GROUPS AND THEIR REPRESENTATIONS - THIRD PILE

KAREN E. SMITH

17. Characters

We now understand, in a sense, *all* complex representations of any finite abelian group, as well as the simplest non-abelian group: they are all direct sums of irreducible representations, and we have explicitly identified the finite list of these. On the other hand, this doesn't help us to find a decomposition of a given representation, or even to recognize when two given representations (say, of an abelian group or S_3) are isomorphic.

Fortunately, there is a very effective technique for decomposing any given finite dimensional representation into its irreducible components. For example, we can tell at a glance—or at least easily program a computer to—whether two very large dimensional representations of a finite group are isomorphic or not. The secret is *character theory.*

In our analysis of the representations of S_3, the key was to study the eigenvalues of the actions of individual elements of S_3. This is the starting point of character theory. Finding individual eigenvalues, however, is difficult. Luckily, it is sufficient to consider their sum, the *trace,* which is much easier to compute.

Definition 17.1. Let $\phi : V \to V$ be a linear transformation of a finite dimensional vector space. The *trace* of ϕ is the sum of the diagonal entries $a_{11} + a_{22} + \cdots + a_{nn}$ of a matrix representing ϕ in any fixed basis for V. This is independent of the choice of basis, and can also be defined as sum of the roots of its characteristic polynomial, counting multiplicity.

Definition 17.2. Fix a finite dimensional representation V of a group G, say, over $\mathbb{C}$. The *character* of the representation is the complex valued function

$$\chi_V : G \to \mathbb{C}$$

$$g \mapsto \text{ trace of } g \text{ acting on } V.$$

Of course if V is defined over $\mathbb{R}$ or some other field, then the character takes values in $\mathbb{R}$ or whatever ground field.

The character of a representation is easy to compute. If G acts on an n-dimensional space V, we write each element g as an $n \times n$ matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix for g to get $\chi_V(g)$. For example, the trace of the identity map of an n-dimensional vector space is the trace of the $n \times n$ identity matrix, or n. Thus, for any group, the character of the trivial representation of dimension n is the constant function sending each element of G to n. More generally, $\chi_V(e) = \dim V$ for any finite dimensional representation V of any group.

We often write the values of the character χ_V as a "vector"whose coordinates are indexed by the elements of G.

$$\chi_V := (tr(g_1), tr(g_2), \ldots, tr(g_r)),$$

where $r = |G|$. For example, the character of the n-dimensional trivial representation of G can be written $(n, n, \ldots, n)$ (where the length of the vector here is the order of G). Similarly, the the character of the tautological representation of D_4 is

$$(2, 0, -2, 0, 0, 0, 0, 0).$$

This is simply the list of the traces of the transformations $\{e, r_1, r_2, r_3, H, A, V, D\}$ of symmetries of the square acting on $\mathbb{R}^2$ (Cf. the list of matrices representing these symmetries in Example **??**.)

17.1. **The fixed point theorem.** The character of any permutation representation (see **??**) is easy to compute. Suppose a group G acts on a finite set X, and let V_X be the associated permutation representation. So V_X is a vector space with basis indexed by the elements of X and G acts by permuting the basis vectors according to its action on the indices. In this case, the character is

$$\chi_{V_X}(g) = \text{ the number of elements of } X \text{ fixed by } g.$$

Indeed, if we imagine the basis $\{e_x\}_{x \in X}$ written as column vectors indexed by $x \in X$, the action of g is simply permuting them in some way, and the corresponding matrix is the corresponding (inverse) permutation of the columns of the identity matrix. In particular, each diagonal entry is either 0 or 1. It is 1 if and only if $e_{gx} = e_x$— that is, if and only if g fixes x—and zero otherwise. Thus the trace of the action of g on V_X is the number of elements of x fixed by g.

17.2. **The character of the regular representation.** Let R be the regular representation of a finite group G—which is to say, the permutation representation of G induced by the action of G on itself by left multiplication. The character χ_R can be computed using the fixed point theorem. Indeed, since left multiplication by a non-identity element fixes *no element* of G, the trace of every $g \neq e$ is zero. On the other hand, of course e acts by the identity on this vector space of dimension G. Thus

$$\chi_R = (|G|, 0, 0, \dots, 0).$$

17.3. **Characters of S_3.** Let us compute the characters of the three irreducible representations of S_3 identified in the last lecture.

