GROUPS AND THEIR REPRESENTATIONS - SECOND PILE

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6. Group Actions

Let $(G, \star)$ be a group, and X any set (finite or infinite).

Definition 6.1. An *action* of G on X is a map

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

which satisfies

(1) $g_1 \cdot (g_2 \cdot x) = (g_1 \star g_2) \cdot x$ for all $g_1, g_2 \in G$ and all $x \in X$.
(2) $e_g \cdot x = x$ for all $x \in X$.

Intuitively, an action of G on X is a way to assign to each g in G some transformation of X, compatibly with the group structure of G. Formally, the action of a group G on a set X is *equivalent to* a homomorphism of groups

$$\rho : G \to \text{Aut } X,$$

defined by sending each g in G to the set map

$$\rho_g : X \to X$$
$$x \mapsto g \cdot x.$$

This map ρ_g really is a bijection (that is, an element of Aut X) because its inverse is explicitly seen to be $\rho_{g^{-1}}$. The fact that ρ is a group homomorphism is essentially a restatement of condition (1): unraveling the meaning of $\rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2)$ we arrive at $g_1 \cdot (g_2 \cdot x) = (g_1 \star g_2) \cdot x$ for all $x \in X$. Sometimes this group homomorphism is called the *associated representation* of the action or the *permutation representation* of the action. Despite the fancy name, it is really just a different packaging of the same idea: the action of a group G on a set X is *tautologically equivalent to* a homomorphism $G \to \text{Aut } X$.

The most obvious example of a group action is as follows: Let X be any set, and take $G = \text{Aut } X$. By definition, G acts on X via $(\phi, x) = \phi(x)$. The homorphism ρ is the identity map. To be even more specific, take $X = \{1, 2, \dots, n\}$. Then S_n acts (tautologically) on X by permutations.

Another easy example is the group $G = GL_n(\mathbb{R})$ of invertible $n \times n$ matrices acting on the space $\mathbb{R}^n$ of column matrices by left multiplication. This is equivalent to giving a group homomorphism

$$G \to \text{ Aut}\,(\mathbb{R}^n).$$

Of course, the image of this representation is the group $\text{GL}(\mathbb{R}^n)$ of all *(invertible) linear transformations* of $\mathbb{R}^n$, so we in fact have a group homomorphism

$$G \to \text{ GL}\,(\mathbb{R}^n).$$

As you know, this is the isomorphism obtained by identifying linear transformations with matrices via the choice of the standard basis of unit column vectors in $\mathbb{R}^n$.

In nature, groups act naturally on sets which often have some additional structure—for example, a vector space structure or a manifold structure. Often, we are mostly interested in actions that respect this additional structure. For example, in the example above, the group $G = GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ in way that preserves the vector space structure of $\mathbb{R}^n$. That is, each element of G gives rise to a bijective *linear* transformation of $\mathbb{R}^n$, which is of course a very special kind of bijective self-map. We say that $GL_n(\mathbb{R}^n)$ acts *linearly* $\mathbb{R}^n$.

Definition 6.2. A *linear representation* of a group G on a vector space V is an action of G on the underlying set V which respects the vector space structure. More precisely, the corresponding group homomorphism

$$G \to \quad \text{Aut } V$$

has image in the subgroup $GL(V)$ of *linear transformations* of V.

More succiently put, a linear representation of a group G on a vector space V is a homomorphism $G \to GL(V)$. When the field is not implicitly understood, we qualify by its name: For example, a complex representation of a group G is a group homomorphism $G \to GL(V)$ where V is a *complex* vector space.

Example 6.3. Let us consider some different actions of the dihedral group D_4 of the square. Perhaps the simplest is the action of D_4 on the set of vertices of the square. If we label the vertices $\{1, 2, 3, 4\}$, say, in clockwise order from the top right, this gives an action of D_4 on the set $\{1, 2, 3, 4\}$. The corresponding homomorphism to Aut $\{1, 2, 3, 4\}$ gives a map

$$\rho : D_4 \to S_4$$

sending for example r_1 to the 4-cycle (1234) and the reflection H to the permutation (12)(34). The group homomorphism ρ is *injective*—a non-trivial symmetry of the square cannot fix all vertices. This is an example of a *faithful action:*

Definition 6.4. A action of a group G on a set X is *faithful* if each non-identity element of G gives rise to a non-trivial transformation of X. Equivalently, an action is faithful if the corresponding group homomorphism $G \to \text{Aut } X$ is injective.

Another easy example is the tautological action of D_4 on the Euclidean plane, inspired by the very definition of D_4 as the rigid motions of the plane which preserve the square. This action is also faithful. These two different action give two different ways of viewing D_4 as a transformation group—the first identifies D_4 with a subgroup of S_4, and the second identifies D_4 with a subgroup of Aut $(\mathbb{R}^2)$.

Let us consider this tautological linear representation

$$\rho : D_4 \to \text{ Aut } (\mathbb{R}^2)$$

in detail. Fix coordinates so that the square is centered at the origin. Then ρ sends the rotation r_1 to the corresponding rotation of $\mathbb{R}^2$, and so on. Because both rotations and reflections are *linear transformations* of $\mathbb{R}^2$, the image of ρ actually lies in $GL(\mathbb{R}^2)$. Identifying elements of $\mathbb{R}^2$ with column vectors so that linear transformations are given by left multiplication by 2×2 matrices, the elements r_1, r_2, r_3 respectively are sent to the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

respectively, whereas the reflections H, V, D and A are sent to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

respectively. Put differently, we can identify the symmetry group of the square with the matrix group consisting of these seven 2×2 matrices and the identity matrix.

The action of F^* on a vector space. Let V be a vector space over a field F. The multiplicative group F^* of the field acts on the set V by scalar multiplication. In fact, as an exercise, show that a set V is an F-vector space if and only if V is an abelian group with an F^* action.

The actions of a group on itself by left multiplication. Let $(G, \star)$ be any group, and write X for a second copy of the underlying set of G. We have an action

$$G \times X \to X$$
$$(g, x) \mapsto g \star x$$

of G on itself (that is, $X = G$) by left multiplication.

The action of G on itself is *faithful.* Indeed, if $g \in G$ acts trivially on *every* $x \in X$, then $gx = x$ for all $x \in X = G$. But this implies that $g = e_G$, so the corresponding group homomorphism

$$\rho : G \to \text{Aut}G$$
$$g \mapsto [G \to G;\ x \mapsto gx]$$

is injective.

Thus, *any group G is isomorphic to its image under this representation ρ*. That is, we have proved

Theorem 6.5 (Cayley's Theorem)**.** *Every group $(G, \star)$ is a transformation group. Specifically, G is isomorphic to a subgroup of Aut G.*

As a corollary, this says that every group of (finite) order n is isomorphic to a subgroup of S_n. Thus abstract groups always have concrete realizations as permutation groups.

The action of group on itself by conjugation. Let G be any group, and let X denote the underlying set of G. We have an action of G on itself:

$$G \times X \to X$$
$$(g, x) \mapsto g \star x \star g^{-1}.$$

This is quite different from the previous action of G on itself. For example, it is not usually faithful! Indeed, an element $g \in G$ fixes all $x \in X$ if and only if $g \star x \star g^{-1} = x$ for all $x \in X$. Equivalently, g acts trivially on X if and only if g commutes with all $x \in G$, that is, if and only if g in in the *center* of G.

Exercise 6.6. Show the the kernel of the conjugation representation

$$\rho : G \to \mathrm{Aut}G$$
$$g \mapsto \left[G \to G;\ x \mapsto gxg^{-1}\right]$$

is the center of G.

In this example (and the previous), the set X on which G acts has more structure than a mere set: it is of course a group! Do these actions preserve the group structure of X? That is, is the induced map $\rho(g) : G \to G$ actually a map of *groups*, ie, a group homomorphism?

For the action of G on itself by left multiplication, the group structure is *not* preserved. Indeed, the identity element is not even sent to the identity element. However, the conjugation action *does* preserve the group structure:

$$G \to G$$
$$x \mapsto gxg^{-1}$$

is easily checked to be a *group homomorphism.* Thus this is a special type of representation of G. The image of the corresponding map

$$\rho : G \to \mathrm{Aut}G$$

actually lands in the subgroup of *group automorphisms,* or self-isomorphisms of G, denoted $\mathrm{Aut}_{Grp}G$. For this reason, we can expect this representation to play an especially important role in the theory of groups.

Exercise 6.7. Suppose a group G acts on a set X. The stabilizer of a point $x \in X$ is the set $\mathrm{Stab}(x) = \{g \in G | g \cdot x = x\}$.

(1) Show that $\mathrm{Stab}(x)$ is a subgroup of G.
(2) Compute the stabilizer in D_4 of a vertex of the square, under the tautological action.
(3) Characterize a faithful action in terms of stabilizers.

Exercise 6.8. Suppose a group G acts on a set X. The orbit of a point $x \in X$ is the set $\mathrm{O}(x) = \{y \in G \mid \text{there exists } g \in G \text{ such that } y = g \cdot x\}$.

(1) Compute the orbit of a vertex of the square under the tautological action of D_4.

(2) Compute the orbit of a point $x \in G$ under the action of G on itself by left multiplication.
(3) Compute the orbit of a point $z \in \mathbb{C}$ under the action of the circle group $U(1)$ by multiplication.

7. Linear Representations

We repeat the fundamental concept: a linear representation of G on a vector space V is a way of assigning a *linear transformation* of V to each element of G, compatible with the group structure of G. More formally, We

Definition 7.1. A *linear representation of a group* G *on a vector space* V is a group homomorphism

$$G \to GL(V).$$

We will usually omit the word "linear"and just speak of a *representation* of a group on a vector space, unless there is a chance of confusion. There are many ways to refer to this fundamental idea; in addition to the ways already described, we sometimes also say "V is a G-representation"or V is a G-module.

We say that a representation has dimension d if the vector space dimension of V is d. (Some books call this the degree of d). The representation is *faithful* if this group map is injective.

For example, **The tautological representation of** D_n on $\mathbb{R}^2$ induced by the action of D_n on the plane by linear transformations is a two-dimensional representation of D_n. We explicitly described the map $D_n \to GL(\mathbb{R}^2)$ in the previous lecture for $n = 4$. The tautological representation is *faithful* since every non-identity element of D_4 is some non-trivial transformation of the plane.

Every group admits a **trivial representation** on every vector space. The trivial representation of G on V is the group homomorphism $G \to GL(V)$ sending every element of G to the identity transformation. That is, the elements of G all act on V trivially—by doing nothing.

Suppose a group G acts on a set X. There is an associated (linear) **permutation representation** defined over any field. Consider the F-vector space on basis elements e_x indexed by the elements x of X. Then G acts by permuting the basis elements via its action on X. That is

$g \cdot e_x = e_{g \cdot x}$. For example, S_3 has a three-dimensional representation defined by

$$S_3 \to GL(\mathbb{R}^3)$$

sending, for example,

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \ (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \ (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An important type of permutation representation is the **Regular Representation** induced by the action of a group G on itself by left multiplication. The regular representation over the field $\mathbb{F}$, for example is the group homomorphism

$$G \to GL(\mathbb{F}^{|G|})$$

$$g \mapsto \left[T : \mathbb{F}^{|G|} \to \mathbb{F}^{|G|};\ e_h \mapsto e_{gh}\right].$$

For example, we have a representation of D_4 on $\mathbb{R}^8$ induced this way. The regular representation is defined even when G is infinite, but we won't usually use it except for finite groups. The regular representation is faithful, since *every* non-identity element of G moves each basis element to some other basis element: $g \cdot e_h = e_{gh}$ for all g and h in G. In particular, the orbit of any basis vector is the full set of basis vectors $\{e_g\}_{g \in G}$. That is, G acts transitively on our chosen basis.

A **subrepresentation** of a representation of G on a vector space V is a subvector space which is also a G-representation under the *same* action of G. Put differently, a subspace W of V is a subrepresentation if W is invariant under G—that is, if $g \cdot w \in W$ for all $g \in G$ and all $w \in W$. In terms of the group homomorphism

$$\rho : G \to GL(V),$$

W is a subrepresentation if and only if ρ factors through the subgroup

$$G_W = \{\phi \in GL(V) \mid \phi(W) \subset W\}$$

of $GL(V)$ of linear transformations stabilizing W.

For example, every subspace is a subrepresentation of the trivial representation on any vector space, since the trivial G action obviously takes every subspace back to itself. At the other extreme, the tautological representation of D_4 on $\mathbb{R}^2$ has no proper non-zero subrepresentations: there is no line taken back to itself under *every* symmetry of the square, that is, there is no line left invariant by D_4.

