, mapping $0 \mapsto p$ and having derivatives

$$d_t\gamma_X : \mathbb{R} \to T_p(M) : s \mapsto (\gamma(t), sX(\gamma(t)),$$

in particular

$$"1 \stackrel{d_t\gamma}{\mapsto} X \in T_{\gamma(t)}".$$

Such an integral curve always exists and is unique on a small enough interval $]-\epsilon,\epsilon[$, but for a general vector field the interval cannot be chosen to be large let alone all of $\mathbb{R}$.[13]

Let us sketch the next step: In a Lie group integral curves of left invariant vector fields can be extended to all of $\mathbb{R}$ and becomes a group homomorphism $\gamma : (\mathbb{R},+) \to G$. This is done by "translating" in G. Before doing this, we use the result to finish the construction of the exponential mapping: We define it to be

$$\exp : \mathcal{G} \to G : X \mapsto \gamma_X(1).$$

We have to notice that this mapping obviously satisfies the conditions (1) ... (3) and is, by uniqueness of the integral curves, the only mapping doing so.

What is left of the construction is the promised extension of the integral curve of a left invariant vector field. Let the integral curve be defined on $]-\epsilon,\epsilon[$- By uniqueness of the integral curves the "homomorphism formula" $\gamma_X(s+t) = \gamma_X(s)\cdot\gamma_X(t)$ holds at least for all $|t| \le \frac{\epsilon}{2}$ and $|s| \le \frac{\epsilon}{2}$, since for fixed s both sides are, as functions of t, integral curves of the same vector field X in the neighbourhood of the same point $\gamma_X(s)$. Let us verify this in detail:

Introduce some short hand notation: $\gamma = \gamma_X$, $\alpha(t) = \gamma_X(s+t)$ and $\beta(t) = \gamma_X(s)\cdot\gamma_X(t)$. Obviously both have the same initial value $\alpha(0) = \beta(0) = \gamma_X(s)$. furthermore both are integral curves of the vector field X, since by the definition **??** of an integral curve

$$d_t\gamma : \mathbb{R} \to T_{\gamma(t)}(M) : r \mapsto rX(\gamma(t))$$

which, expressed as a formula for the basis vector 1 of $\mathbb{R}$ is equivalent to:

$$d_t\gamma : \mathbb{R} \to T_{\gamma(t)}(M) : 1 \mapsto X(\gamma(t))$$

and gives for α:

$$\begin{aligned} d_t\alpha : &\mathbb{R} \overset{Id=d_t(r\mapsto s+r)}{\longrightarrow} \mathbb{R} \overset{d_{s+t}(\gamma)}{\longrightarrow} T_{\alpha(t)}G \\ &1 \mapsto d_{t+s}(\gamma) = X(\gamma(s+t)) = X(\alpha(t)). \end{aligned}$$

and for $\beta = m_{\gamma(s)} \circ \gamma$:

$$\begin{aligned} d_t\beta : &\mathbb{R} \overset{d_t\gamma}{\longrightarrow} T_{\gamma(t)}(G) \overset{d_{\gamma(t)}(m_{\gamma(s)})}{\longrightarrow} T_{\beta(t)}(G) \\ &1 \mapsto X(\gamma(t)) \mapsto d_{\gamma(t)}(m_{\gamma(s)})(X(\gamma(t))) \overset{vas.inv!}{=} X(\gamma(s)\cdot\gamma(t)) = X(\beta(t)). \end{aligned}$$

[13] Counterexample: a constant vector field in an open circle $\subset \mathbb{R}^2$.

Define for all $r \in]-2\epsilon, 2\epsilon[$

$$\gamma_X(r) = \gamma_X(\tfrac{r}{2} + \tfrac{r}{2}) = \gamma_X(\tfrac{r}{2}) \cdot \gamma_X(\tfrac{r}{2}).$$

We have already proved that this extension is not anywhere in conflict with the original definition. Also, it should be clear (exercise?) that the extended curve γ_X is an integral curve of X with the same initial point. The continuation process can be repeated to find an integral curve γ_X defined on $]-2^n\epsilon, 2^n\epsilon[$ and all in all in $\mathbb{R}$.

$\square$