The trivial representation of S_3 is one dimensional and takes the value 1 for each of the six elements of S_3. Its character is therefore

$$\chi_E : G \to \mathbb{C}; \;\; g \mapsto 1; \quad \text{or } (1, 1, 1, 1, 1, 1).$$

The alternating representation is also one dimensional, but takes the value 1 on the even permutations ($e, (123)$ and 132), and -1 on the odd permutations (the transpositions $(12), (23)$ and (13)). Thus

$$\chi_A : (1, -1, -1, -1, 1, 1).$$

The character of the standard representation could be found by writing out the matrix for the action of each of the six elements of S_3, with respect to some basis, perhaps the one already identified in **??**. However, we prefer to make use of the following helpful fact:

Proposition 17.3. *Let V and W be finite dimensional representations of a group G. Then*

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

as functions on G.

Now, because the permutation representation of S_3 decomposes as a sum of the trivial and the standard representations, we can compute the character of the standard representation using Proposition 18.3. This has the advantage of being easy: the matrices for the permutation action are simply the permutation matrices, so we see immediately that the identity element has trace 3, each of the transpositions has trace 1, and each of the 3-cycles has trace 0. Subtracting the character of the trivial representation, we conclude that the standard representation has character

$$\chi_W = (2, 0, 0, 0, -1, -1).$$

This information can be displayed in a character table

	e	(12)	(13)	(23)	(123)	(132)
trivial	1	1	1	1	1	1
alternating	1	−1	−1	−1	1	1
standard	2	0	0	0	−1	−1

whose rows are the characters of the prescribed representations. The character of any complex representation V of S_3 can be obtained from these three, by decomposing V into irreducibles $V = E^a \oplus A^b \oplus W^c$, and then using Proposition 18.3 to conclude that

$$\chi_V = a\chi_E + b\chi_A + c\chi_W.$$

18. Statements of the main theorems of character theory

Obviously, isomorphic representations have the same character, but remarkably, the character *completely determines the representation* up to isomorphism, at least over $\mathbb{C}$:

Theorem 18.1. *Two finite dimensional complex representations of a finite group G are isomorphic if and only if they have the same character.*

Indeed, much more is true. Before stating general theorems, we first "discover"them by taking a close look at the characters of S_3.

First: note that the rows of the character table for S_3 are all *orthogonal* with respect to the standard (Hermitian) inner product. Also, each has length $\sqrt{|S_3|}$. In other words, the characters of the irreducible representations are *orthonormal* with respect to the inner product on the space of $\mathbb{C}$-valued functions of G

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}. \tag{1}$$

Amazingly, this is a general feature of the characters of any finite group!

Theorem 18.2. *For any finite group G, the characters for the irreducible complex representations are orthonormal under the inner product (1) on the space of all $\mathbb{C}$-valued functions of G.*

Put differently, if χ_V and χ_W are characters of irreducible representations, then (χ_V, χ_W) is either 1 or 0, depending on whether $V \cong W$ or not.

Theorem 19.2 already implies that there are finitely many isomorphism classes of irreducible representations. Orthonormal vectors are independent, so clearly there are at most $|G|$ irreducible representations for G.

In fact, there is a better bound on the number of irreducible representations a finite group. Note another interesting feature of the character table of S_3: each character takes the same value on all transpositions, and also takes the same value on each of the three-cycle. This, too, is a general feature of the character of any finite group.

Proposition 18.3. *The character of a representation is constant on conjugacy classes of G. That is, for any finite dimensional representation V of a group G,*

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

for all $g, h \in G$.

Todistus. This is more or less obvious, since the action of h can be considered a change of basis for V. Since the trace does not depend on the basis—that is, conjugate (similar) matrices have the same trace—the character must be constant on conjugacy classes of G. □

For example, S_3 has three conjugacy classes—the identity, the transpositions, and the three-cycles—and we have seen that the character of any representation is constant on each of these.

To avoid redundancy, we usually think of the character of a representation as a function the set of *conjugacy classes* of G. For example, the character table of S_3 could be compactified to

	e	(12)	(123)
trivial	1	1	1
alternating	1	-1	1
standard	2	0	-1

Here the elements e, (12) and (123) are representatives for their respective conjugacy classes. Some authors include another row above the first row to indicate the number of elements in each conjugacy class;

this is helpful in computing the inner product, since that sum is still taken over all the elements of G.

We postpone the proof of Theorem 19.2 in order to summarize some of its amazing consequences.

