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ON THE CONFORMAL MARTIN BOUNDARY OF DOMAINS IN METRIC SPACES

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ABSTRACT. Using Cheeger's differentiability theorem for Lipschitz functions on metric measure spaces, we construct a conformal analogue of the Martin boundary for relatively compact domains in locally compact metric measure spaces which are locally Q-regular for some $Q > 1$ and support a $(1, p)$-Poincaré inequality for some $p < Q$. For relatively compact uniform domains which have uniformly Q-fat complement we show that the conformal Martin boundary maps surjectively onto the topological boundary. We also investigate the behavior of the conformal Martin boundary under conformal and quasiconformal maps. Our results are new even in the setting of Euclidean domains.

1. Introduction

The Martin boundary was first studied in [M] in the context of minimal positive harmonic functions in classical potential theory. Some recent studies (see [AS] and [An2]) on the Martin boundary of Riemannian manifolds have focused on the relationship between the Martin and geometric boundaries; see also [HW], [JK], and [Ai] for the Euclidean setting. The Martin boundary of a (non-parabolic) Riemannian manifold M is defined as follows. Fix $x_0 \in M$. Given a sequence of points (x_n) in M, we say that the sequence is fundamental if the sequence has no accumulation point in M and the sequence of functions

$$\frac{G(\cdot, x_n)}{G(x_0, x_n)}$$

is locally uniformly convergent, where $G(\cdot, y)$ is the Green function for the Laplace-Beltrami operator Δ on M with singularity at y. Two fundamental sequences are said to be equivalent if the corresponding sequences of functions converge to the same limit. The collection of all equivalence classes of fundamental sequences is the Martin boundary of M. If $M \subset \mathbb{R}^n$ is a bounded Lipschitz domain in the Euclidean space,

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the corresponding limit functions agree with the usual Poisson kernel; see [M] and [HW].

According to a result of Bonk, Heinonen, and Koskela, if Ω and Ω' are quasiconformally equivalent Euclidean domains (in any dimension $n \geq 2$) which are Gromov hyperbolic in the quasihyperbolic metric and whose complements satisfy a density condition for the variational 2-capacity (so-called *uniform 2-fatness*), then each quasiconformal map from Ω to Ω' extends to a homeomorphism of the respective Martin boundaries; see [BHK, Theorem 9.8 and Theorem 8.15]. In the planar case the two conditions on the domain and its complement are naturally quasiconformally equivalent. In higher dimensions only the Gromov hyperbolicity remains quasiconformally invariant; the density condition for the 2-capacity no longer transforms appropriately under quasiconformal maps. While a simple and direct proof of the result of Bonk, Heinonen, and Koskela can be given for conformally equivalent planar domains, the proof which they give (which is valid in all dimensions and for arbitrary quasiconformal maps) is indirect. Indeed, they show first that the quasiconformal map extends to a quasisymmetry between the Gromov boundaries, and then they use a result of Ancona [An2] to identify the Gromov and Martin boundaries.

As the reader is doubtless aware, the correct potential theory to consider in connection with quasiconformal maps in higher dimensions is that associated with the *n-Laplacian*

$$\Delta_n u = -\operatorname{div}(|\nabla u|^{n-2}\nabla u),$$

where n is the dimension of the underlying space. For the nonlinear potential theory associated with this and more general quasilinear elliptic operators, we refer to the comprehensive text by Heinonen, Kilpeläinen and Martio [HKM]. In this paper, we initiate a study of the *conformal Martin boundary*, defined using singular solutions to the n-harmonic equation $\Delta_n u = 0$. Taking advantage of recent developments in the abstract theory of analysis and first-order differential calculus on metric spaces (see [HeK1], [He], [HKShT], [Ch]), we work throughout in the general setting of metric measure spaces of locally Q^--bounded geometry, that is, locally Q-regular spaces supporting a local $(1,p)$-Poincaré inequality for some $1 \leq p < Q$. The first two authors have established the existence of singular solutions to the Q-harmonic equation $\Delta_Q u = 0$ (interpreted in a suitable weak sense) on such a metric measure space X; see [HoSh]. Using the differentiation theorem of Cheeger [Ch], the discussion in [Ho] shows the uniqueness of such solutions; see Theorem 3.2 below. We are thus in a position to define the conformal Martin boundary $\partial_{cM}\Omega$ for relatively compact domains $\Omega \subset X$. As in the classical case, $\partial_{cM}\Omega$ comes equipped with a natural topology for which $\Omega \cup \partial_{cM}\Omega$ is a compactification of Ω.

