JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Undergraduate Representation Theory 2010 **Exercise Set 4**
author Karen Smith
space-time coordinates monday Feb. 8 at 12-14 in MaA 203
Reading: Review of linear algebra if needed (construction of tensor products, etc.)

Problem 1. Consider a group G acting on a set X.

(1) Prove the relation $x \sim y$ if and only if x is in the G-orbit of y defines an equivalence relation on X.
(2) Explicitly compute the orbits of a group G acting on itself by conjugation in the following cases: G is $\mathbb{Z}_4$, G is D_4 and G is S_4.
(3) How can you generalize (2) for other groups acting on themselves by conjugation?

Problem 2: Building representations from others. Let G be a group acting on a finite dimensional (complex, say) vector spaces V and W.

(1) Show that there is a natural G-action induced on $V \oplus W$.
(2) Show that there is a natural G-action induced on $V \otimes W$.
(3) Show that there is a natural G-action induced on V^*, the space of linear functional $V \to \mathbb{C}$, defined as follows: for $\phi : V \to \mathbb{C}$, $g \in G$ acts by $g \cdot \phi : V \to \mathbb{C}$ sending $v \mapsto \phi(g^{-1} \cdot v)$. Why can't we use g instead of g^{-1} in this expression?
(4) Show that the G-action defined on V^* respects the natural pairing between a vector space and its dual. That is: $g \cdot \phi(g \cdot v) = \phi(v)$ for all $v \in V$ and all $\phi \in V^*$.
(5) Show that there is a natural G-action induced on the symmetric powers of V.
(6) Show that there is a natural G action induced on the space $\mathrm{Hom}(V, W)$ of linear transformation from V to W.
(7) Show that there is a natural G-action induced on the exterior powers of V.
(8) If we take G to be $GL_n(\mathbb{F}_p)$ acting on the vector space of column matrices $\mathbb{F}_p^n$, describe explicitly the induced action on $\bigwedge^n \mathbb{F}_p^n$.

Problem 3· Consider the group S_3 of permutations of three objects.

(1) Show that $S_3 \cong D_3$ of symmetries of an equilateral triangle.
(2) Let $D_3 \to GL(\mathbf{R^2})$ be the tautological representation of D_3. Explicitly describe the images of the elements of D_3 as matrices (fixing the standard basis for $\mathbb{R}^2$). Is this representation irreducible?
(3) Let $S_3 \to GL(\mathbf{R^3})$ be the representation induced by the action of S_3 on a basis indexed by the vertices of an equilateral triangle. Show that this representation is not *irreducible* by showing that the subspace consisting of vectors (x_1, x_2, x_3) with $x_1+x_2+x_3 = 0$ is a subrepresentation, called the *standard representation* of S_3.
(4) Identifying S_3 with D_3 using the isomorphism from (1), prove or disprove that the tautological and standard representations are isomorphic.

KÄÄNNÄ

Problem 4. Consider the group $(\mathbb{Z}_8, +)$.

(1) List all one-element generating sets of $\mathbb{Z}_8$.
(2) Prove that a group isomorphism (or group *automorphism*) $\mathbb{Z}_8 \to \mathbb{Z}_8$ is determined by the image a single generator.
(3) Explicitly list all group isomorphisms $\mathbb{Z}_8 \to \mathbb{Z}_8$.
(4) Show the set of group automorphisms of $\mathbb{Z}_8$ forms a subgroup of $\mathrm{Aut}_{\mathrm{set}}(\mathbb{Z}_8)$.[1] Let us denote this subgroup by $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{Z}_8)$.
(5) What is the order of $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{Z}_8)$? Is it abelian?
(6) Describe the structure of $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{Z}_8)$, for example, by expressing it as a direct sum of cyclic groups, and/or identifying it with some easily understood subgroup of S_8.
(7) How much of this can you generalize to arbitrary cyclic groups $\mathbb{Z}_n$?
(8) If you are familar with *rings,* repeat 3-7 in the category of rings, that is, looking at automorphisms of $\mathbb{Z}_8$ which preserve the *ring structure.*

[1] By $\mathrm{Aut}_{\mathrm{set}}(\mathbb{Z}_8)$ we mean simply $\mathrm{Aut}(\mathbb{Z}_8)$ but we are emphasizing in the notation that we are looking only at bijective self-maps of the set $\mathbb{Z}_8$, regardless of whether not they respect the group structure.