Corollary 18.4. *For finite dimensional complex representations of a finite group G we have:*

(1) *There are at most t irreducible representations of G, where t is the number of conjugacy classes of G.*
(2) *The multiplicity of an irreducible representation W in a representation V is (χ_W, χ_V).*
(3) *Each representation is determined (up to isomorphism) by its character.*
(4) *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

Remark 18.5. In fact, the characters of the irreducible representations actually span the t-dimensional vector space of all functions on G constant on conjugacy classes, so there are *exactly* t distinct isomorphism classes of irreducible representations of G. We will outline the proof of this in the exercises.

Proof of Corollary. (1). The characters of the different representations of G live in the t-dimensional space of functions on G which are constant on conjugacy classes (for example, by listing the values at each conjugacy class, we get a "vector"of t complex numbers). Since the characters of the irreducible representations are orthonormal, they are independent, and hence there can be at most t of them. This means there are at most t isomorphism classes of irreducible representations.

(2). Now, suppose $V \cong W_1^{a_1} \oplus \cdots \oplus W_t^{a_t}$ is a decomposition of V into irreducibles. Using Proposition 18.3, $\chi_V = a_1\chi_{W_1} + \cdots + a_t\chi_{W_t}$. So using the bilinearity of the inner product and the orthonormality of the χ_{W_i}, we conclude that

$$(\chi_V, \chi_{W_i}) = a_i.$$

(3). Suppose V and U are two representations with the same character. Decomposing each,

$$V \cong W_1^{a_1} \oplus \cdots \oplus W_t^{a_t}, \quad U \cong W_1^{b_1} \oplus \cdots \oplus W_t^{b_t}$$

so that if $\chi_V = \chi_U$, then

$$a_1\chi_{W_1} + \cdots + a_t\chi_{W_t} = b_1\chi_{W_1} + \cdots + b_t\chi_{W_t}.$$

But now because the χ_{W_i} are independent, we see that $a_i = b_i$ for each i, and so $V \cong U$.

(4). Decompose V into irreducibles $W_1^{a_1} \oplus \cdots \oplus W_t^{a_t}$, so that $\chi_V = a_1\chi_{W_1} + \cdots + a_t\chi_{W_t}$. Then using the orthonormality of the χ_{W_i}, we see that

$$(\chi_V, \chi_V) = a_1^2 + a_2^2 + \cdots + a_t^2.$$

Since the a_i are all non-negative integers, we see that $(\chi_V, \chi_V) = 1$ if and only if exactly one of the a_i is 1, and the others are zero, that is, if and only V is irreducible. □

19. Using Character theory to decompose representations.

We now have a very powerful tool[1] for analyzing complex representations of a finite group. For example, let us decompose the regular representation R of S_3 into its irreducible components. (Recall that the regular representation of S_3 is the six-dimensional representation with a basis indexed by the elements of S_3, where S_3 acts by left multiplication on these indices.) We have proved that the only irreducible representations of S_3 are the trivial E, the alternating A and the standard W. Thus we have a decomposition

$$R \cong E^a \oplus A^b \oplus W^c$$

for some non-negative integers a, b and c. This produces the following relation on the characters:

$$\chi_R = a\chi_E + b\chi_A + c\chi_W.$$

The character of the regular representation is $(6, 0, 0)$, and easy computation carried out in 18.2. Thus we have a system of linear equations in three unkowns,

$$(6, 0, 0) = a(1, 1, 1) + b(1, -1, 1) + c(2, 0, -1)$$

which is easy to solve: $(a, b, c) = (1, 1, 2)$. So the regular representation decomposes as

$$R \cong E \oplus A \oplus W^2.$$

[1]though we haven't proven it yet

19.1. **The decomposition of the regular representation.** A similarly beautiful picture emerges for the regular representation of any finite group:

Corollary 19.1. *The regular representation R of any finite group G decomposes (over $\mathbb{C}$) as*

$$R \cong W_1^{\dim W_1} \oplus W_2^{\dim W_2} \oplus \cdots \oplus W_t^{\dim W_t}$$

with every irreducible representation W_i appearing exactly $\dim W_i$ *times.*

In particular,

Corollary 19.2. *For any finite group G*

$$|G| = \sum_{W_i} (\dim W_i)^2, \tag{2}$$

where the sum is taken over the (isomorphism classes of) irreducible complex representations of G.