The permutation representation of D_4 on $\mathbb{R}^4$ induced by the action of D_4 on a set of four basis elements $\{e_1, e_2, e_3, e_4\}$ indexed by the vertices of a square *does have* a proper non-trivial subrepresentation. For example, consider the one dimensional subspace spanned by $e_1 + e_2 + e_3 + e_4$. When D_4 acts on this, it simply permutes the indices of the e_i, and there sum remains unchanged. Thus for all $g \in G$, we have $g \cdot (\lambda, \lambda, \lambda, \lambda) = (\lambda, \lambda, \lambda, \lambda)$ for all vectors in this one-dimensional subspace of $\mathbb{R}^4$. That is, D_4 acts trivially on this one-dimensional subrepresentation. In particular, trivial representations can be non-trivial subrepresentations of a non-trivial representation of G.

Another subrepresentation of the permutation representation of D_4 on $\mathbb{R}^4$ is the subspace $W \subset \mathbb{R}^4$ of vectors whose coordinates sum to 0. Clearly, when D_4 acts by permuting the coordinates, it leaves their sum unchanged. For example H sends $(1, 2, 3, -6)$ to $(-6, 3, 2, 1)$ if the vertices are labelled counterclockwise from the upper right. The space W is a three dimensional subrepresentation of the permutation representation of D_4 on $\mathbb{R}^4$. Note that is also *non-trivial;* the elements of G *do* move around the vectors in the space W.

Direct Sums of Representations.

Suppose a group G acts on the vector spaces V and W (over the same field, $\mathbb{F}$). We can define an action of G "coordinate-wise"on their direct sum as follows:

$$g \cdot (v, w) = (g \cdot v, g \cdot w) \in V \oplus W.$$

Note that if V has basis $v_1, \dots, v_n$ and W has basis $w_1, \dots w_m$, then the direct sum has basis $v_1, \dots, v_n, w_1, \dots w_m$ (where v_i is interpreted to mean $(v_i, 0)$, etc). With this choice of basis for the direct sum, the matrix of every g acting on $V \oplus W$ will be the block diagonal matrix

$$\begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

obtained from the $n \times n$ matrix $\rho_1(g)$ describing the action of g on V in given basis, and the $m \times m$ matrix $\rho_2(g)$ describing the action of g on W in its given basis.

For example, we have a three-dimensional real representation of D_4 defined as follows:

$$g \cdot (x, y, z) = (g \cdot (x, y), z),$$

where $g\dot{(}x, y)$ denotes the image of (x, y) under the tautological action of D_4 on $\mathbb{R}^2$. In the standard basis, for example, the element $r_1 \in D_4$ acts by the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This representation is the *direct sum* of the tautological and the one-dimensional trivial representations of D_4.

Of course, we also have a representation of D_4 on $\mathbb{R}^3$ where D_4 acts trivially in the x-direction and by the tautological action in the yz plane direction: $g \cdot (x, y, z) = (x, g \cdot (y, z))$. This is also a direct sum of the trivial and tautological representation. In fact, just thinking about these actions there is a very strong sense in which they are "the same."We are led to the following definition:

Definition 7.2. Two linear representations V and W of G (over the same field) are *isomorphic* if there is a vector space isomorphism between them that preserves the G-action—that is, if there exists an isomorphism $\phi : V \to W$ such that $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ and all $g \in G$.

The idea that ϕ preserves the G action is also expressed by the following commutative diagram, which mush hold for all $g \in G$:

For example, the two representation of D_4 discussed above are isomorphic under the isomorphism which switches the x and z coordinates.

There are many different words for the notion of a homomorphism of representations of a group G, probably because representation theory is used in so many different places in mathematics and science. You may here the terms "G-linear mapping", "G-module mapping,"or even just "a map of representations of G-–they all mean the same thing. The adjective "linear"is often suppressed: if they representations are linear, so also are the homorphisms between them assumed to be *linear* homomorphisms. But, depending on the context, it may also be interesting to study group actions on, say, a topological space, in which case the corresponding representations should be *continuous,* and the maps between them as well should be continuous.

Of course, we can also define a notion of mapping or *homomorphism* for G-representations:

Definition 7.3. A homomorphism of G-representations is a map $\phi : V \to W$ which preserves both the vector space structure and the G-action. That is, it is a linear map ϕ of vector spaces (over the same field) satisfying $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ and all $g \in G$.

An isomorphism of G-representations is simply a bijective homomorphism.

The following lecture is in Finnish. The English version - **covering this again** - continues on page 13.

Definition 7.4. Lineaaristen esitysten homomorfismi on lineaarikuvaus, joka säilyttää G:n toiminnan. Tarkemmin sanoen: Olkoot V ja W saman ryhmän $\mathbb{F}$-lineaarisia esityksiä. Lineaarikuvaus $\phi : V \to W$ on näiden esitysten homomorfismi, jos $\varphi(g{\cdot}v) = g{\cdot}\varphi(v)\,\forall v \in V$, $\forall g \in G$, toisin sanoen $\varphi : V \to W$ on homomorfismi, jos *diagramma*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{\varphi} & W \end{array}$$

kommutoi.

Example 7.5. Esimerkkejä:

On helppo tarkastaa, että isomorfismi on sama asia kuin sellainen homomorfismi, jolle myös käänteinen lineaarikuvaus $\phi^{-1} : W \to V$ on esitysten homomorfismi.

Aliesitys antaa homomorfismin: Olkoot W ja V esityksiä kuten yllä ja $W \subset V$. Silloin inkluusioikuvaus $\phi : W \to V : x \mapsto x$ on injekstiivinen lineaarikuvaus ja samalla G-esityshomomorfismi, onhan $\varphi(g) \cdot w) = g \cdot \varphi(w)$.

Periaatteessa jokainen *injektiivinen esityshomomorfinsmi* on tällainen, sillä injektiivinen lineaarikuvaus on vektoriavaruuksien isomorfiaa vaille sama asia kuin inkluusiokuvaus.

Vastaavasti saadaan esimerkki *surjektiivisesta esityshomomorfismista* tarkastelemalla vektoriavaruuksien suoran summan projektiokuvausta toiselle komponentille eli lineaarikuvausta $\varphi = \pi : W \oplus V \to V : (x, y) \mapsto y$. Selvästi π on surjektio ja $\pi(g \cdot w, g \cdot v) = g \cdot v$ ja myös $g \cdot \pi(w, v) = g \cdot v$.

Esitysavaruuksien W ja V välinen lineaarikuvaus ei tietenkään yleensä ole esityshomomorfismi. Kahden eri esityksen välillä on sentään aina olemassa homomorfismi, nimittäin nollakuvaus, mutta niiden välillä ei tarvitse olla olemassa injektiivistä eikä surjektiivista homomorfismia, vaikka vektoriavasruuksien välillä aina on joko injektiivinen tai surjektiivinen lineaarikuvaus ulotteisuuksista riippuen. Esimerkkinä tarkastellaan kahta dihedraalisen ryhmän D_4 esitystä, joista ensimmäinen olkoon tautologinen toiminta, ts. $W = \mathbb{R}^2$ja D_4 toimii kuten määritelmässään. VIITE Olkoon toisena D_4:n esityksenä permutaatioesitys avaruudessa $V = \mathbb{R}^4$ eli neliön nurkilla indeksoitujen kantavektorien permutointi. VIITE Kysytään, onko ollenkaan olemassa injektiivistä esityshomomorfismia näiden välillä? Voisi kokeilla umpimähkään esimerkiksi kuvaamalla $\varphi : (x, y) \mapsto (x, y, 0, 0)$. Tämä ei ainakaan ole esityshomomorfismi, koska $f_x \circ \phi$ ei ole sama kuin $\varphi \circ f_x$, sillä[1] esimerkiksi $\phi \circ f_x(x, y) = \phi(x, -y) = (x, -y, 0, 0)$, mutta $f_x \circ \phi : (x, y) = (0, 0, y, x)$. Tässä siis lineaarikuvaus, joka ei säilytä esitystä. Onko siis ollenkaan olemassa lineaarista injektiota $\pi : \mathbb{R}^2 \to \mathbb{R}^4$, joka esitykset? Vastaus saadaan myöhemmin.

Huomautus ja määritelmä: Tarkastellaan ryhmän $GL_n(\mathbb{R})$ tautologista toimintaa avaruudessa R^n eli, kun kanta on valittu, $n \times n$-matriiseina. Aliryhmä $SL_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R}) \mid \det M = 1\}$ toimii samalla tavalla. Tämä ei tietenkään voi olla aliesitys.Esitetävänä on kokonaan eri ryhmä! Tällaiselle tilanteelle annetaan eri nimi. Yleisesti kysymys on siitä, että jokainen G esitys määrittelee aliryhmän $H \subset G$ esityksen, joka on nimeltään alkuperäisen *esityksen rajoittuma* (NÄINKÖ, VIITE) aliryhmään H. Tätä ideaa voi vielä yleistää: Jokaiseen ryhmähomomorfismiin $f : H \to G$ ja esitykseen $\rho : G \to GL(\mathbb{F})$ liittyy esitys $f^*(\rho) = \rho \circ f : H \to GL(\mathbb{F})$, joka on nimeltään esityksen ρ *pull-back*[2].

Kiinteään vektoriavaruuteen V ja ryhmähomomorfismiin $F : H \to G$ liittyy näin *pullback-funktori*, joka liittää jokaiseen ryhmän G lineaariseen esitykseen $\rho : G \to GL(V)$:ssä ryhmän H esityksen $f^*(\rho) : H \to GL(V)$. Siis:

$$f^* : \{G\text{:n esitykset } V \text{ ssä}\} \to \{H\text{:n esitykset } V \text{ ssä}\} : \rho \mapsto \rho \circ f.$$

On syytä tarkastaa, että $\rho \circ f$ todella on ryhmän H esitys. Sitä varten on todettava kaksi ominaisuutta: (VIITE)

(1) $e_H \overset{f}{\mapsto} e_g \overset{\rho}{\mapsto} = Id_V$.
(2) ja $(\rho \circ f)(h_1 \cdot h_2) = \rho(f(h_1 \cdot h_2)) = \rho(f(h_1)) \circ \rho(f(h_2)) =$
$= (\rho \circ f)(h_1) \circ (\rho \circ f)(h_2)$.

[1] f_x on heijastus $(a, b) \mapsto (a, -b)$.
[2] Hyvää suomalaista nimeä ei taida olla keksitty vielä.

English covering the same

An isomorphism of G-representations is simply a bijective homomorphism. On the other hand, it might be better to think of an isomorphism as a homomorphism $\phi : V \to W$ which has an inverse, because this way of thinking is valid in any *category.*[3] But for representations, you can check that if $\phi : V \to W$ is a bijective linear transformation, then the inverse map $\phi^{-1} : W \to V$ is also a linear map, and that if ϕ preserves the group action, so does ϕ^{-1}: $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ if and only if $g^{-1} \cdot \phi^{-1}(w) = \phi^{-1}(g^{-1} \cdot w)$ for all $w \in W$.

The **inclusion** of any subrepresentation W of a representation V of a group G provides an example of an injective homomorphism of representations: $\phi : W \hookrightarrow V$ obviously satisfies $g\phi(w) = \phi(gw)$ because, by definition of subrepresentation, the action of G is the same on elements of W whether we think of them in W or V.

The **projection** $\pi : V \oplus W \to V$ from a direct sum of G-representations onto either factor is an example of a surjective homomorphism. Again, that this map is G-linear is easy to verify directly from the definitions (do it!).

Although it is far from obvious, these examples—isomorphism, inclusion, and projection—form the *only* homomorphisms between representations of a fixed group G in a certain sense. Perhaps this is not surprising: a similar fact is true for vector spaces. If $\phi : V \to W$ is a linear map of vector spaces, then because $V \cong im\,\phi \oplus ker\,\phi$, every mapping of vector spaces essentially factors as projection onto the image followed by inclusion of the image into the target. On the other hand, nothing like this is remotely true for most sorts of maps between mathematical objects: there are all sorts of group homomorphisms, such as $\mathbb{Z} \to \mathbb{Z}_n$, which are not of this sort.