As far as we are aware, Martin boundaries associated with the n-Laplacian, $n > 2$, have not been treated before in the literature, even in the setting of $\mathbb{R}^n$. Also we want to emphasize that our theory applies, apart from the Euclidean and Riemannian settings, to the Carnot groups as well, and to the highly non-classical spaces constructed by Bourdon and Pajot [BP] and Laakso [La].

To begin with, we seek to understand the relationship between the conformal Martin boundary and the topological boundary. Our first result provides a partial answer:

Theorem 1.1. *Let Ω be a relatively compact uniform domain in a metric measure space X of locally Q^--bounded geometry, $Q > 1$. Suppose also that the complement* *of Ω is uniformly Q-fat. Then there exists a surjective continuous map from the conformal Martin boundary $\partial_{cM}\Omega$ onto the topological boundary $\partial\Omega$.*

For the definitions of uniformity of a domain and uniform Q-fatness; see Section 3.

Theorem 1.1 does not hold true if the uniformity assumption is relaxed, see Example 4.6. However, it is possible that the exterior condition (uniform Q-fatness of the complement) is not needed in this result. See Question 3 in Section 3.

Observe that by [BHK, Theorem 3.6 and Proposition 3.12], a uniform domain Ω, equipped with the quasihyperbolic metric k_Ω, is Gromov hyperbolic. Moreover, there is a natural quasisymmetric equivalence between the Gromov (i.e., geometric) boundary $\partial_G(\Omega, k_\Omega)$ and $\partial\Omega$. We conjecture that if Ω is a relatively compact uniform domain, then the Gromov (geometric) boundary maps surjectively onto $\partial_{cM}\Omega$ and hence all three boundaries are homeomorphic. For the corresponding result for the classical (2-)Martin boundary in the Euclidean setting, combine Theorems 8.15 and 3.6 of [BHK].

Recall that a homeomorphism $f : Y_1 \to Y_2$ between two metric spaces (Y_1, d_1) and (Y_2, d_2) is said to be *quasiconformal* if there exists a constant $H \geq 1$ so that

$$\limsup_{r\to 0} \frac{\sup_{d_1(x,y)\leq r} d_2(f(x), f(y))}{\inf_{d_1(x,y)\geq r} d_2(f(x), f(y))} \leq H$$

for every $x \in Y_1$. Furthermore, a homeomorphism $f : Y_1 \to Y_2$ is a *quasisymmetry* if there exists an increasing homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$\frac{d_2(f(x), f(y))}{d_2(f(x), f(z))} \leq \eta\left(\frac{d_1(x,y)}{d_1(x,z)}\right)$$

for every triple of points $x, y, z \in Y_1$. By the results in [HKShT], a quasiconformal map between two metric spaces of locally Q^--bounded geometry is locally quasisymmetric. Observe that the conditions in the above theorem are quasisymmetrically invariant: if X, X' are two metric measure spaces of locally Q^--bounded geometry, and if $\Omega \subset X$ satisfies the conditions in Theorem 1.1 and $f : X \to X'$ is a quasisymmetric map, then $\Omega' = f(\Omega)$ also satisfies those conditions.

Our second result describes the relationship between the conformal Martin boundaries of conformally equivalent domains.

Theorem 1.2. *Let $\Omega \subset X$ and $\Omega' \subset Y$ be conformally equivalent domains in metric measure spaces of locally Q^--bounded geometry, $Q > 1$. Then each conformal map $f : \Omega \to \Omega'$ extends as a homeomorphism to the respective conformal Martin boundaries.*

See Section 4 for the definition of conformality. Note that there are no conditions on the domain Ω. If Ω does not support Q-singular functions, its conformal Martin boundary is empty. In this case Theorem 1.2 states that the conformal Martin boundary of Ω' also must be empty. This fact follows also from [HoSh, Theorem 4.5].

For quasiconformal maps we prove a version of Theorem 1.2 by making use of Martin boundaries defined relative to general variational kernels of the type considered in [HKM, Section 5]. Our results here make use of fundamental analytic properties of quasiconformal maps in spaces of local Q^--bounded geometry; see [HKShT, Section 10] and [C, Section 4]. However, we leave open the question of when the topological type of the "standard" conformal Martin boundary $\partial_{cM}\Omega$ is a quasiconformal invariant. There are plane domains D_1 and D_2 and a quasiconformal map of D_1 onto D_2 which does not extend to a homeomorphism between their Martin compactifications; see [Seg] and [SegT].

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2. Definitions and Notations

We assume throughout the paper that X is a path connected, locally compact, non-compact metric measure space equipped with a metric d and a non-trivial Borel regular measure μ supported on all of X. We furthermore assume that the measure is *locally Q-regular* for some $Q > 1$. By this we mean that there exists a constant $C \geq 1$ so that each point in X has a neighborhood U such that

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q \tag{1}$$

for all balls $B(x,r) \subset U$. The index Q is called the *regularity dimension* of X; we emphasize that it need not be an integer.

