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Undergraduate Representation Theory 2010 **Exercise Set 2**
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time monday Jan. 25 at 12-14 in MaA 203 (!!!)
Reading: Dummit and Foote pp 23–27, 29–32.

Problem 1: A study of S_n. Let S_n denote the permutation group on n objects.

a. Show that S_n has exactly $n!$ elements.
b. Show that every permutation $\sigma \in S_n$ can be written as a composition of disjoint cycles $\sigma_1 \circ \cdots \circ \sigma_t$ where the σ_i are cyclic permutations of some subset of the n objects. Show that this representation is unique, up to reordering the cycles.
c. Show that every permutation is a composition of transpositions (that is, 2-cycles). Are the transpositions unique?
d. Show that there is a way to interpret D_n in a natural way as a subgroup of S_n.
e. Find (all) subgroups of S_n isomorphic to S_k for all $k \leq n$.
f. Show that if $k + m \leq n$, then $S_k \times S_m$ is isomorphic to a subgroup of S_n. Can you count the number of subgroups of S_n isomorphic to $S_k \times S_m$?

Problem 2: Cyclic Groups. A group is cyclic if it can be generated by a single element.

a. Prove that every cyclic group is abelian.
b. Prove that every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.
c. Prove that every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +)$, for some n.
d. List all cyclic subgroups of D_4.
e. How many cyclic subgroups does D_p have, when p is prime?
f. Find a formula for the number of cyclic subgroups of D_n, in terms of (the prime factorization of) n.

Problem 3: Products of Cyclic Groups.

a. Show that $\mathbb{Z}_4$ is *not* isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
b. Show that $\mathbb{Z}_6$ *is* isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.
c. Can you conjecture a precise condition for when $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times Z_n$.
d. Can you prove it?

Problem 4: Generators and Relations for D_n. Consider the group D_n of symmetries of the regular n-gon. Let r be the counterclockwise rotation though the angle $\frac{2\pi}{n}$ and let s be reflection over a line through the center of the n-gon and any one fixed vertex.

a. Show that r and s generate D_n.
b. Show that $srs = r^{n-1}$.
c. Show that every element of D_n can be written *uniquely* in the form $s^k r^i$ where $k = 0$ or 1 and $i = 0, \ldots, n-1$.

d. Is any group generated by two elements x and y, satisfying $x^2 = e, y^n = e$ and $xy = y^{-1}x$ is isomorphic to D_n?

Problem 5: The order of subgroups.

a. Describe all subgroups of D_{12}. Note their orders, in relation to the order of D_{12}.
b. Make a conjecture about the orders of subgroups of a fixed group. If you already know the theorem, try to prove it. (We will state and prove such a theorem in class eventually).

Problem 6: Classification of Small Order Groups.

a. Show that every group of order three or less is isomorphic to $\mathbb{Z}_n$.
b. Show that every group of order four is abelian.
c. Show that there are, up to isomorphism, exactly two groups of order four.
d. Show that there is, up to isomorphism, exactly one group of order five.