7.1. **A non-Example.** Let us consider two representations of D_4: the tautalogical representation T (which is the vector space $\mathbb{R}^2$, with the tautological D_4 action) and the vertex permutation representation V (namely, the vector space $\mathbb{R}^4$ with basis elements indexed by the vertices of a square, with the action of D_4 given by permuting the vertices according to the action of D_4 on the vertices.) Of course, there are

[3]For example, an isomorphism in the category of topological spaces—or homeomorphism— is not the same as a bijective continuous map: we also require the inverse to be continuous.

many linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^4$. Do any of these preserve the group action? That is, are any of them homomorphism of the D_4-representations $T \to V$. Of course, stupidly, we could take the zero-map $T \to V$ mapping every element of $\mathbb{R}^2$ to $\mathbb{R}^4$. This preserves the group action, but it is not interesting. Can we find any injective maps?

Let us consider whether the inclusion

$$i : \mathbb{R}^2 \to \mathbb{R}^4$$

sending

$$(x, y) \mapsto (x, y, 0, 0)$$

is D_4-linear. For example, does this preserve the action of the element r_1 in D_4. We have that $r_1 \cdot (x, y) = (-y, x)$ for the representation T, which yields $(-y, x, 0, 0)$ under the inclusion map i, where as $r_1 \cdot (x, y, 0, 0) = (0, x, y, 0)$ for the representation V. This means that i *does not* respect the group structure— the linear map i is *not* a map of representations of D_4.

English continuing with new

7.2. **Searching for sub-representations.** Can there be *any* injective map of representations $T \to V$? If so, then the image of T would be a two-dimensional subspace of the four-dimensional space V where the the vertex permutation representation restricts to the tautological representation. Is this possible? Essentially, we are asking whether there is any two-dimensional square in four space on which the vertex permutation action of D_4 (given by permuting the coordinates as D_4 permutes the vertices of a square) agrees with the usual symmetries of that square. Clearly, this can happen only if there is a set of four points in $\mathbb{R}^4$ spanning a two-dimensional subspace invariant under this action.

To understand the permutation action of D_4 on $\mathbb{R}^4$ it is helpful to identify D_4 with the subgroup of S_4 generated by the 4-cycle (1234) (the rotation r_1) and the transposition (13) (the reflection A). Formally, we are embedding D_4 in S_4 by sending each symmetry to the corresponding permutation of the vertices. The 4-cycle acts by cycling the coordinates (one spot to the right, say), and the transposition acts by switching the first and third coordinates. Since D_4 is generated by these two elements, a subspace is invariant under D_4 if and only if it is invariant under these two elements (Prove it!).

To see whether the permutation representation has any subrepresentation isomorphic to T, then, we are looking for four points in $\mathbb{R}^4$ spanning a two-dimensional subspace, and invariant under these two transformations, which can serve as the vertices of a square. Can one of these points be, say $(1, 0, 0, 0)$? Note that when r_1 acts on this vector in V, it moves it to $(0, 1, 0, 0)$. When it acts again—in other words when r_2 acts on $(1, 0, 0, 0)$— we get $(0, 0, 1, 0)$. Again, and we get $(0, 0, 0, 1)$, and then we cycle back to $(1, 0, 0, 0)$. Thus, any vector subspace of V containing $(1, 0, 0, 0)$ and invariant under D_4[4] must contain each of the four standard basis elements—that is, it must be V, since these span V. Indeed, it is easy to check that this sort of thing happens for "most vectors:" typically, the orbit of a randomly chosen vector in V will consisting of (eight) vectors which span V.

To find interesting subrepresentations of V, we can look for non-zero vectors whose orbits span proper subspaces of V. One way is to find vectors with small orbits. For example, the vector $(1, 1, 1, 1)$ is fixed by D_4. It spans a one dimensional sub-representation of V where D_4 acts trivially, as we have already noted.

Another vector with a small orbit is $w = (-1, 1, -1, 1)$. Note that r_1 acts on w_1 to produce $(1, -1, 1, -1)$, which is $-w$. Also, the reflection A acts by permuting the first and third coordinates, which means it fixes w. Since *every* element in D_4 is an iterated composition of the generators r_1 and A, we see that w spans a one-dimensional sub-representation of V. This is *not* a trivial representation—some element acts by multiplication by -1.

How does this help us find two dimensional sub-representations of V isomorphic to T, or show non any exist? Well, clearly if one does exist, it can not contain $(1, 1, 1, 1)$, since is a faithful representation and *no* subspaces are pointwise fixed. Nor could it contain $(1, -1, 1, -1)$ since r_1 acts there by multiplication by -1 but r_1 is *never* scalar multiplication on T. Thus, we need to look for a vector in V whose orbit does not contain either of these two special vectors (nor anything in their span).

Consider the vector $(1, 1, -1, -1)$. Its orbit produces the four points

$$(1, 1, -1, -1),\ \ (-1, 1, 1, -1),\ \ (-1, -1, 1, 1),\ \ (1, -1, -1, 1),$$

[4]Or even the smaller group of rotations R_4.

and so it *does* generate a two-dimensional sub-representation T' of V (which has basis, for example, $(1, 1, -1, -1), (-1, 1, 1, -1)$.) It is easy to check that the vertex permutation action of D_4 on T' *does restrict* to the tautological representation of D_4 on this two-plane. Indeed, the four points described above serve nicely as the vertices of a square on which D_4 acts by the usual symmetry actions. So V *does contain* a sub-representation isomorphic to T.

Finally, it is not hard to check that V decomposes as the direct sum of the three sub-representations:

$$V \cong [\mathbb{R}\langle(1, 1, -1, -1), (-1, 1, 1, -1)\rangle] \bigoplus \mathbb{R}(1, 1, 1, 1) \bigoplus \mathbb{R}(1, -1, 1, -1),$$

where the first summand is isomorphic to the tautological representation of D_4 and the second summand is a trivial representation of D_4, but the third is not isomorphic to either of these. Furthermore, none of these three sub-representations can be further decomposed. Although it is not obvious, we will soon prove that *every representation* of a finite group on a complex (say) vector space *decomposes as a direct sum of irreducible representations.*

We isolate for future reference a simple idea used in the previous example:

Lemma 7.6. *Let V be a linear representation of a group G. Let W be a sub-vector space of V, spanned by the elements $w_1, \dots, w_t$. Then W is a subrepresentation if and only if $g \cdot w_i \in W$ for all $i = 1, \dots, t$ and each g in one fixed generating set for G.*

Proof. We leave this as an easy exercise, but point out the only slightly more subtle point: If an element g leaves W invariant, then also g^{-1} does. Indeed, the linear transformation $g : V \to V$ is invertible, which means that restricted to W, it is also invertible. So g^{-1} defines its inverse, also on W. □

7.3. **The kernel, image and cokernel of a map of representations.** The category of G-representations and their maps is very nice from an algebraic point of view: we can form kernel representations, image representations, quotient representations and cokernel representations.

Proposition 7.7. *Let $\phi : V \to W$ be a homomorphism of linear representations of a group G. Then the kernel is a subrepresentation of V and the image is a subrepresentation of W.*

Proof. Since the kernel and image of ϕ are subvector spaces, we only have to check that each is invariant under the G-action. Suppose that $v \in V$ is in the kernel of V, and $g \in G$. We need to check that $g \cdot v$ is in the kernel of ϕ. But $\phi(g \cdot v) = g\phi(v)$ because ϕ is G-linear, and $g\phi(v) = g \cdot 0 = 0$ since v is in the kernel of ϕ and g acts by linear transformations (so preserves the zero).

The proof for the image is also straightforward: $g \cdot \phi(v) = \phi(g \cdot v)$ is in the image of ϕ. □

Whenever there an inclusion of linear G-representations $W \subset V$, the **quotient representation** V/W can be defined. Indeed, we define V/W as the quotient vector space with the G-action defined by

$$g \cdot \overline{v} = \overline{g \cdot v}$$

where v is any representative of the class. This does not depend on the choice of representative, since if $\overline{v} = \overline{v}'$, then $v - v' \in W$, and whence $g \cdot v - g \cdot v' \in g \cdot W \subset W$, which of course means $g \cdot v$ and $g \cdot v'$ represent the same class of V/W.

In particular, the cokernel $V/\mathrm{im}\phi$ of any homomorphism $\phi : V \to W$ of representations of G is also a representation of G.

A lecture in Finnish - the same reappears below in English, p 21

8. Redusoituvuus ja redusoitumattomuus

Seuraavassa ryhmän *esityksellä* tarkoitetaan, kun ei toisin sanota, $\mathbb{F}$-lineaarista esitystä jossain $\mathbb{F}$-vektoriavaruudessa. Tässä $\mathbb{F}$ on kunta, usein $\mathbb{R}$ tai $\mathbb{C}$.

Definition 8.1. Esitys V on *redusoitumaton*, jos sillä ei ole yhtään aitoa nollasta eroavaa aliesitystä.

Example 8.2. . (0) Jokainen yksiulotteinen esitys on tietenkin redusoitumaton.

(1) Dihederaalisen ryhmän D_4 tautologinen esitys avaruudessa $\mathbb{R}^2$ on redusoitumaton, sillä kaksiulotteisella avaruudella $\mathbb{R}^2$ ei ole muita aitoja nollasta eroavaia aliavaruuksia kuin yksiulotteiset, eikä niistä mikään ole invariantti edes 90 asteen kierrossa saati koko ryhmän D_4 kaikkien alkioiden vaikutuksissa.

Dihederaalisen ryhmän D_4 permutaatioesitys avaruudessa $(2)\mathbb{R}^4$ on redusoitumaton, sillä ainakin vektorin

$$v = e_1 + e_2 + e_3 + e_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

virittämä yksiluotteinen aliavaruus $W_1 =< v >= \{\lambda v \mid \lambda \in \mathbb{R}\} \subset \mathbb{R}^4$ on luonnollisesti invaritantti jopa kaikkien kantavektoriehin e_1, e_2, e_3, e_4 kohdistuvien permutaatioiden suhteen. Edellisessä luvussa huomattiin, että tietysti myös aliavaruus $W_2 =< e_1 - 2_2 + e_3 + e_3 >$ on invariantti D_4:n, vaikka ei kaikkien permutaatioiden suhteen. Lisäksi ryhmän D_4 esityksellä on myös kaksiulotteinen invariantti aliavaruus, nimittäin $W_3 < e_1 - 2_2 + e_3 - e_3, e_1 - 2_2 - e_3 + e_3 >$, minkä välitön keksiminen ei ehkä ole aivan helppoa. Huomasimme edellä myös, että tämä kaksiulotteinen aliesitys on isomorfinen D_4,n tautologisen esityksen kanssa ja siis edellisen esimerkin nojalla redusoitumaton. Voi pohtia, onko $\mathbb{R}^4$:llä mitään muuta kaksiulotteista invarianttia aliavaruutta, tai jopa toista sellaista, jossa aliesitys olisi isomorfinen tautologisen esityksen kanssa. (Ei ole!)

Kiinnitetään vielä huomiota siihen, että W_3 näyttää olevan kahden aikaisemman esityksen suoran summan $< e_1+e_2+e_3+e_3, e_1-2_2+e_3-e_3 >$ ortogonaalinen komplementti avaruuden $\mathbb{R}^4$ tavallisen sisätulon mielessä, toisin sanoen $(v|u) = 0$ aina, kun $v \in W_1 \oplus W_2$ ja $u \in W_3$. Ainakin on selvää, että löytämämme aliavaruudet ovat lineaarisesti riippumattomia ja virittävät siis yhteensä 1+1+2 -ulotteisen avaruuden eli koko esitysavaruuden R^4, joka näin on niiden suora summa. Tutkimamme permutaatioesitys on siis kolmen redusoitumattoman aliesityksensä suora summa.

(3) Äärettömälle additiiviselle ryhmälle $(\mathbb{R}, +)$ voi määritellä esityksen avaruudessa $\mathbb{R}^2$ asettamalla

$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix}$$

eli kuvaamalla $\lambda \mapsto \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in GL(\mathbb{R}^2)$.

PIIRRÄN KUVAN!

Tällä esityksellä on invariantti aliavaruus eli aliesitys, nimittäin vektorin $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ virittämä suora eli x-akseli. Asia on helppo tarkastaa huomaamalla, että peräti $v \mapsto v \in < v >$. Muita epätriviaaleja invariantteja aliavaruuksia ei siten olekaan, sillä kaksiulotteisen esityksen epätriviaalit aliesitykset ovat tietenkin yksiulotteisia, eikä mikään x-akselista eroava suora kuvaudu itselleen edes kuvauksessa $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ saati kaikissa $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$.

