Undergraduate Representation Theory 2010 **Exercise Set 11**
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space-time coordinates *Wednesday Apr. 7 (2 weeks!) MaD 380 at 8.20*

Problem 1: Unitary Groups. Fix a positive definite Hermitian inner product Q on an n-dimensional complex vector space V.

(1) Show that the linear transformations $\phi \in GL(V)$ that respect Q (meaning $Q(\phi(v), \phi(w)) = Q(v, w)$ for all vectors v, w in V) form a subgroup of $GL(V)$. We denote this group $U_Q(V)$.
(2) When $n = 1$, describe $U(1)$ explicitly. What familiar lie group is it isomorphic to?
(3) When Q is the standard Hermitian inner product on $\mathbb{C}^n$, show that $U_Q(V)$ can be identified with the set $U(n)$ of all $n \times n$ complex matrices A such that $\overline{A}^t A = \mathbf{I}_n$.
(4) Prove that $U(n)$ is a Lie group, and compute its dimension.
(5) Note that $U_n(\mathbb{C})$ sits inside the complex manifold $GL_n(\mathbb{C})$. Is it a complex manifold?

Problem 2. Let V be a finite dimensional representation of a group G and fix a basis $\{e_1, \ldots, e_n\}$ of V, so that the representation is given by a group homomorphism $G \to GL_n(\mathbb{R})$. For any positive integer $d \leq n$, explicitly describe the induced group map

$$G \to GL_{\binom{n}{d}}(\mathbb{R})$$

corresponding to the induced representation on $\Lambda^d V$ with respect to the basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}\}$.

Problem 3. Let $\mathbb{F}$ be any field such that the element $2 = 1 + 1$ has a multiplicative inverse (if this bothers you, take $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.)

(1) Show that a bilinear map $B : V \times V \to P$ is alternating (meaning $B(v, v) = 0$) if and only if it is skew symmetric (meaning $B(v, w) = -B(w, v)$).
(2) Show that for a finite dimensional vector space over $\mathbb{F}$, an alternating bilinear form

$$V \times V \to \mathbb{F}$$

can always be described by giving a matrix A such that $B(v, w) = v^t A w$. How is the skew-symmetry of B reflected in the matrix A?
(3) Prove that there exists a *non-degenerate* alternating bilinear form on a finite dimensional V if and only if V has even dimension.

Problem 4. Fix a representation of a group G on a vector space V of dimension n over a field $\mathbb{F}$.

(1) Show that the representation $V \otimes V$ decomposes (as a representation!) as $S^2 V \oplus \Lambda^2 V$.

(2) Consider the permutation action of S_3 on $V = \mathbb{C}^3$. Compute the character of the representations $V^{\otimes 2}, S^2V$ and $\Lambda^2 V$.
(3) Decompose the nine dimensional representation $V^{\otimes 2}$ into irreducible representations (up to isomorphism).

Problem 5. Let $V = \mathbb{R}^3$.

(1) Show that there is a unique skew-symmetric bilinear map
$$B : V \times V \to V$$
satisfying
$$B(e_1, e_2) = e_3, \;\; B(e_2, e_3) = e_1, \;\; B(e_3, e_1) = e_2,$$
where e_1, e_2, e_3 is the standard basis.
(2) Find a formula for $B(v, w)$ in terms of the coordinates of v and w in the standard basis.
(3) Prove that $B(v, w)$ is the standard "cross product"from physics: $B(v, w) = |v||w| \sin\theta \; \hat{n}$ where θ is the angle between v and w in the plane they span, and $\hat{n}$ is the (right handed) unit normal vector to this plane.
(4) Show that B satisfies the Jacobi identity, and thus gives $\mathbb{R}^3$ the structure of a Lie algebra with Lie bracket B.
(5) Consider the map
$$\mathbb{R}^3 \to M_{3\times 3}(\mathbb{R})$$
sending
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$
Show that this induces an isomorphism of Lie algebras, with the standard lie bracket on $M_{3\times 3}(\mathbb{R})$ (given by $[X, Y] = XY - YX$.)