JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Undergraduate Representation Theory 2010 **Exercise Set 3**

author Karen Smith

space-time coordinates monday Feb. 1 at 12-14 in MaA 203

Reading: Dummit and Foote pp 36–39, 46-48, 54–59, 61–64, 66-71, 73-85,

Problem 1: Normal Subgroups. A subgroup H of G is *normal* if $gHg^{-1} \subset H$ for all $g \in G$, or equivalently if $gH = Hg$ for all $g \in G$.

(1) Show that every subgroup of an abelian group is normal.
(2) Which subgroups of D_4 are normal?
(3) A group G is *simple* if it has no non-trivial proper normal subgroups. Find all simple cyclic groups.
(4) The *index* of a subgroup K of G is the number of distinct (right) cosets of K. Prove that a subgroup of index two is always normal.
(5) Prove or disprove: D_n is simple.

Problem 2: Homomorphisms. Let $\phi : (G, \circ_G) \to (H, \circ_H)$ be a group homomorphism. (By definition, ϕ is a map of sets satisfying $\phi(g_1 \circ_G g_2) = \phi(g_1) \circ_H \phi(g_2)$.) Show that:

a. Homomorphisms preserve the identity: $\phi(e_G) = e_H$.
b. Homomorphisms preserve inverses: $\phi(g^{-1}) = [\phi(g)]^{-1}$.
c. The kernel of a homomorphism is a normal subgroup of G: that is, $\ker \phi = \{g \in G \mid \phi(g) = e_H\}$ is a normal subgroup of G.
d. Prove that ϕ is injective if and only if $\ker \phi$ is the trivial subgroup $\{e_G\}$ of G.
e. The image of a homomorphism is a subgroup: that is, $\text{im } \phi \quad = \{h \in H \mid \text{ there exists } g \in G \text{ with } \phi(g) = h\}$ is a subgroup of H.
f. Prove that the quotient group $G/\ker\phi$ is isomorphic to $\text{im } \phi$.
g. Prove that a subgroup H of G is normal if and only if it is the kernel of some group homomorphism.

Problem 3. Prove a criterion (in terms of m and n) for the existence of an isomorphism $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{mn}$.

Problem 4. Consider the vector space $\mathbb{R}^2$ as an additive group. Let L be a one dimensional subspace.

(1) Note that L is subgroup, and explicitly (geometrically) describe the cosets of L in $\mathbb{R}^2$. Illustrate.
(2) Note that $\mathbb{Z}^2$ is also a subgroup, and explicitly (geometrically) describe its cosets.

Problem 5. True or False? For each statement, prove or find a (non-trivial) counterexample. Let $\phi : G \to H$ be a homomorphism of groups.

(1) If G is abelian, then H is abelian.
(2) If G is abelian and ϕ is surjective, then H is abelian.
(3) If G is abelian and ϕ is injective, then H is abelian.

(4) If H is abelian and ϕ is surjective, then G is abelian.
(5) If G is cyclic and ϕ is surjective, then H is cyclic.
(6) If $g \in G$ has order n, then $\phi(g)$ has order n.

Problem 6: The center. The *center* of a group G is the set Z of elements which commute with all elements of G: $Z = \{z \in G \mid gz = zg \text{ for all } g \in G\}$.

(1) Prove that the center is a normal subgroup.
(2) Find the center of $\mathbb{Z}_n$.
(3) Find the center of D_4. Of D_n.
(4) Find the center of S_n.