Tutkittava esitys osoittautui siis redusoituvaksi, mutta siinä mielessä erilaiseksi kuin edellinen esimerkki, että nyt redusoitumattomia aliesityksiä on niin vähän, että koko esitysavaruutta, joka tässä oli $\mathbb{R}^2$, ei voikaan lausua niiden summana.

Definition 8.3. Olkoon V ryhmän G esitys. Joukon $S \subset V$ *virittämä aliesitys* on pienin aliesitys $W \subset V$, joka sisältää joukon S. Erityisesti, jos S on äärellinen joukko $S = \{x_1, \dots\}$, niin joukon S virittämää aliesitystä sanotaan *vektorien* $x_1, \dots$ *virittämäksi aliesitykseksi.*

Remark 8.4. Joukon $S \subset V$ virittämä aliesitys on aina olemassa, sillä mielivaltaisen monen, myös äärettömän monen, esityksen leikkaus on aina esitys, joten pienin aliesitys $W \subset V$, joka sisältää joukon S on

$$\bigcap \{W' \mid S \subset W' \text{ ja } W' \text{ on } V\text{:n aliesitys.}\}$$

Remark 8.5. Esitys V on redusoitumaton, jos ja vain jos se on jokaisen vektorin $v \in V$ virittämä aliesitys eli, jos jokaisen vektorin $v \in V$ rata virittää lineaarisesti koko vektoriavaruuden V.

Theorem 8.6. Täydellinen redusoituvuus *Olkoon äärellisen ryhmän G esitys V äärellisulotteinen $\mathbb{F}$-vektoriavaruus, missä $bF = \mathbb{R}$ tai $\mathbb{C}$. Silloin V on redusoitumattomien aliesityksiensä suora summa. (Lause pätee itse asiassa yleisemmällekin kunnalle $\mathbb{F}$, kunhan sen* karakteristika *ei ole 2, vaan $2 \neq 0$. Esimerkiksi kunta $\mathbb{Z}_2$ ei todellakaan kelpaa.)*

The same lecture in English

9. Irreducible Representations.

Definition 9.1. A representation of a group on a vector space is *irreducible* if there are no proper non-trivial subrepresentations.

For example, the tautological representation T of D_4 is irreducible: if there were some proper non-zero subrepresentation, it would have to be one dimensional, but clearly no line in the plane is left invariant under the symmetry group of the square. Indeed, the plane $\mathbb{R}^2$ is irreducible even under the action of the subgroup R_4 of rotations.

On the other hand, the vertex permutation representation is *not* irreducible. For example, the line spanned by the vector $(1, 1, 1, 1)$ or either of the two other subspaces described above in Example 13.1 are non-zero proper subspaces.

Another example of an irreducible representation is the tautological representation of $GL(V)$ on a vector space V. Indeed, given any non-zero vector $v \in V$, we can always find a linear transformation taking v to any other non-zero vector. In other words, $GL(V)$ acts transitively on the set of non-zero vectors, so there can be no *proper* subset of V left invariant under this action, other than the single element set $\{0\}$.

On the other hand, the action of $GL_n(\mathbb{R})$ on the space of $n \times m$ real matrices by left multiplication (in other words, by row operations) is *not irreducible.* For example, the subspace of $n \times m$ matrices whose last column is zero certainly invariant under row operations.

To check whether or not a representation is irreducible, it is helpful to think about *sub-representations generated by certain elements:*

Definition 9.2. Let V be any representation of a group G, and S be any sub*set*[5] of V. The *sub-representation* generated by S is the smallest sub-representation of V containing the set S, that is

$$\bigcap_{W \text{sub-rep of } V \text{containing } S} W.$$

[5]emphasis: S need be a sub*set* only, not necessarily a sub-representation or even a subspace

Put differently, the sub-representation of a G-representation V generated by a subset S is the vector space spanned by the vectors in the G-orbits of all the elements in S. For example, the sub-representation of the vertex permutation representation V of D_4 generated by $(1,0,0,0)$ is the whole of V, since r_1 takes $(0,1,0,0)$, r_2 takes $(1,0,0,0)$ to $(0,0,1,0)$ and r_3 takes $(1,0,0,0)$ to $(0,0,0,1)$.

The following easy fact is more or less obvious:

Proposition 9.3. *A representation is irreducible if and only if it is generated (as a representation) by* **any one** *non-zero vector.*

Example 9.4. Consider the additive group $G = (\mathbb{R}, +)$. This has a representation on $\mathbb{R}^2$ given by the group homomorphism

$$(\mathbb{R},+) \to GL_2(\mathbb{R})$$

$$\lambda \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Explicitly, the element λ in G acts by sending a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x+\lambda y \\ y \end{bmatrix}$. Clearly, the x-axis is a G-invariant subspace—-indeed G acts trivially on the one-dimensional subspace of vectors of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$. In particular, this is *not* an irreducible representation of $(\mathbb{R}, +)$.

Are there any other interesting sub-representations? Let us take any vector not in this sub-representation already identified, say $\begin{bmatrix} a \\ b \end{bmatrix}$ where $b \neq 0$. When some non-trivial element λ in G acts on $\mathbb{R}^2$, it sends this element to $\begin{bmatrix} a+\lambda b \\ b \end{bmatrix}$, which is obviously *not* a scalar multiple of $\begin{bmatrix} a \\ b \end{bmatrix}$ since $\lambda \neq 0$. Thus, these two vectors span the whole of $\mathbb{R}^2$. This shows that no vector off the x-axis can generate a proper sub-representation— the x-axis is the *only* proper non-zero sub-representation of V. Thus, this two-dimensional representation of the additive group $(\mathbb{R}, +)$ has exactly one irreducible non-zero proper sub-representation. In particular, it can not be decomposed into a direct sum of irreducible representations!

Example 9.5. Whether or not a representation is irreducible depends on the *field* over which in is defined. For example, we have already observed that the tautological representation of the rotation group R_4

on $\mathbb{R}^2$ is irreducible. Explicitly, this is the representation

$$R_4 \to GL_2(\mathbb{R})$$

$$r_1 \mapsto \begin{pmatrix} 0 & -1\lambda \\ 1 & 0 \end{pmatrix}; \quad r_2 \mapsto \begin{pmatrix} -1 & 0\lambda \\ 0 & -1 \end{pmatrix}; \quad r_3 \mapsto \begin{pmatrix} 0 & 1\lambda \\ -1 & 0 \end{pmatrix}; \quad e \mapsto r_1 \mapsto \begin{pmatrix} 1 & 0\lambda \\ 0 & 1 \end{pmatrix}.$$

Of course, we can also think of this as a representation of the group $\mathbb{Z}_4$, in which case, we will call it the **rotation representation** of $\mathbb{Z}_4$. Of course, the elements $1, -1$ and 0 make sense in any field $\mathbb{F}$, so we can consider the "same" action of R_4 on $\mathbb{F}^2$, for any field! For example, the group map above can be taken to have target say, $GL_2(\mathbb{C})$, instead of $GL_2(\mathbb{R})$.

Let us consider whether the complex rotation representation has any interesting sub-representations. If so, there must be some complex vector $\begin{bmatrix} a \\ b \end{bmatrix}$ which generates a one-dimensional sub-representation of $\mathbb{C}^2$. In particular, we must have

$$r_i \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \lambda_i \begin{bmatrix} a \\ b \end{bmatrix}$$

for each $r_i \in R_4$ and some complex scalar λ_i, depending on r_i. Since R_4 is generated by r_1, we see that it is enough to check this condition just for r_1: the condition for r_2 will follow with λ_2 taken to be λ_1^2, and the condition for r_3 will follow with $\lambda_3 = \lambda_1^3$ (of course, the condition for the identity element e holds in any case, with the corresponding λ taken to be 1).

That is, the one-dimensional subspace of $\mathbb{C}^2$ spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$ is invariant under R_4 if and only if $\begin{bmatrix} a \\ b \end{bmatrix}$ is an *eigenvector* of the linear transformation r_1. In this case, λ_1 is the corresponding eigenvalue.

Over $\mathbb{C}$, of course, every linear transformation has eigenvectors! Indeed, we can find them by computing the zeros of the characteristic polynomial. For the transformation given by r_1, which is represented by the matrix $\begin{pmatrix} 0 & -1\lambda \\ 1 & 0 \end{pmatrix}$, the characteristic polynomial is

$$\chi(T) = \det \begin{pmatrix} T & 1\lambda \\ -1 & T \end{pmatrix} = T^2 + 1.$$

It's roots are the complex numbers i and $-i$, and we easily compute the corresponding eigenvectors to be

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Thus, the two dimensional complex rotation representation of R_4 decomposes into the two one-dimensional representations

$$\mathbb{C}\begin{bmatrix} i \\ 1 \end{bmatrix} \oplus \mathbb{C}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

The idea of a *irreducible object* comes up in other algebraic categories as well.[6] For example, we can ask what *groups* have the property that they contain no (normal, say) proper non-trivial subgroups: these are called the *simple* groups. Mathematicians have spend the better part of the last century classifying all the finite simple groups. In other settings, this task is much easier: for example, a vector space has proper non-trivial subspaces if and only if it has dimension one.

Of course, once one understands all the simple groups (or the "simplest objects" in any category), the next step is to understand how *every group* can be built up from these simple ones. Again, for vector spaces, this is quite easy to understand: every (finite dimensional) vector space is a direct sum of "simple"—meaning one-dimensional—vector spaces. For groups, the story is more complicated. For example, the group $\mathbb{Z}_4$ is *not* simple, as the element $\{\overline{0}, \overline{2}\}$ forms a proper subgroup. However, nor can $\mathbb{Z}_4$ be isomorphic to the direct sum of two smaller groups in a non-trivial way (prove it!). There is a whole theory of *extensions* for groups which treats the question of how arbitrary groups can be build up from simple ones. This is a beautiful story, but not one we digress to discuss here.

Returning to our main interest: what happens for representations? Can every representation of a group be built up out of irreducible ones in some easy to understand way? The answer is as nice as possible, at least for finite groups over familiar fields:

[6]The language of category theory is so ubiquitous in algebra that, even though we do not need it, it is probably a good idea for the student to start hearing it, at least informally.

Continuation in English

Theorem 9.6. *Every representation of a finite group on vector space over $\mathbb{R}$ or $\mathbb{C}$ is (isomorphic to) a direct sum of irreducible sub-representations.*

In Example 13.1, we explictly found such a decomposition for the vertex permutation representation V for D_4. Let us now prove the general case.

Proof. Let V be a representation of a finite group G over $\mathbb{R}$ or $\mathbb{C}$. If V has no proper non-zero sub-representations, we are done. Otherwise, suppose that W is such a sub-representation. By fixing any *vector space complement* U for W inside V, we can decompose V as a direct sum of vector space $V \cong W \oplus U$. Of course U may not be invariant under the G-action, so there is still work to be done.

Using this vector space decomposition of V, we get a projection $\pi : V \to W$ onto the first factor. Although this linear map is *not* necessarily a homomorphism of G-representations, we will "average it over G" to construct a G-linear map from it. To this end, define

$$\phi : V \to W$$

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v).$$

Although it looks complicated, this is really just a simple projection of V onto W. Indeed, we are simply summing a finite collection of linear maps $V \to W$, since the composition $g \circ \pi \circ g^{-1}$ breaks down as a linear map of V, followed by the projection onto W, followed again by the action of g on W, taking us back to W. Also, since π is the *identity* map on W, it is easy to check that $g \circ \pi \circ g^{-1}$ is the identity map of W for each g: summing over all the elements of G and then dividing by the order, of course, it follows that ϕ is the identity map on the subspace W.

The linear projection ϕ is much better than the arbitrary π we began with: it is a homomorphism of G-representations! To check this, we just compute, for any $h \in G$,

$$h \cdot \phi(v) = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot \pi(g^{-1} \cdot v))$$

whereas
$$\phi(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot (h \cdot v)).$$
Unravelling the meaning of these expressions, we see that they are the same: we are simply summing up the expressions of the form $g \cdot (\pi(g^{-1} \star h) \cdot v)$ over the elements g of G in two different orders. In other words, for any $h \in G$, we have $h \cdot \phi(v) = \phi(h \cdot v)$, so that the linear map ϕ is a homomorphism of representations of G.

Now, because the kernel of a G-representation map is a G-representation, it follows that there is a subspace W' in V on which G acts and the decomposition
$$V \cong W \oplus W'$$
is a decomposition of G-representations. Indeed, $g \in G$ acts on $v = w + w'$ by $g \cdot v = g \cdot w + g \cdot w'$. Since both $g \cdot w$ and $g \cdot w'$ belong to the respective subspaces W and W', the G-action is "coordinatewise" as needed. □

Let us try to understand this proof in some examples we have already studied. Let V be the vertex permutation representation of D_4 and consider the sub-representation L spanned by $(-1, 1, -1, 1)$. How can we construct a D_4-representation complement?