Todistus. We know that

$$R \cong W_1^{a_1} \oplus W_2^{a_2} \oplus \cdots \oplus W_t^{a_t}$$

where the W_i range through all the irreducible representations of G, and the a_i are some non-negative integers. We have already computed that $\chi_R = (|G|, 0, 0, \dots, 0)$ (see Example 18.2), so Corollary 19.4 (2), we have

$$a_i = (\chi_{W_i}, \chi_R) = \frac{1}{|G|} \sum_{g \in G} \chi_{W_i}(g) \overline{\chi_R(g)} = \frac{1}{|G|} \chi_{W_i}(e) |G| = \dim W_i.$$

□

Formula (2) can be very helpful in unraveling the mysteries of the representations of a particular group. For example,

Exercise 19.3. Describe all the irreducible complex representations of D_4.

Solution: The group D_4 has exactly five conjuagy classes: $\{I\}, \{r_2\}, \{r_1, r_3\}, \{H, V\}$ and $\{A, D\}$. Therefore D_4 has at most five irreducible representations. We have already found three in Example **??**: the trivial, the tautological, and the one we called L, where r_1 acts by -1 and A acts by 1. These have dimensions 1, 2, and 1, respectively. (To verify that the tautological representation is irreducible over $\mathbb{C}$—we checked this only over $\mathbb{R}$— we can observe that its character is $(2, -2, 0, 0, 0, 0, 0, 0)$, which indeed has length 1 in the inner product.) By formula (2), we see that the as-yet-unidentified representations must have dimensions whose squares sum to 2. This means that there must be precisely two more irreducible representations of D_4, both of dimension one. To find these, we must consider group homomorphisms

$$D_4 \to GL(\mathbb{C}) = \mathbb{C}^*.$$

The trivial homomorphism gives the trivial representation, and the homomorphism sending generators r_1 and A to -1 and 1, respectively, defines L. We also get well-defined homomorphisms ϕ defined by $\phi(A) = -1$ and $\phi(r_1) = 1$, and ψ defined by $\psi(A) = \psi(r_1) = -1$. These give two more distinct representations of D_4. (This last one can be viewed as the pull-back of the alternating representation of S_4 to the subgroup D_4.)

Exercise 19.4. For any finite dimensional representations V and W of a finite group G, show

(1) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
(2) $\chi_{V^*} = \frac{1}{\chi_V}$. In particular, over $\mathbb{C}$, $\chi_{V^*} = \overline{\chi_V}$, the complex conjugate.
(3) $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \overline{\chi_V}\chi_W$.

20. The Proof of orthonormality

20.1. **Another nice property of the character.** Let V be any finite dimensional (real or complex) representation of a finite group G. The trivial part of V—that is, the sub-representation V^G where G acts trivially—can be split off from V using the projection

$$\begin{aligned} \pi : \ & V \to V \\ & v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v. \end{aligned}$$

The linear map π is easily seen to be a homomorphism of G-representations with image V^G. Because π is the identity on V^G and zero on its complement, the trace of π is simply the dimension of V^G. We conclude that

$$\dim V^G = \text{trace } \pi = \frac{1}{|G|} \sum_{g \in G} \text{trace of } g \text{ acting on } V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

In particular, if V is irreducible but not trivial, we have $V^G = 0$, so we conclude

Proposition 20.1. *For finite dimensional complex representations of a finite group G*

(1) *The sum of the values of the character of a non-trivial irreducible representation is zero.*
(2) *For any representation V, the multiplicity of the trivial representation in a decomposition into irreducibles is*

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

Proof of Theorem 19.2. Let W and V be irreducible complex representations of a finite group G. We want to show that

$$(\chi_W, \chi_V) = \quad 1 \quad \text{or} \quad 0,$$

depending on whether $V \cong W$ or not. Writing out the meaning of this, we want to show that

$$\frac{1}{|G|}\sum_{g\in G}\chi_W(g)\overline{\chi_V(g)} = \quad 1 \quad \text{or} \quad 0,$$

depending on whether $V \cong W$ or not. Taking a clue from Exercise 20.4, we consider

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)} = \chi_{V^*} \otimes \chi_W,$$

and observe that

$$(\chi_W, \chi_V) := \frac{1}{|G|}\sum_{g\in G}\chi_W(g)\overline{\chi_V(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g),$$

which, according to Proposition 21.1 should be equal to the multiplicity of the trivial representation in $\mathrm{Hom}_{\mathbb{C}}(V, W)$. How can we compute this multiplicity?

