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Undergraduate Representation Theory 2010 **Exercise Set 10**
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space-time coordinates *Wednesday Mar. 24 MaD 380 at MORNING TIME 8.20*

Problem 1: Borel Groups. Fix a finite dimensional real vector space V of dimension n and a *full flag* of subspaces:

$$V = V_n \supset V_{n-1} \supset V_{n-2} \supset \dots V_2 \supset V_1 \supset V_0 = 0$$

where each V_i is a subspace of dimension i. Consider the subset B of $GL(V)$ of linear transformations that preserve this flag:

$$B = \{g \in GL(V) \mid gV_i \subset V_i \text{ for all } i\}.$$

(1) Show that B is a subgroup of $GL(V)$.
(2) By fixing appropriate bases for the V_i, show that B can be identified with the subgroup of invertible upper triangular matrices.
(3) Prove that B is a Lie group of dimension $\frac{1}{2}(n+1)(n)$.
(4) Now fix a partial flag

$$V = V_n \supset V_{d_t} \supset V_{d_t} \supset \dots V_{d_2} \supset V_{d_1} \supset V_0 = 0$$

where V_{d_i} is a subspace of dimension d_i. Let $P \subset GL(V)$ be the subset of linear transformations preserving this flag. What can you say about P? Is is also a Lie Group, and if so, can you find its dimension?

Problem 2.

(1) Show that conjugation by elements of $SL_2(\mathbb{R})$ induces a four dimensional smooth representation of $SL_2(\mathbb{R})$ on the space of all 2×2 matrices.
(2) Find a one dimensional subrepresentation. What is the action there?
(3) Show that the subspace of trace zero matrices forms an *irreducible* subrepresentation of dimension three.
(4) Decompose the four-dimensional space of all matrices (with conjugation action as above) into irreducible representations.
(5) Show that the three dimensional component is isomorphic to V_2 from Exercise 2. (Hint: index the rows and columns of the matrices by the variables defining the V_d).

Problem 3. Let M and N be smooth manifolds of dimensions e and d.

(1) Show that $M \times N$ is a smooth manifold of dimension $d + e$.
(2) if p, q are points on M, N, show that there is a natural isomorphism $T_{(p,q)}(M \times N) \cong T_pM \times T_qN$.

Problem 4. For the group $G = GL_2(\mathbb{R})$, explicitly compute the derivative, at the identity, of the multiplication map. That is, what is the induced map

$$d_e\mu : T_eG \times T_eG \to T_eG$$

where $\mu : G \times G \to G$ is multiplication?

Problem 5. Fix a vector space V of dimension n over a field $\mathbb{F}$.

(1) Prove that $\Lambda^d V$ is a vector space of dimension $\binom{n}{d}$.
(2) Prove that $S^d V$ is a vector space of dimension $\binom{n+d-1}{d-1}$.
(3) If a group G acts on V by linear transformations, prove that there is an induced G action also on $\Lambda^d V$ and $S^d V$ by linear transformations.
(4) Let V be an irreducible representation of a group G over $\mathbb{R}$ or $\mathbb{C}$. Show that the representation $V \otimes V$ decomposes (as a representation!) as $S^2 V \oplus \Lambda^2 V$.
(5) Consider the tautological representation of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ (ie, acting by linear transformations). Explicitly describe the induced one dimensional representation $\Lambda^n \mathbb{R}^n$.
(6) Consider the permutation action of S_3 on $V = \mathbb{C}^3$. Compute the character of the representations $V^{\otimes 2}, S^2 V$ and $\Lambda^2 V$. Use this to explicitly decompose the nine dimensional representation $V^{\otimes 2}$ into irreducible representations.

Problem 6: Compact Groups A topological space is compact if every open cover has a finite subcover. For a subspace X of Euclidean space, X is compact if and only if X is closed and bounded. Which of the following Lie Groups are compact?

$$GL_n(\mathbb{R}),\ SL_n(\mathbb{R}),\ S^1,\ \mathbb{R}^n,\ \mathbb{R}^* \times \mathbb{R}^*,\ \mathbb{R}^2/\mathbb{Z}^2,\ S^1 \times S^1,\ SO_n,\ SO(k,l)$$