First, take any vector space complement U—say U consists of the vectors in $\mathbb{R}$ whose last coordinate is zero. The induced projection $\pi : V \to L$ maps
$$\pi : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} -x_4 \\ x_4 \\ -x_4 \\ x_4 \end{bmatrix}.$$
Now averaging over the eight elements of D_4 we have
$$\phi : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \frac{1}{8} \sum_{g \in D_4} g \cdot \pi(g^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}).$$
Thinking about the action of D_4 on the sub-representation L of V, we see that the element r_1, r_3, H and V all act by multiplication by -1, where as the others act by the identity. Thus this expression (with the

elements ordered as in the table on page 3) simplifies as $\frac{1}{8}$ of

$$\pi\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}-\pi\begin{bmatrix}x_2\\x_3\\x_4\\x_1\end{bmatrix}+\pi\begin{bmatrix}x_3\\x_4\\x_1\\x_2\end{bmatrix}-\pi\begin{bmatrix}x_4\\x_1\\x_2\\x_3\end{bmatrix}-\pi\begin{bmatrix}x_2\\x_1\\x_4\\x_3\end{bmatrix}+\pi\begin{bmatrix}x_3\\x_2\\x_1\\x_4\end{bmatrix}-\pi\begin{bmatrix}x_4\\x_3\\x_2\\x_1\end{bmatrix}+\pi\begin{bmatrix}x_1\\x_4\\x_3\\x_2\end{bmatrix}.$$

Now applying the map π and simplifying further, we see that

$$\phi : V \to L$$

sending

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\mapsto\frac{1}{4}(x_2+x_4-x_1-x_3)\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$$

is a D_4 linear map. [Of course, we have proven this in general in our proof of Theorem 15.1, but you should verify explicitly for yourself that ϕ respects the action of D_4.]

Now, the kernel of ϕ, like any homomorphisms of G-representations, will be a G-representation. Indeed, it is the three-dimensional representation $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 + x_3 = x_2 + x_4\} \subset \mathbb{R}^4$. Thus $V \cong L \oplus W$ *as representations* of G. Now, since W has an invariant subspace, namely the space L spanned by $(1, 1, 1, 1)$, we can repeat this process to construct a G-representation complement to L in W. You should do it, and verify that this produces the sub-representation T' we found above, spanned by $(1, 1, -1, -1)$ and $(1, -1, -1, 1)$.

Again, we have decomposed the four-dimensional G-representation V as the direct sum $T' \oplus L \oplus \mathbb{R}(1, 1, 1, 1)$. This is essentially an algorithm for decomposing any representation: we keep choosing invariant subspaces and constructing their G-invariant complements until none of the sub-represents we construct this way has any invariant subspaces. Thus we have proved:

Theorem 9.7. *Any finite dimensional real or complex representation of a finite group can be decomposed into a direct sum of irreducible sub-representations.*

Such a representation is said to be *completely reducible* or *semi-simple.*

The theorem does not say anything about *what* irreducible representations appear in the decomposition of a given one, and whether or not there might be more than one to decompose in this way. Indeed, we made several choices along the way to proving it: a choice of an original invariant subspace, then the choice of a vector space complement. Might these result in different decompositions? Or, might all choices lead ultimately to the same decomposition—might there be a unique way to decompose into irreducibles? The rigidity in Example 13.1 makes this plausible, perhaps.

Unfortunately, it is too optimistic to hope for a unique decomposition. Indeed, consider the trivial representation of any group G on a two-dimensional vector space V. Now *any* decomposition of V as a vector space will also be a decomposition of V as a G-representation, and clearly, there are many ways to decompose V into two one-dimensional subspaces. On the other hand, these one-dimensional sub-representations are all isomorphic to eachother, so it is true that the ismorphism-types of G-representations and the number of times each appears is the same for any decomposition. We will soon prove a theorem asserting something like this very generally.

Remark 9.8. Why the restriction on the field? The restriction to $\mathbb{R}$ or $\mathbb{C}$ is not really necessary, but is just to keep everything as familiar as possible. Indeed, the statement is valid for any field of characteristic zero—-for example, the rational numbers $\mathbb{Q}$, the field of Laurent polynomials $\mathbb{C}((t))$ or many others. The only restriction we have is that "division by $|G|$ makes sense." For example, if G where a group with p elements for some prime number p, we could not consider vector spaces over the finite $\mathbb{F}_p$ of p elements. On the other hand, we can state that *Every finite dimensional representation over a field $\mathbb{F}$ of a finite group G decomposes as a sum of irreducible representations, provided the order of G is invertible in $\mathbb{F}$.*

9.1. **A digression on some uses of representation theory.** Suppose that a group acts on a set X, and that set X has

Put differently, if X belongs to some interesting *category* of mathematical objects, we often look at automorphisms of X in that category instead of merely the automorphisms of X as a set. The term category can be taken here in its precise technical meaning or more informally, depending on the reader's background and inclination.[7] In the category of vector spaces, an automorphism is simply an invertible linear map, so another notation for subgroup $GL(X)$ of Aut X that preserve the vector space structure of X is $\mathrm{Aut}_{vect.sp}(X)$. The notation $\mathrm{Aut}_{vect.sp}(X)$ indicates that we are considering automorphisms in the category of vector spaces.

Roughly speaking, a *category* is a collection of mathematical objects with some common structure, and a notion of mappings between them respecting that structure. For example, we have the category of groups (with group homomorphisms), the category of vector spaces over a fixed field (with linear mappings), the category of topological spaces (with continuous mappings), the category of smooth manifolds (with smooth mappings), and the category of sets (with set mappings), to name just a few familiar categories.

In any category (whether groups, topological spaces, etc), there is a notion of "sameness" for our objects: groups are "the same" if they are isomorphic, topological spaces are "the same" if they are homeomorphic, and so on. In general, $X \to Y$ and $Y \to X$ which are mutually inverse. The *automorphisms,* or self-isomorphisms, of a fixed object X in any category form a group under composition. For example, the automorphisms of a set X is simply the group Aut X already discussed, and the automorphisms of a vector space V is the group $\mathrm{GL}(V)$ of linear transformations of V. Likewise, if X is a topological space, the group $\mathrm{Aut}_{top}(X)$ of self-homeomorphisms from to itself is a group, which may be very large in general.

A **functor** is a mapping from one category to another, which of course, preserves the structures. For example, a functor Γ from the category of topological spaces to the category of groups is a gadget which assigns to each topological space X, some group $\Gamma(X)$, *and to each continuous mapping of topological spaces* $X \to Y$ some corresponding group homomorphism $\Gamma(X) \to \Gamma(Y)$. For example, the assignment of the fundamental group to each topological space is a functor from the category $\{\mathbf{Top}\}$ to $\{\mathbf{Gp}\}$. Naturally, the assignment must satisfy some basic properties in order to be a functor: it should send the identity map in one category to the identity map in the other for example, and

[7]The language of category theory is so ubiquitous in algebra that, even though we do not need it, it is probably a good idea for the student to start hearing it, at least informally.

it is should preserve compositions. The reader can look up the precise definition of a functor in any book on category theory (or most graduate level texts on algebra).

Representation theory is so useful in physics and mathematics because group actions—symmetries— are everywhere, on all sorts of structures from individual molecules to space-time. We have invented nice functors to transform these actions into actions on vector spaces—that is, into linear representations of groups. And finally, linear algebra is something we have plenty of tools for—even computers can be easily programmed to do linear algebra. So although linear representations of groups on vector spaces may seem quite abstract and algebraic, it is an excellent way of understanding symmetry at the micro-and-macroscopic levels throughout the universe.

10. Sub-representations.

Unless otherwise explicitly stated, we now consider only finite dimensional *linear representations.*

A sub-representation is a subvector space which is also a G-representation under the *same* action of G. More precisely

Definition 10.1. Let V be a linear representation of a group G. A subspace W of V is a sub-representation if W is invariant under G—that is, if $g \cdot w \in W$ for all $g \in G$ and all $w \in W$.

In terms of the group homomorphism

$$\rho : G \to GL(V),$$

W is a sub-representation if and only if ρ factors through the subgroup

$$G_W = \{\phi \in GL(V) \mid \phi(W) \subset W\}$$

of linear transformations stabilizing W.

For example, every subspace is a sub-representation of the trivial representation on any vector space, since the trivial G action obviously takes every subspace back to itself. At the other extreme, the tautological representation of D_4 on $\mathbb{R}^2$ has no proper non-zero sub-representations: there is no line taken back to itself under *every* symmetry of the square, that is, there is no line left invariant by D_4.

The vertex permutation representation of D_4 on $\mathbb{R}^4$ induced by the action of D_4 on a set of four basis elements $\{e_1, e_2, e_3, e_4\}$ indexed by the vertices of a square *does have* a proper non-trivial sub-representation. For example, the one dimensional subspace spanned by $e_1+e_2+e_3+e_4$ is fixed by D_4— when D_4 acts, it simply permutes the e_i so their sum remains unchanged. Thus for all $g \in G$, we have $g \cdot (\lambda, \lambda, \lambda, \lambda) = (\lambda, \lambda, \lambda, \lambda)$ for all vectors in this one-dimensional subspace of $\mathbb{R}^4$. That is, D_4 acts trivially on this one-dimensional sub-representation.

Another sub-representation of the vertex permutation representation of D_4 on $\mathbb{R}^4$ is the subspace $W \subset \mathbb{R}^4$ of vectors whose coordinates sum to 0. Clearly, when D_4 acts by permuting the coordinates, it leaves their sum unchanged. For example H sends $(1, 2, 3, -6)$ to $(-6, 3, 2, 1)$ if the vertices are labelled counterclockwise from the upper right. The space W is a three dimensional sub-representation of the permutation representation of D_4 on $\mathbb{R}^4$. Note that W is a *non-trivial* sub-representation the elements of G *do* move around the vectors in the space W.

10.1. **The Standard Representation of S_n.** One important representation is the standard representation of S_n, which is defined as a sub-representation of the permutation representation of S_n. Let S_n act on a vector space of dimension n, say $\mathbb{C}^n$, by permuting the n vectors of a fixed basis (say, the standard basis of unit column vectors in $\mathbb{C}^n$). Note that the subspace spanned by the sum of the basis elements is fixed by the action of S_n—that is, it is a sub-representation on which S_n acts trivially. But more interesting, the $n-1$-dimensional subspace

$$W = \{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid \sum x_i = 0\} \subset \mathbb{C}^n$$

is also invariant under the permutation action. This is called the **standard representation** of S_n.

11. Direct Sums of Representations.

Suppose a group G acts on the vector spaces V and W (over the same field, $\mathbb{F}$). We can define an action of G "coordinate-wise" on their direct sum of follows:

$$g \cdot (v, w) = (g \cdot v, g \cdot w) \in V \oplus W.$$

Note that if V has basis $v_1, \dots, v_n$ and W has basis $w_1, \dots w_m$, then the direct sum has basis $v_1, \dots, v_n, w_1, \dots w_m$ (where v_i is interpreted to mean $(v_i, 0)$, etc). With this choice of basis for the direct sum, the matrix of every g acting on $V \oplus W$ will be the block diagonal matrix

$$\begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

obtained from the $n \times n$ matrix $\rho_1(g)$ describing the action of g on V in given basis, and the $m \times m$ matrix $\rho_2(g)$ describing the action of g on W in its given basis.

For example, we have a three-dimensional real representation of D_4 defined as follows:

$$g \cdot (x, y, z) = (g \cdot (x, y), z),$$

where $\dot{g}(x, y)$ denotes the image of (x, y) under the tautological action of D_4 on $\mathbb{R}^2$. In the standard basis, for example, the element $r_1 \in D_4$ acts by the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This representation is the *direct sum* of the tautological and the one-dimensional trivial representations of D_4.

Of course, we also have a representation of D_4 on $\mathbb{R}^3$ where D_4 acts trivially in the x-direction and by the tautological action in the yz plane direction: $g \cdot (x, y, z) = (x, g \cdot (y, z))$. This is also a direct sum of the trivial and tautological representation. In fact, just thinking about these actions there is a very strong sense in which they are "the same." We are led to the following definition:

Definition 11.1. Two linear representations V and W of G (over the same field) are *isomorphic* if there is a vector space isomorphism between them that preserves the G-action—that is, if there exists an isomorphism $\phi : V \to W$ such that $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ and all $g \in G$.

