Exercise help set 3/2011 **Number Theory**

1. *Why is*

$$a) \quad \zeta(2) = \pi^2/6\,?$$

1. reason: Well known formula (Our college shirt!)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$e^{i\pi} = -1 \text{ and}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \quad \Box$$

2. reason: http://mathworld.wolfram.com/RiemannZetaFunctionZeta2.html
The numerical solution is easy.

2. *Prove the* **second** *Möbius inversion equivalence:*

$$f(x) = \sum_{n\le x} g\left(\frac{x}{n}\right) \qquad (x \ge 1),$$

$$g(x) = \sum_{n\le x} \mu(n) f\left(\frac{x}{n}\right)$$

Interpretation: $n \in \mathbb{N}$, f and g defined in a suitable set like $\mathbb{R}$. Kiinteällä $x \in \mathbb{R}$ the function $n \mapsto f(\frac{x}{n})$ is a number theoretic function ie defined in $\mathbb{N}$. Begin with (1) and calculate the right side of (2):

$$\begin{aligned}\sum_{n\le x} \mu(n) f\left(\frac{x}{n}\right) &= \sum_{n=1}^{\lfloor x\rfloor} \mu(n) f\left(\frac{x}{n}\right) \\ &= \sum_{n=1}^{\lfloor x\rfloor} \mu(n) \sum_{m=1}^{\lfloor \frac{x}{n}\rfloor} g\left(\frac{x}{nm}\right) \\ &= \sum_{k=1}^{\lfloor x\rfloor} g\left(\frac{x}{k}\right) \sum_{n\mid k} \mu(n) \\ &= \sum_{k=1}^{\lfloor x\rfloor} g\left(\frac{x}{k}\right) (E*\mu)(k) \\ &= \sum_{k=1}^{\lfloor x\rfloor} g\left(\frac{x}{k}\right) (E_0(k)) \qquad = g\left(\frac{x}{1}\right) = g(x).\end{aligned}$$

Similarly in reverse.

3. *Prove*

$$\sum_{n\le x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1.$$

In the previous, take $f(x) = \lfloor x \rfloor$ and find:

$$(1) \quad \lfloor x \rfloor = \sum_{n \le x} g\left(\frac{x}{n}\right) \quad (x \ge 1),$$

$$(2) \quad g(x) = \sum_{n \le x} \mu(n) \lfloor \frac{x}{n} \rfloor$$

Choose for all $x \in \mathbb{R}$ $g(x) = 1$ and find that the following are equivalent

$$(1) \quad \lfloor x \rfloor = \sum_{n \le x} 1 \quad (x \ge 1),$$

$$(2) \quad 1 = \sum_{n \le x} \mu(n) \lfloor \frac{x}{n} \rfloor$$

Jutila mentions that the following equation is equivalent to the "Prime number theorem". (At the moment, I don't know which version of the theorem he refers to.)

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

4. The sieve principle. *Let $\mathcal{A} \subset \mathbb{Z}$ be a finite set and $P \in \mathbb{Z}$, $P \neq 0$. Define $N = \#\{a \in \mathcal{A} \mid (a, P) = 1\}$ and $A_d = \#\{a \in \mathcal{A} \mid d \mid a\}$.*

Prove: $N = \sum_{d|P} \mu(d) A_d\}$.

$$N = \sum_{\substack{a \in \mathcal{A} \\ (a,P)=1}} 1 = \sum_{a \in \mathcal{A}} \sum_{d|(a,P)} \mu(d) = \sum_{d|P} \mu(d) \sum_{\substack{a \in \mathcal{A} \\ d|a}} 1 = \sum_{d|P} \mu(d) A_d. \quad \square$$

5. *Definition:* von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p, & \textit{if } n = p^m, \\ 0 & \textit{else}; \end{cases}$$

here $p \in \mathbb{P}$ and $m \ge 1$.

Prove

$$\log n = (\Lambda * E)(n).$$

$$(\Lambda * E)(n) = \sum_{d|n} E\left(\frac{n}{d}\right)\Lambda(d) = \sum_{d|n} \Lambda(d) = \sum_{p \in \mathbb{P}} \sum_{d|n, d=p^a} \Lambda(p^a) = \sum_{p \in \mathbb{P}} \sum_{d|n, d=p^a} \log p = \log n.$$

6. *Prove*

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d$$

By the previous exercise:

$$\begin{aligned}\Lambda(n) &= (\Lambda * E_0)(n) = (\Lambda * E * \mu)(n) = (\log *\mu)(n)\\ &= \sum_{d|n}\mu(d)\log(n/d) = \sum_{d|n}\mu(d)(\log(n) - \log(d)) =\\ &= \sum_{d|n}\mu(d)\log n - \sum_{d|n}\mu(d)\log d\\ &= \log n\sum_{d|n}\mu(d) - \sum_{d|n}\mu(d)\log d = 0 - \sum_{d|n}\mu(d),\end{aligned}$$

since $\log n = 0$, *when* $n = 1$ *and* $\sum_{d|n}\mu(d) = E_0(n) = 0$ *for all other* $n \in \mathbb{N}$.

von Mangoldt's function is important among other reasons because also

$$\mathit{lim}_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\Lambda(n) = 1$$

is equivalent to the prime number thm.

7. *Clearly* $641 = 2^4 + 5^4 = 5 \cdot 2^7 + 1$. *We prove* $2^{32} = 641k - 1$ *for some* $k \in \mathbb{N}$ *ie.* $2^{32} \cong -1 \pmod{641}$.

$$\begin{aligned}5 \cdot 2^7 = 640 &\equiv -1 \pmod{641} \ \textit{(above!)}\\ 5^4 \cdot 2^{28} = (5 \cdot 2^7)^4 &\equiv (-1)^4 = 1 \pmod{641}\\ 2^4 &\equiv -5^4 \pmod{641} \ \textit{(above!)}\\ 5^4 \cdot 2^{32} &\equiv -5^4 \pmod{641}\\ 2^{32} &\equiv -1 \pmod{641} \ \textit{(since } (5, 641) = 1)\end{aligned}$$

This implies $F_5 = 2^{2^5} + 1 = 2^{32} + 1 \equiv -1 + 1 = 0 \pmod{641}$.