20.2. **The representation $\mathbf{Hom}_{\mathbb{C}}(V, W)$.** Let V and W be representations of G. The vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ has a natural structure of a representation of G under the action:

$$g \cdot \phi : V \to W$$
$$v \mapsto g \cdot \phi(g^{-1} \cdot v).$$

The G-linear homomorphisms are precisely those linear maps in $\mathrm{Hom}_{\mathbb{C}}(V, W)$ on which g acts trivially. (Prove it!) In particular, the *trivial part* of

$\mathrm{Hom}_{\mathbb{C}}(V, W)$—that is, the sub-representation of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ on which G acts trivially–is precisely the space $\mathrm{Hom}_G(V, W)$ of G-representation homormorphisms.

Now, Schur's lemma tells us that for (*complex*) irreducible representations W and V,

$$\dim \mathrm{Hom}_G(V, W) = 1 \text{ if } V \cong W; \quad 0 \text{ if } V \ncong W,$$

so that multiplicity of the trivial representation in $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is equal to one or zero, depending on whether $V \cong W$ or not. The proof of Theorem 19.2 is complete. □

Remark 20.2. It is not much harder to show that the characters of the irreducible representations span the space of functions constant on conjugacy classes, which is to say, they form an orthonormal basis. However, since the main conclusions follow already from the orthonormality, we relegate this stronger statement to the exercises.

Exercise 20.3. Fix a finite group G. Consider the vector space $\mathcal{F}_G$ of all $\mathbb{C}$-valued functions on G, and the subspace $\mathcal{C}$ of those that are constant on conjugacy classes.

(1) Show that $\alpha \in \mathcal{F}_G$ is constant of conjugacy classes if and only if the map

$$\phi_{\alpha,V} : V \to V; \;\; v \mapsto \sum_{g \in G} \alpha(g) g \cdot v$$

is G-linear for all complex representations V.

(2) Show that the trace of $\phi_{\alpha,V}$ is (α, χ_{V^*}) for all $\alpha \in \mathcal{F}$.

(3) Show if $(\alpha, \chi_{V^*}) = 0$ for some irreducible representation V and $\alpha \in \mathcal{C}$, then $\phi_{\alpha,V}$ is the zero map.

(4) Show that if $\alpha \in \mathcal{C}$ is non-zero, then $\phi_{\alpha,R}$ is not zero, where R is the regular representation.

(5) Conclude that the characters of irreducible representations span $\mathcal{C}$.

21. Representations of S_n

21.1. **Conjugacy in S_n.** Recall that every permutation in S_n can be written, uniquely up to order, as a composition of disjoint cycles $\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_t$, where here we even list the 1-cycles (though we ealier agreed that sometimes we drop them from the notation). Say that σ_i is a k_i-cycle, and that we have listed the cycles so that $k_1 \geq k_2 \geq \ldots k_t$.

Note that the cycles (including the 1-cycles) give a perfect partition of the set of n elements into t disjoint sets whose cardinalities are k_i.

Definition 21.1. The *cycle type* of σ is the partition

$$[k_1 k_2 \dots k_t] := \{k_1 \geq k_2 \geq \cdots \geq k_t \geq 1 \mid \sum k_i = n\}$$

of n, where the k_1 are a (weakly) decreasing list of positive natural numbers summing to n.

For example the cycle type of the permutation (1457)(368) in S_8 is $[4, 3, 1]$ because this permutation decomposes as the composition of the cycles $(1457), (368)$ and (2), of lengths $4, 3$, and 1, respectively. The permutation (1234)(567) has the same cycle type.

Exercise 21.2. Two permutations in S_n are conjugate if and only if they have the same cycle type.

The exercise is quite easy if one notes that conjugation can be thought of as a "change of labeling"for the transformations of the set of n-objects. For example, conjugating by the transposition (12) will interchange the roles of 1 and 2 in the permutation of $\{1, 2, \dots, n\}$. For example, the permutations (123)(456) and (623)(451) are conjugate to each other via conjugation by (16):

$$(16)(123)(456)(16) = (623)(451).$$

This makes it easy to list the conjugacy classes of S_n: there is exactly one for each distinct way of partitioning a set of n objects—that is, they are indexed by the *partitions of* n.

For example, there are exactly three conjugacy classes S_3, corresponding to the partitions [3], $[2, 1]$ and $[1, 1, 1]$. These are the classes of 3-cycles $\{(123), (132)\}$, of transpositions $\{(12), (13), (23)\}$, and the identity, respectively.

Likewise, there are five conjugacy classes in S_4, corresponding to the partitions $[4], [3, 1], [2, 2], [2, 1, 1]$, and $[1, 1, 1, 1]$. These correspond to the 4-cycles, the 3-cycles, the pairs of transpositions, the transpositions, and the identity elements, respectively.