The idea that ϕ preserves the G action is also expressed by the following commutative diagram, which must hold for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ \phi \downarrow & & \phi \downarrow \\ W & \xrightarrow{g} & W \end{array}.$$

For example, the two representation of D_4 discussed above are isomorphic under the isomorphism which switches the x and z coordinates.

We are especially interested in decomposing representations in direct sums of sub-representations. For example, the permutation representation of S_n on $\mathbb{C}^n$ is easily seen to be isomorphic to the direct sum of the trivial and standard sub-representations discussed in 10.1 above:

$$\mathbb{C}^n \cong \mathbb{C} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \bigoplus W,$$

where W is the subspace of vectors whose coordinates sum to zero.

12. Mappings of Representations.

Given two representations of a fixed group G over the same field, we define a mapping between them as follows:

Definition 12.1. A *homomorphism* of (linear) G-representations is a map $\phi : V \to W$ which preserves both the vector space structure and the G-action. That is, it is a linear map ϕ of vector spaces (over the same field) satisfying $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ and all $g \in G$.

There are many different words for the notion of a homomorphism of representations, probably because representation theory is used in so many different places in mathematics and science. The terms "G-linear mapping", "G-module mapping," or even just "a map of representations of G" are common. The adjective "linear" is often suppressed: if the representations are linear, so also are the homorphisms between them assumed to be *linear* homomorphisms. But, depending on the context, it may also be interesting to study group actions on, say, a topological space, in which case the corresponding representations should be *continuous,* and the maps between them as well should be continuous.

An isomorphism of G-representations is simply a bijective homomorphism. On the other hand, it might be better to think of an isomorphism as a homomorphism $\phi : V \to W$ which has an inverse homomorphism, because this way of thinking is valid more broadly throughout mathematics.[8] But for representations, you can check that if $\phi : V \to W$ is a bijective linear transformation, then the inverse map $\phi^{-1} : W \to V$ is also a linear map, and that if ϕ preserves the group action, so does ϕ^{-1}. To wit: $g \cdot \phi(v) = \phi(g \cdot v)$ for all $v \in V$ if and only if $g^{-1} \cdot \phi^{-1}(w) = \phi^{-1}(g^{-1} \cdot w)$ for all $w \in W$.

The **inclusion** of any sub-representation W of a representation V of a group G provides an example of an injective homomorphism of representations. Indeed, $\phi : W \hookrightarrow V$ obviously satisfies $g\phi(w) = \phi(gw)$ because, by definition of sub-representation, the action of G is the same on elements of W whether we think of them in W or V.

The **projection** $\pi : V \oplus W \to V$ from a direct sum of G-representations onto either factor is an example of a surjective homomorphism. Again, that this map is G-linear is easy to verify directly from the definitions (do it!).

Although it is far from obvious, these examples—isomorphism, inclusion, and projection—are virtually the *only* homomorphisms between representations, at least for finite dimensional representations of finite groups G. Perhaps this is not surprising: a similar fact is true for vector spaces. If $\phi : V \to W$ is a linear map of vector spaces, then because $V \cong im\,\phi \oplus ker\,\phi$, every mapping of vector spaces essentially factors as projection onto the image followed by inclusion of the image into the target. On the other hand, nothing like this is remotely true for most sorts of maps between mathematical objects: there are all sorts of group homomorphisms, such as $\mathbb{Z} \to \mathbb{Z}_n$, which are not of this sort.

13. Homomorphisms of Representations are rare.

Let us consider two representations of D_4: the tautalogical representation T (which is the vector space $\mathbb{R}^2$, with the tautological D_4 action) and the vertex permutation representation V (namely, the vector space $\mathbb{R}^4$ with basis elements indexed by the vertices of a square, with the action of D_4 given by permuting the vertices according to the action of D_4 on the vertices.) Of course, there are many linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^4$. Do any of these preserve the group action? That is, are any

[8]Indeed, this is how the notion of "same-ness" or isomorphism can be defined in any category. For example, an isomorphism in the category of topological spaces— or homeomorphism— is a continuous map which has a continuous inverse; it is not enough to have a bijective continuous map.

of them homomorphisms of the D_4-representations $T \to V$. Of course, stupidly, we could take the zero-map $T \to V$ mapping every element of $\mathbb{R}^2$ to $\mathbb{R}^4$. This preserves the group action, but it is not interesting. Can we find any injective maps?

Let us consider whether the inclusion

$$i : \mathbb{R}^2 \to \mathbb{R}^4$$

sending

$$(x, y) \mapsto (x, y, 0, 0)$$

is D_4-linear. For example, does this preserve the action of the element r_1 in D_4. We have that $r_1 \cdot (x, y) = (-y, x)$ for the representation T, which yields $(-y, x, 0, 0)$ under the inclusion map i, where as $r_1 \cdot (x, y, 0, 0) = (0, x, y, 0)$ for the representation V. This means that i *does not* respect the group structure— the linear map i is *not* a map of representations of D_4.

13.1. **Searching for sub-representations.** Can there be *any* injective map of representations $T \to V$? If so, then the image of T would be a two-dimensional subspace of the four-dimensional space V where the the vertex permutation representation restricts to the tautological representation. Is this possible? Essentially, we are asking whether there is any two-dimensional square in four space on which the vertex permutation action of D_4 (given by permuting the coordinates as D_4 permutes the vertices of a square) agrees with the usual symmetries of that square. Clearly, this can happen only if there is a set of four points in $\mathbb{R}^4$ spanning a two-dimensional subspace invariant under this action.

To understand the permutation action of D_4 on $\mathbb{R}^4$ it is helpful to identify D_4 with the subgroup of S_4 generated by the 4-cycle (1234) (the rotation r_1) and the transposition (13) (the reflection A). Formally, we are embedding D_4 in S_4 by sending each symmetry to the corresponding permutation of the vertices. The 4-cycle acts by cycling the coordinates (one spot to the right, say), and the transposition acts by switching the first and third coordinates. Since D_4 is generated by these two elements, a subspace is invariant under D_4 if and only if it is invariant under these two elements (Prove it!).

To see whether the permutation representation has any sub-representation isomorphic to T, then, we are looking for four points in $\mathbb{R}^4$ spanning a two-dimensional subspace, and invariant under these two transformations, which can serve as the vertices of a square. Can one of these points be, say $(1, 0, 0, 0)$? Note that when r_1 acts on this vector in V, it moves it to $(0, 1, 0, 0)$. When it acts again—in other words when r_2 acts on $(1, 0, 0, 0)$— we get $(0, 0, 1, 0)$. Again, and we get $(0, 0, 0, 1)$, and then we cycle back to $(1, 0, 0, 0)$. Thus, any vector subspace of V containing $(1, 0, 0, 0)$ and invariant under D_4 must contain each of the four standard basis elements—that is, it must be V, since these span V. Indeed, it is easy to check that this sort of thing happens for "most vectors:" typically, the orbit of a randomly chosen vector in V will consisting of (eight) vectors which span V.

To find interesting sub-representations of V, we can look for non-zero vectors whose orbits span proper subspaces of V. One way is to find vectors with small orbits. For example, the vector $(1, 1, 1, 1)$ is fixed by D_4. It spans a one dimensional sub-representation of V where D_4 acts trivially, as we have already noted.

Another vector with a small orbit is $w = (-1, 1, -1, 1)$. Note that r_1 acts on w to produce $(1, -1, 1, -1)$, which is $-w$. Also, the reflection A acts by permuting the first and third coordinates, which means it fixes w. Since *every* element in D_4 is an iterated composition of the generators r_1 and A, we see that the D_4-orbit of w is the two element set $\{w, -w\}$. Thus the one-dimensional subspace spanned by w is a sub-representation of V. This is *not* a trivial representation—some element acts by multiplication by -1.

How does this help us find two dimensional sub-representations of V isomorphic to T, or show none exist? Well, clearly if any such two-dimensional sub-representation exists, it can not contain $(1, 1, 1, 1)$, since T fixes *no* subspace. Nor could it contain $(1, -1, 1, -1)$ since r_1 acts there by multiplication by -1 but r_1 is *never* scalar multiplication on T. Thus, we need to look for a vector in V whose orbit does not contain either of these two special vectors (nor anything in their span).

Consider the vector $(1, 1, -1, -1)$. Its orbit produces the four points

$$(1, 1, -1, -1), \ (-1, 1, 1, -1), \ (-1, -1, 1, 1), \ (1, -1, -1, 1),$$

and so it *does* generate a two-dimensional sub-representation T' of V (which has basis, for example, $(1, 1, -1, -1), (-1, 1, 1, -1)$.) It is easy to check that the vertex permutation action of D_4 on T' *does restrict* to the tautological representation of D_4 on this two-plane. Indeed, the four points described above serve nicely as the vertices of a square on which D_4 acts by the usual symmetry actions. So V *does contain* a sub-representation isomorphic to T, embedded in a rather special way as a skew plane.

Finally, since these three sub-representations span V, there is a direct sum decomposition of representations:

$$V \cong T' \bigoplus \mathbb{R}(1,1,1,1) \bigoplus \mathbb{R}(1,-1,1,-1),$$

where the first summand has basis $\{(1,1,-1,-1),(-1,1,1,-1)\}$ and is isomorphic to the tautological representation of D_4 and the second summand is a trivial representation of D_4, but the third is not isomorphic to either of these. Furthermore, none of these three sub-representations can be further decomposed. Although it is not obvious, we will soon prove that *every representation* of a finite group on a real or complex vector space *decomposes as a direct sum of irreducible representations.*

We isolate for future reference a simple idea used in the previous example:

Lemma 13.1. *Let V be a linear representation of a group G. Let W be a sub-vector space of V, spanned by the elements $w_1, \dots, w_t$. Then W is a sub-representation if and only if $g \cdot w_i \in W$ for all $i = 1, \dots, t$ and each g in one fixed generating set for G.*

Proof. We leave this as an easy exercise. One slightly subtle point that is needed: If an element g leaves W invariant, then also g^{-1} does. Indeed, the linear transformation $g : V \to V$ is invertible, which means that restricted to W, it is also invertible. So g^{-1} defines its inverse, also on W. □

13.2. **The kernel, image and cokernel of a map of representations.** The category[9] of G-representations and their maps is very nice from an algebraic point of view: we can form kernel representations, image representations, quotient representations and cokernel representations.

Proposition 13.2. *Let $\phi : V \to W$ be a homomorphism of linear representations of a group G. Then the kernel is a sub-representation of V and the image is a sub-representation of W.*

[9] The reader can take the word *category* in an informal sense, or in its full mathematically technical sense here.

Proof. Since the kernel and image of ϕ are subvector spaces, we only have to check that each is invariant under the G-action. Suppose that $v \in V$ is in the kernel of V, and $g \in G$. We need to check that $g \cdot v$ is in the kernel of ϕ. But $\phi(g \cdot v) = g\phi(v)$ because ϕ is G-linear, and $g\phi(v) = g \cdot 0 = 0$ since v is in the kernel of ϕ and g acts by linear transformations (so preserves the zero).

The proof for the image is also straightforward: $g \cdot \phi(v) = \phi(g \cdot v)$ is in the image of ϕ. □

Whenever there an inclusion of linear G-representations $W \subset V$, the **quotient representation** V/W can be defined. Indeed, we define V/W as the quotient vector space with the G-action defined by

$$g \cdot \overline{v} = \overline{g \cdot v}$$

where v is any representative of the class. This does not depend on the choice of representative, since if $\overline{v} = \overline{v}'$, then $v - v' \in W$, and whence $g \cdot v - g \cdot v' \in g \cdot W \subset W$, which of course means $g \cdot v$ and $g \cdot v'$ represent the same class of V/W.

In particular, the **cokernel** $V/\text{im}\phi$ of any homomorphism $\phi : V \rightarrow W$ of representations of G is also a representation of G.

14. Irreducible Representations

Definition 14.1. A representation of a group on a vector space is *irreducible* if it has no proper non-trivial sub-representations.

For example, the tautological representation T of D_4 is irreducible: if there were some proper non-zero sub-representation, it would have to be one dimensional, but clearly no line in the plane is left invariant under the symmetry group of the square. Indeed, the plane $\mathbb{R}^2$ is irreducible even under the action of the subgroup R_4 of rotations.

On the other hand, the vertex permutation representation is *not* irreducible. For example, the line spanned by the vector $(1, 1, 1, 1)$ or either of the two other subspaces described above in Example 13.1 are non-zero proper sub-representations.

