Undergraduate Representation Theory 2010 **Exercise Set 12**
space-time coordinates *Wednesday Apr. 21* ***(2 weeks again!)*** *MaD 380, 8.20*

Problem 1: Induced representations on Lie algebras. Let G be a Lie group, with lie algebra $\mathcal{G}$, and let $G \to GL(V)$ be a smooth finite dimensional representation of G.

(1) Prove that there is an induced lie algebra representation $\mathcal{G} \to gl(V)$.
(2) Let G be $\mathbb{R}^*$ and let $V = \mathbb{R}^2$. For each pair of integer a, b, show that the action of $\lambda \in G$ on $\mathbb{R}^2$ by $\lambda \cdot (x, y) = (\lambda^a x, \lambda^b y)$ is a smooth representation of G, and explicitly describe the corresponding lie algebra representation.
(3) For $G = GL_2(\mathbb{R})$, explicitly describe the lie algebra representation corresponding to the one dimensional "determinantal"representation of G (that is, for the representation $\bigwedge^2 V$ where V is the tautological representation for G.)

Problem 2: Special Linear Group and its lie algebra. Let $G = SL_2(\mathbb{R})$. Recall that its lie algebra $sl_2(\mathbb{R})$ is naturally identified with the lie algebra of trace zero 2×2 matrices (with the "usual bracket").

(1) Show that for any $A \in sl_2(\mathbb{R})$, we have $A^2 = -\det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
(2) For each of the three cases $\det A = 0$, $\det A > 0$ and $\det A < 0$, compute exp^A explicitly, using the classical definition of the exponential map in terms of power series of matrices. (Hint: it might help to refresh yourself on the series for trigonometric and hyperbolic trigonometric functions.)
(3) Verify that $exp(A) \in SL_2$ for all $A \in sl_2$. That is, verify that the exponential map takes sl_2 to SL_2.
(4) Show that diagonal elements in SL_2 are not in the image of the exponential map—so the exponential map need not be surjective.

Problem 3: Tangent vectors as velocity vectors of curves. A curve at x in manifold M is a smooth map $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$, where $(-\epsilon, \epsilon)$ is an open interval in the real line. Its *velocity vector* is the tangent vector $d_0\gamma(1) \in T_x M$. Prove that every vector in $T_x M$ is the velocity vector of some curve. (Hint: check it first for open sets in $\mathbb{R}^d$; now use charts).

Problem 4: Derivations and Vector Fields. Let A be any associative algebra (over, say $\mathbb{R}$).[1] A derivation of A is a linear map $\partial : A \to A$ which satisfies $\partial(fg) = \partial(f)g + f\partial(g)$.

(1) If A is the algebra of all (real valued) smooth functions on some open set U in $\mathbb{R}^d$, show that the differential operators $\frac{\partial}{\partial x_i}$ are derivations, and also that any A-linear combination of $\frac{\partial}{\partial x_i}$ is a derivation.
(2) For any algebra A, prove that the set $Der(A)$ of all derivations of A is a subspace of the space $Hom(A, A)$ of *all* linear self-maps of A.

[1]This means simply that A is a ring containing the real numbers as a subring.

(3) Prove that $Der(A)$ is a lie-subalgebra of $Hom(A, A) = gl(A)$ (where the lie algebra structure on the latter is the standard one given by the "commutator"bracket $[X, Y] = XY - YX$ for linear self maps X and Y of A.)
(4) Now for $A = C^\infty(U)$ where U is an open set in euclidean space, find an explicit example to show that the composition of two derivations (as linear maps) need *not* be a derivation (although you just proved above that their commutator is).
(5) Now let $A = C^\infty(M)$ where M is any manifold. Show that there is a natural injective linear map

$$\mathcal{V}ect(M) \to Der(A)$$

where from the space $\mathcal{V}ect$ of all vector fields on M. (Hint: send a vector field ∇ to the directional derivative of f in the direction of ∇— that is, to the linear map ∇ whose value on $f \in A$ is the smooth function assigning a point $p \in M$ to the real number $d_p f(\nabla(p)) \in \mathbb{R}$.)[2]

Problem 5: The lie algebra as Invariant Vector Fields. Let G be a Lie group with lie algebra $\mathcal{G}$, and let $\mathcal{V}ect(G)$ denote the space of all smooth vector fields on G.

(1) Verify that the formula $a \cdot \nabla(b) = d_{ab}(m_{a^{-1}})(\nabla(ab))$ for all $a, b \in G$ and $\nabla \in \mathcal{V}ect(G)$ defines an action of G on $\mathcal{V}ect(G)$.
(2) Show that ∇ is fixed by G if and only ∇ is of the form X^L where $X \in \mathcal{G}$. (By definition, recall that X^L is the vector field assigning to $g \in G$ the vector $d_e(m_g)(X)$ where m_g is left multiplication by g.)
(3) Prove that the map $\mathcal{G} \to \mathcal{V}ect(G)$ sending $X \mapsto X^L$ identifies the lie algebra of G with the subspace of G-invariant vector fields (that is, with the subspace of the representation $\mathcal{V}ect(G)$ where G acts trivially).
(4) Show that if we compose the linear maps from Exercises 4 and 5

$$\mathcal{G} \to \mathcal{V}ect(G) \to Der(C^\infty(G))$$

we get an embedding *of the lie algebra* $\mathcal{G}$ in the lie algebra $Der(C^\infty(G))$. (That is, this linear map is injective *and* respects the lie bracket. Here the lie bracket on $\mathcal{G}$ is the one defined in class, $[X, Y] = ad_X Y$, and the lie bracket on derivations is as defined above in exercise 4.)
(5) Use this to show that the lie bracket on $\mathbb{G}$ defined as $[X, Y] = ad_X Y$ is skew symmetric and satisfies the Jacobi identity.

Problem 6: Eigenvalues. Let A, B and U be $n \times n$ invertible matrices with U invertible (all over $\mathbb{R}$, say, or $\mathbb{C}$ if you prefer).

(1) Show that

$$e^{UAU^{-1}} = Ue^A U^{-1}.$$

(2) Show that if λ is an eigenvalue for A, then e^λ is an eigenvalue for e^A.
(3) Show that $\det e^A = e^{tr(A)}$.
(4) Show that if $AB = BA$, then $e^{A+B} = e^A e^B$.

[2]It is actually true, and not so hard to show, that this map is an isomorphism. It is common in geometry to identify vector fields with derivations. Try proving it if you have time; start with seeing what needs to happen for open sets in Euclidean space.