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Undergraduate Representation Theory 2010 **Exercise Set 5**
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space-time coordinates monday Feb. 15 at 12-14 in MaA 203
Reading: Review of linear algebra if needed (construction of tensor products, etc.)

Problem 1: The permutation representation of S_3. Consider the permutation representation of S_3 acting by permuting the elements of a basis for $\mathbb{C}^3$.

(1) Show that the span of $(1, 1, 1)$ is a subrepresentation of S_3.
(2) Explicitly decompose $\mathbb{C}^3$ into irreducible representations.

Problem 2: Which of the following representations are irreducible?

(1) The tautological representation of D_n on $\mathbb{R}^2$?
(2) The action of $U(1)$ on $\mathbb{C}$ by multiplication?
(3) The tautological action of $GL(V)$ on V over a field F.
(4) The group homormorphism $(\mathbb{Q}, +) \to GL(\mathbb{Q}^2)$ given by $\lambda \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.
(5) The permutation representation of S_n on $\mathbb{C}^n$.
(6) The regular representation of $\mathbb{Z}_4$.
(7) The action of $SL_2(\mathbb{R})$ on the space of all 2×2 real matrices by left multiplication.
(8) The action of $SL_2(\mathbb{R})$ on space of all 2×2 real matrices by conjugation.
(9) The representation of $GL(V)$ induced on $\Lambda^{dim\ V}$ by the tautological action of $GL(V)$ on V.

Problem 3: Explicitly decompose the following representations into irreducibles.

(1) The regular representation of $G = \mathbb{Z}_4$.
(2) The regular representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.
(3) The representation on $\mathbb{Z}_4$ on $\mathbb{R}^2$ induced by restricting the tautological representation of D_4 to the subgroup of rotations, identified with $\mathbb{Z}_4$ by sending r_i to $\bar{i}$.

Can you make any generalizations?

Problem 4. Let G be a finite group, and let G^* be the set of all complex valued functions on G.

(1) Show that G^* has a natural $\mathbb{C}$-vector space structure.
(2) Show G^* has a natural G-representation structure defined by $g \cdot \phi(h) = \phi(hg^{-1})$.
(3) Prove that G^* is isomorphic to R, the regular representation of G (as a *representation of* G). (Hint: think of e_g as the characteristic function of $g \in G$.)

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Problem 5: Homomorphisms of Representations. Let G be a group acting on finite dimensional (complex, say) vector spaces V and W.

(1) Show that G acts on the vector space of linear maps $Hom_{\mathbb{C}}(V, W)$ by $g \cdot \phi(v) = g \cdot \phi(g^{-1} \cdot v)$ for all $g \in G$ and all $v \in V$.

(2) Explain why the set of all G-representation homomorphisms from V to W can be viewed as a subset of the set $Hom_{\mathbb{C}}(V, W)$ of vector space maps from V to W. Is it is a subvector space?

(3) Show that the set of G-representations homomorphisms of V to W can indentified with the set of linear transformations in $Hom_{\mathbb{C}}(V, W)$ fixed by every element of G under the action described in (1)..