Another example of an irreducible representation is the tautological representation of $GL(V)$ on a vector space V. Indeed, given any non-zero vector $v \in V$, we can always find a linear transformation taking v

to any other non-zero vector. In other words, $GL(V)$ acts transitively on the set of non-zero vectors, so there can be no *proper* subset of V left invariant under this action, other than the single element set $\{0\}$.

On the other hand, the action of $GL_n(\mathbb{R})$ on the space of $n \times m$ real matrices by left multiplication (in other words, by row operations) is *not irreducible.* For example, the subspace of $n \times m$ matrices whose last column is zero certainly invariant under row operations.

To check whether or not a representation is irreducible, it is helpful to think about *sub-representations generated by certain elements:*

Definition 14.2. Let V be any representation of a group G, and S be any sub*set*[10] of V. The *sub-representation* generated by S is the smallest sub-representation of V containing the set S, that is

$$\bigcap_{W \text{ sub-rep of } V \text{containing } S} W.$$

Put differently, the sub-representation of a G-representation V generated by a subset S is the vector space spanned by the vectors in the G-orbits of all the elements in S. For example, the sub-representation of the vertex permutation representation V of D_4 generated by $(1,0,0,0)$ is the whole of V, since r_1 takes $(0,1,0,0)$, r_2 takes $(1,0,0,0)$ to $(0,0,1,0)$ and r_3 takes $(1,0,0,0)$ to $(0,0,0,1)$.

The following easy fact is more or less obvious:

Proposition 14.3. *A representation is irreducible if and only if it is generated (as a representation) by* **any one** *non-zero vector.*

Example 14.4. Consider the additive group $G = (\mathbb{R},+)$. This has a representation on $\mathbb{R}^2$ given by the group homomorphism

$$(\mathbb{R},+) \to GL_2(\mathbb{R})$$
$$\lambda \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Explicitly, the element λ in G acts by sending a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x+\lambda y \\ y \end{bmatrix}$. Clearly, the x-axis is a G-invariant subspace—indeed G acts trivially on the one-dimensional subspace of vectors of the form

[10] emphasis: S need be a sub*set* only, not necessarily a sub-representation or even a subspace

$\begin{bmatrix} a \\ 0 \end{bmatrix}$. In particular, this two dimensional representation of $(\mathbb{R}, +)$ is *not* irreducible.

Are there any other interesting sub-representations? Let us take any vector not on the x-axis, say $\begin{bmatrix} a \\ b \end{bmatrix}$ where $b \neq 0$. When some non-trivial element λ in G acts on $\mathbb{R}^2$, it sends this element to $\begin{bmatrix} a + \lambda b \\ b \end{bmatrix}$, which is obviously *not* a scalar multiple of $\begin{bmatrix} a \\ b \end{bmatrix}$ since $\lambda \neq 0$. Thus, these two vectors span the whole of $\mathbb{R}^2$. This shows that no vector off the x-axis can generate a proper sub-representation— the x-axis is the *only* proper non-zero sub-representation of V. Thus, this two-dimensional representation of the additive group $(\mathbb{R}, +)$ has exactly one irreducible non-zero proper sub-representation. In particular, it can not be decomposed into a direct sum of irreducible representations!

Example 14.5. Caution! Whether or not a representation is irreducible may depend on the *field* over which it is defined. For example, we have already observed that the tautological representation of the rotation group R_4 on $\mathbb{R}^2$ is irreducible. Explicitly, this is the representation

$$R_4 \to GL_2(\mathbb{R})$$

$$r_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad r_2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad r_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Of course, we can also think of this as a representation of the group $\mathbb{Z}_4$, in which case, we will call it the **rotation representation** of $\mathbb{Z}_4$. Of course, the elements $1, -1$ and 0 make sense in any field $\mathbb{F}$, so we can consider the "same" action of R_4 on $\mathbb{F}^2$, for any field! For example, the group map above can be taken to have target say, $GL_2(\mathbb{C})$, instead of $GL_2(\mathbb{R})$.

Let us consider whether the complex rotation representation has any interesting sub-representations. If so, there must be some complex vector $\begin{bmatrix} a \\ b \end{bmatrix}$ which generates a one-dimensional sub-representation of $\mathbb{C}^2$. In particular, we must have

$$r_t \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \lambda_t \begin{bmatrix} a \\ b \end{bmatrix}$$

for each $r_t \in R_4$ and some complex scalar λ_t, depending on r_t. Since R_4 is generated by r_1, we see that it is enough to check this condition just for r_1: the condition for r_t will follow with λ_t taken to be λ_1^t (of course, the condition for the identity element e holds in any case, with the corresponding λ taken to be 1).

That is, the one-dimensional subspace of $\mathbb{C}^2$ spanned by $\begin{bmatrix} a \\ b \end{bmatrix}$ is invariant under R_4 if and only if $\begin{bmatrix} a \\ b \end{bmatrix}$ is an *eigenvector* of the linear transformation r_1. In this case, λ_1 is the corresponding eigenvalue.

Over $\mathbb{C}$, of course, every linear transformation has eigenvectors! Indeed, we can find them by computing the zeros of the characteristic polynomial. For the transformation given by r_1, which is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the characteristic polynomial is

$$\chi(T) = \det \begin{pmatrix} T & 1 \\ -1 & T \end{pmatrix} = T^2 + 1.$$

Its roots are the complex numbers i and $-i$, and we easily compute the corresponding eigenvectors to be

$$\begin{bmatrix} i \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Thus, the two dimensional complex rotation representation of R_4 decomposes into the two one-dimensional representations

$$\mathbb{C} \begin{bmatrix} i \\ 1 \end{bmatrix} \oplus \mathbb{C} \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

The idea of a *irreducible object* comes up in other algebraic categories as well.[11] For example, we can ask what *groups* have the property that they contain no (normal, say) proper non-trivial subgroups: these are called the *simple* groups. Mathematicians have spend the better part of the last century classifying all the finite simple groups. In other settings, this task is much easier: for example, a vector space has proper non-trivial subspaces if and only if it has dimension one.

[11] The language of category theory is so ubiquitous in algebra that, even though we do not need it, it is probably a good idea for the student to start hearing it, at least informally.

Of course, once one understands all the simple groups (or the "simplest objects" in any category), the next step is to understand how *every group* can be built up from these simple ones. Again, for vector spaces, this is quite easy to understand: every (finite dimensional) vector space is a direct sum of "simple"—meaning one-dimensional—vector spaces. For groups, the story is more complicated. For example, the group $\mathbb{Z}_4$ is *not* simple, as the element $\{\overline{0}, \overline{2}\}$ forms a proper subgroup. However, nor can $\mathbb{Z}_4$ be isomorphic to the direct sum of two smaller groups in a non-trivial way (prove it!). There is a whole theory of *extensions* for groups which treats the question of how arbitrary groups can be build up from simple ones. This is a beautiful story, but not one we digress to discuss here.

Returning to our main interest: what happens for representations? Can every representation of a group be built up out of irreducible ones in some easy to understand way? The answer is as nice as possible, at least for finite groups over familiar fields:

15. Complete Reducibility

Theorem 15.1. *Every finite dimensional representation of a finite group over the real or complex numbers decomposes into a direct sum of irreducible sub-representations.*

We explictly found such a decomposition for the vertex permutation representation V for D_4. (Example 13.1). The key to proving Theorem 15.1 is the following fact, valid even for infinite dimensional representations:

Theorem 15.2. *Every sub-representation of a real or complex representation of a finite group has a representation complement. That is, if W is a sub-representation of V, then there exists another sub-representation W' of V such that $V \cong W \oplus W'$ as representations of G.*

Remark 15.3. The restriction to $\mathbb{R}$ or $\mathbb{C}$ is not really necessary. Indeed, our proof is valid for any field of characteristic zero—for example, the rational numbers $\mathbb{Q}$ or the field of Laurent polynomials $\mathbb{C}((t))$—or more generally over any field in which $|G|$ is non-zero. However, if for example, G is a group of order p (prime), our proof fails (as does the theorem). Representation theory over fields of prime characteristic is tremendously interesting and important, especially in number theory, but in this course our ground field will be usually assumed to be $\mathbb{R}$ or $\mathbb{C}$, the cases of primary interest in geometry and physics.

The proof we given is not valid for infinite groups. However, we will later show that for many of the most interesting infinite groups, a similar statement holds.

Proof. Fix any *vector space complement* U for W inside V, and decompose V as a direct sum of vector spaces $V \cong W \oplus U$. Of course, if U happens to be G-invariant, we are done. But most likely it is not, so there is work to be done.

This vector space decomposition allows us to define a projection $\pi : V \to W$ onto the first factor. The map π is a surjective linear map which restricts to the identity map on W. Although π is *not* necessarily a homomorphism of G-representations, we will "average it over G" to construct a G-linear projection. To this end, define

$$\begin{aligned} \phi : V &\to W \\ v &\mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v). \end{aligned}$$

Although it looks complicated, this is really just a simple projection of V onto W. Indeed, we are simply summing a finite collection of linear maps $V \to W$, since the composition $g \circ \pi \circ g^{-1}$ breaks down as a linear map of V, followed by the projection onto W, followed again by the linear map g of W. Also, since π is the *identity* map on W, it is easy to check that $g \circ \pi \circ g^{-1}$ is the identity map of W for each g: summing over all the elements of G and then dividing by the order, of course, it follows that ϕ is the identity map on the subspace W.

The projection ϕ is much better than the arbitrary π we began with: it is a homomorphism of G-representations! To check this, we just compute, for any $h \in G$,

$$h \cdot \phi(v) = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot \pi(g^{-1} \cdot v))$$

whereas

$$\phi(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot (h \cdot v)).$$

Unravelling the meaning of these expressions, we see that they are the same: we are simply summing up the expressions of the form $g \cdot (\pi(g^{-1} \star h) \cdot v)$ over the elements g of G in two different orders. In other words, for any $h \in G$, we have $h \cdot \phi(v) = \phi(h \cdot v)$, so that the linear map ϕ is a homomorphism of representations of G.

Now, the advantage of a G-representation map is that its kernel is a G-representation. Let W' be the kernel of ϕ. Then G acts on W' and the vector space decomposition

$$V \cong W \oplus W'$$

is a decomposition of G-representations. Indeed, $g \in G$ acts on $v = w + w'$ by $g \cdot v = g \cdot w + g \cdot w'$. Since both $g \cdot w$ and $g \cdot w'$ belong to the respective subspaces W and W', the G-action is "coordinatewise" as needed. □

Let us try to understand this proof in some examples we have already studied. Let V be the vertex permutation representation of D_4 and consider the sub-representation L spanned by $(-1, 1, -1, 1)$. How can we construct a D_4-representation complement?

First, take any vector space complement U—say U consists of the vectors in $\mathbb{R}$ whose last coordinate is zero. The induced projection $\pi : V \to L$ maps

$$\pi : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \begin{bmatrix} -x_4 \\ x_4 \\ -x_4 \\ x_4 \end{bmatrix}.$$

Now averaging over the eight elements of D_4 we have

$$\phi : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \frac{1}{8} \sum_{g \in D_4} g \cdot \pi(g^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}).$$

Thinking about the action of D_4 on the sub-representation L of V, we see that the element r_1, r_3, H and V all act by multiplication by -1, where as the others act by the identity. On the other hand, on the four-dimensional representation V, the rotations act by cyclicly permuting the coordinates and the reflections by interchanging the respective coordinates, as discussed in Subsection 13.1. Thus this expression (with the elements ordered as in the table on page 3) simplifies as $\frac{1}{8}$ of

$$\pi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \pi \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix} + \pi \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix} - \pi \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} - \pi \begin{bmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{bmatrix} + \pi \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_4 \end{bmatrix} - \pi \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} + \pi \begin{bmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{bmatrix}.$$

Now applying the map π and simplifying further, we see that

$$\phi : V \to L$$

sending

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mapsto \frac{1}{4}(x_2 + x_4 - x_1 - x_3) \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

is a D_4 linear map. [Of course, we have proven this in general in our proof of Theorem 15.1, but you should verify explicitly for yourself that ϕ respects the action of D_4.]

Now, the kernel of ϕ, like any homomorphisms of G-representations, will be a G-representation. Indeed, it is the three-dimensional representation $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 + x_3 = x_2 + x_4\} \subset \mathbb{R}^4$. Thus $V \cong L \oplus W$ *as representations* of G. Now, since W has an invariant subspace, namely the space L spanned by $(1, 1, 1, 1)$, we can repeat this process to construct a G-representation complement to L in W. You should do it, and verify that this produces the sub-representation T' we found above, spanned by $(1, 1, -1, -1)$ and $(1, -1, -1, 1)$.

Again, we have decomposed the four-dimensional G-representation V as the direct sum $T' \oplus L \oplus \mathbb{R}(1, 1, 1, 1)$. The procedure we used is essentially an algorithm for decomposing any representation finite dimensional representation: we keep choosing invariant subspaces and constructing their G-invariant complements until none of the sub-representations we construct this way has any invariant subspaces. Thus we have proved:

Theorem 15.4. *Any finite dimensional real or complex representation of a finite group can be decomposed into a direct sum of irreducible sub-representations.*

Such a representation is said to be *completely reducible* or *semi-simple.*

The theorem does not say anything about *what* irreducible representations appear in the decomposition of a given one, or whether there might be more than one such decomposition. Indeed, we made several choices along the way: a choice of an original invariant subspace, then the choice of a vector space complement. Might these result in different decompositions? Or, might all choices lead ultimately to the same decomposition—might there be a unique way to decompose into irreducibles? At least the vertex permutation representation of D_4 appears to decompose in *only one way* into irreducibles.

Unfortunately, it is too optimistic to hope for a unique decomposition. Indeed, consider the trivial representation of any group G on a two-dimensional vector space V. Now *any* decomposition of V as a vector space will also be a decomposition of V as a G-representation, and clearly, there are many ways to decompose V into two one-dimensional subspaces. On the other hand, these one-dimensional sub-representations are all isomorphic to eachother, so it is true that the isomorphism-types of G-representations and the number of times each appears is the same for any decomposition. As we will soon prove, this degree of "uniqueness of the decomposition" *does* hold quite generally.

16. Uniqueness of Decomposition into Irreducibles.

Every finite dimensional real or complex representation of a finite group can be decomposed into irreducible representations (Theorem 15.1). Is this decomposition unique? In Example 13.1, we saw explicitly that *there is only one way* to decompose the vertex permutation representation into its three non-zero proper sub-representations. Might it be true in general that the decomposition of a representation into irreducible sub-representations is unique?

Naively posed, the answer is "NO." For example, *every* decomposition of a two-dimensional vector space into two one-dimensional subspaces will be a decomposition of the trivial representation of a group G. Of course, there are many different choices of vector-space decompositions! On the other hand, in this example, the isomorphism types of the indecomposable sub-representations, and the number of them appearing, is the same in every decomposition. A similar fact holds in general:

Theorem 16.1. *Let V be a finite dimensional real or complex representation of a finite group G. Then V has a unique decomposition into sub-representations*

$$V = V_1 \oplus \cdots \oplus V_t$$

where each V_i is isomorphic to a direct sum of some number of copies of some fixed irreducible representations W_i, with $W_i \ncong W_j$ unless $i = j$. That is, given two different decompositions of V into non-isomorphic *irreducible sub-representations*

$$W_1^{a_1} \oplus \cdots \oplus W_t^{a_t} = U_1^{b_1} \oplus \cdots \oplus U_r^{b_r},$$

where the W_i (respectively U_i) are all irreducible and non-isomorphic, then after relabling, $t = r$, $a_i = b_i$, the sub-representations $W_i^{a_i}$ equal the $U_i^{b_i}$ for all i, and corresponding W_i are isomorphic to U_i for all i.

In other words, the irreducible sub-representations that appear as summands *are uniquely determined up to isomorphism,* as is their *multiplicity* (or number of them appearing) in V. Furthermore, the summands V_i consisting of the span of all the vectors in the subrepresentations W_i are *uniquely determined* sub-representations of V, although the decomposition of V_i into the components isomorphic W_i may not be.

The vertex permutation representation of D_4 discussed in Example 13.1 admits a completely unique decomposition because the three irreducible sub-representations we identified are all non-isomorphic and the multiplicity of each is one. According to Theorem 16.1, this is therefore the only decomposition.

To prove Theorem 16.1, we first need the following, quite general, lemma.

Lemma 16.2. *A homomorphism of irreducible representations is either zero or an isomorphism.*

Proof. Consider a homomorphism $V \to W$ of irreducible representations. Since the kernel is a sub-representation of V, we see that the kernel is either 0 or all of V. Likewise, since the image is a subrepresentation of W, it is either zero or all of W. Thus, a non-zero homomorphism between irreducible representations must be both injective and surjective. The lemma follows. □

Proof of Theorem. Suppose that $V = W_1^{a_1} \oplus \cdots \oplus W_t^{a_t} = U_1^{b_1} \oplus \cdots \oplus U_r^{b_r}$ are two different decompositions of V into irreducible representations of G. The composition of the inclusion of W_1 in V followed by projection onto U_i

$$W_1 \hookrightarrow V \to U_i$$

is a G-linear map of irreducible representations, so must be either zero or an isomorphism. It can not be the zero map for all i, so some U_i— after relabling say U_1— is isomorphic to W_1. Repeating this argument for W_2 we see that $W_2 \cong U_2$ and so on until each W_j is paired with some U_i. Reversing the roles of the U_j and W_i we see that the isomorphism types appearing in both decompositions are precisely the same.

Now it remains only to show that $W_i^{a_i}$ and $U_i^{b_i}$ are precisely the *same* sub-representations of V (not just isomorphic—but literally the same subspaces). For this, we again consider the composition of inclusion with projection:

$$W_1^{a_1} \hookrightarrow V \to U_2^{b_2} \oplus \cdots \oplus U_t^{b_t},$$

where the second map is projection onto all the summands complementary to $U_1^{b_1}$. It is easy to see that this must be the zero map. (If not, then by restricting and projecting onto selected factors, we'd have a non-zero map $W_1 \to U_i$, for $i \neq 1$.) This means that $W_1^{a_1}$ is contained in the kernel of the projection, in other words,

$$W_1^{a_1} \subset U_1^{b_1}.$$

Reversing the roles of U and W, we get the reverse inclusion. It follows that $W_1^{a_1} = U_1^{b_1}$, and since U_1 and W_1 have the same dimension, also $a_1 = b_1$. Clearly, we can apply this argument for each index $i = 2, \ldots, n$, and so the theorem is proved. □

17. Irreducible representations over the complex numbers.

We have seen that irreducible representations are quite rigid. This is even more true for representations over the complex numbers.

Lemma 17.1 (Schur's Lemma)**.** *The only self-isomomorphisms of a finite dimensional irreducible representation of a group G over the complex numbers are given by scalar multiplication.*

Proof. Fix an isomorphism $\phi : V \to V$ of complex representations of G. The linear map ϕ must have an eigenvalue λ over $\mathbb{C}$, and so also some non-zero eigenvector v. But then the G-linear map

$$V \to V$$

$$x \mapsto [\phi(x) - \lambda x]$$

has the vector v in its kernel, which is again a representation of G. Since V is irreducible, the kernel must be all of V. In other words, we have $\phi(x) = \lambda(x)$ for all $x \in V$, which is to say, ϕ is multiplication by λ. □

Schur's Lemma is false over $\mathbb{R}$. For example, rotation through 90^0 (or indeed any angle) is obviously an automorphism of the irreducible representation of the rotation subgroup R_4 acting tautologically on the real plane. The eigenvalues of this rotation map are non-real complex numbers, so we can not argue as above over the reals. Indeed, consider the "same" representation over $\mathbb{C}$, that is, by composing

$$R_4 \subset GL_2(\mathbb{R}) \hookrightarrow GL_2(\mathbb{C})$$

to get a representation of R_4 on $\mathbb{C}^2$. Over $\mathbb{C}$, this representation is not irreducible. Indeed, it decomposes into the two representations spanned by $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} i \\ -1 \end{bmatrix}$ where the generator r_1 acts by multiplication by i and $-i$ respectively. See also Example **??**.

Schur's Lemma has some striking consequences for the classification of representations of finite groups over the complex numbers.

17.1. **Representations of Abelian groups.** Let V be any representation of a group G. Each element $g \in G$ induces a linear map

$$V \to V$$

$$v \to g \cdot v.$$

Is this a homomorphism of representations? Not usually! It is a homomorphism of G-representations if and only if it commutes with the action of each $h \in G$, that is, if and only if

$$h \cdot g \cdot v = g \cdot h \cdot v$$

for all h in G and all $v \in V$. Of course, this rarely happens: imagine that G is $GL_n(\mathbb{R})$, and g and h are non-commuting matrices.

However, if G is abelian, or more generally if g is in its center, then the action of g on V is G-linear. Indeed, then

$$h \cdot (g \cdot v) = (h \star g) \cdot v = (g \star h) \cdot v = g \cdot (h \cdot v)$$

for all $h \in G$ and all $v \in V$.

This observation, together with Schur's lemma, leads to the following striking result:

Proposition 17.2. *Every finite dimensional irreducible complex representation of an abelian group is one-dimensional.*

Proof. Suppose that V is a finite dimensional irreducible representation of an abelian group G. Then the action of G on V is G-linear, so by Schur's lemma, the action of g on V is simply multiplication by some scalar, $\lambda(g)$, which of course, can depend on the element g. In any case, every subspace is invariant under scalar multiplication, so every subspace is a sub-representation. So since V is irreducible, it must have dimension one. □

This does not mean that the representation theory of abelian groups over $\mathbb{C}$ is completely trivial, however. An irreducible representation of an abelian group is a group homomorphism

$$G \to GL_1(\mathbb{C}) = (\mathbb{C}^*, \cdot),$$

and there can be many different such homomorphisms (or elements of the "dual group"). Furthermore, it may not be obvious, given a representation of an abelian group, how to decompose it into one-dimensional sub-representations. Schur's Lemma guarantees that there is a choice of basis for V so that the action of an abelian group G is given by multiplication by

$$\begin{pmatrix} \lambda_1(g) & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2(g) & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_n(g) \end{pmatrix}$$

where the diagonal entries $\lambda_i : G \to \mathbb{C}^*$ are group homomorphisms. But it doesn't tell us how to find this basis or the functions λ_i.

17.2. **Irreducible representations of S_3.** Let us now try the first non-abelian case: can we identify all the irreducible representations of S_3 over $\mathbb{C}$, up to isomorphism?

Suppose V is complex representation of S_3. The group S_3 is generated by $\sigma = (123)$ and $\tau = (12)$, so to find sub-representations, it is enough to find subspaces of V invariant under the action of both σ and τ.

First consider the action of σ on V. Let v be an eigenvector for this action, with eigenvalue θ. Let $\tau v = w$. Because we know that $\tau\sigma\tau = \sigma^2$ in S_3, is easy to check that w is also an eigenvector for σ with eigenvalue θ^2. Indeed:

$$\sigma \cdot w = \sigma\tau \cdot v = \tau\sigma^2 \cdot v = \tau \cdot \theta^2 v = \theta^2 \tau \cdot v = \theta^2 w.$$

It follows that the subspace generated by v and w is invariant under both σ and τ, hence under all of S_3. In particular, an irreducible representation of S_3 over the complex numbers can have dimension no higher than two.

Can we actually identify all the irreducible representations of S_3 over $\mathbb{C}$? Suppose V is irreducible, and again let v be an eigenvector for the action of σ on V, with eigenvalue θ. Since σ^3 is the identity element of S_3, this eigenvalue must satisfy $\theta^3 = 1$. There are essentially two cases to consider: either the eigenvalue is one, or it is a primitive third root of unity.

First we consider the case where the eigenvalue $\theta \neq 1$. In this case, $\theta \neq \theta^2$, so the vectors v and $w = \tau v$ have distinct eigenvalues θ and θ^2, hence are independent. This means that v and w span V, which necessarily has dimension two. In this case, the map

$$V \to W = \{(x_1, x_2, x_3) \mid \sum x_i = 0\} \subset \mathbb{C}^3$$
$$v \mapsto (1, \theta, \theta^2); \;\; w \mapsto (\theta, 1, \theta^2),$$

defines an isomorphism from V to the **standard representation** of S_3 (prove it!).

It remains to consider the case where $\theta = 1$. Now $w = \sigma v = v$, so that the irreducible representation V is one dimensional. We already know that σ acts trivially, so the only issue is how τ might act. But since $\tau^2 = 1$, the only possibilities for the actions of τ are either trivially, or by multiplication by -1. If τ acts trivially, the irreducible representation V is the trivial representation. If τ acts acts by -1, then V is the **alternating representation.** Thus, up to isomorphism, there are precisely three irreducible representations of S_3 over the complex numbers: the trivial, the alternating and the standard representations.