A non-negative Borel measurable function $\rho : X \to [0,\infty]$ is said to be an *upper gradient* of an extended real-valued function u on X if for every rectifiable curve $\gamma : [a,b] \to X$,

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma \rho \, ds.$$

We also assume that X supports a *local (weak) $(1,p)$-Poincaré inequality* for some $1 \leq p < Q$, that is, there exist constants $C > 0$ and $\tau \geq 1$ so that each point in X has a neighborhood U such that

$$\fint_B |u - u_B| \, d\mu \leq Cr \left(\fint_{\tau B} \rho^p \, d\mu \right)^{1/p} \tag{2}$$

whenever $B = B(x,r) \subset U$ is a ball, u is a measurable function on B and ρ is an upper gradient of u. Here $u_B = \mu(B)^{-1} \int_B u \, d\mu$ denotes the mean value of u over the ball B and $\tau B = B(x, \tau r)$. We note that by Hölder's inequality, X then also supports a local $(1,q)$-Poincaré inequality for every $q \geq p$. In particular, X supports a local $(1,Q)$-Poincaré inequality.

Spaces that satisfy (1) and (2) with $p = Q$ have sometimes been called *spaces of locally Q-bounded geometry* in the literature; see [HKShT, Section 9] or [BHK, Section 9]. Consequently, we describe the spaces under consideration in this paper as *spaces of locally Q^--bounded geometry.* Every n-dimensional Riemannian manifold, equipped with the Riemannian volume measure, is locally n-regular and supports a local $(1,1)$-Poincaré inequality. For more exotic examples in all real dimensions $Q > 1$; see [BP], [La], and [S].

Remark 2.1. Under the above assumptions, a suitable local version of the proof of Propositon 4.4 of [HaK] implies that X is locally *quasiconvex*: each point in X has a neighborhood U for which any two points $y, z \in U$ can be joined by a curve of length at most $Cd(y,z)$ for some absolute constant C. Furthermore, Theorems 3.13 and 5.7 of [HeK1] (suitably localized) imply that X is locally *linearly locally connected (LLC)*. By this we mean that there exists a constant $C \geq 1$ so that each point in X has a neighborhood U such that every pair of points x_1, x_2 in an annulus $\bar{B}(z,2r) \setminus B(z,r) \subset U$ can be joined by a curve in $\bar{B}(z,2Cr) \setminus B(z,r/C)$.

The function space that plays the role of classical Sobolev space in this paper is the *Newtonian space* $N^{1,p}(X)$. Specifically, $N^{1,p}(X)$ is the collection of equivalence classes of functions $f \in L^p(X)$ that have an upper gradient $\rho \in L^p(X)$. The equivalence relation used here is $u \sim v$ if and only if $\|u - v\|_{1,p} = 0$, where

$$\|f\|_{1,p} := \|f\|_{L^p(X)} + \inf \|\rho\|_{L^p(X)},$$

the infimum being taken over all upper gradients ρ of f. A set $A \subset X$ is said to have *p-capacity zero* if there exist functions $u \in N^{1,p}(X)$ with arbitrarily small $\|\cdot\|_{1,p}$ norm with $u|A \geq 1$, and a property of points in X is said to hold *p-quasi-everywhere in* X (p-q.e.) if the set of points for which the property does not hold has zero p-capacity. The Sobolev space of functions with zero boundary values on a domain $\Omega \subset X$, denoted $N_0^{1,p}(\Omega)$, is defined to be the collection of elements of $N^{1,p}(X)$ whose representative functions vanish p-q.e. in $X \setminus \Omega$.

In [Ch] Cheeger gives another definition of a Sobolev type space, but for indices $p > 1$ Cheeger's construction yields the same space as $N^{1,p}(X)$; see [Sh1, Theorem 4.10]. It is a deep theorem of Cheeger that if a metric measure space supports a doubling measure and a $(1,p)$-Poincaré inequality, then the corresponding Sobolev-type space is reflexive; see [C, Theorem 4.48]. The results of [Ch] can easily be extended to the case of spaces supporting a locally Q-regular measure and a local $(1,p)$-Poincaré inequality. Using this Sobolev space, properties of p-harmonic functions were investigated in the papers [Sh2] and [KiSh]. The definition of p-harmonic functions used in these papers required such functions to be in the class $N^{1,p}(X)$. In this paper we consider a less restrictive definition of p-harmonic functions; however, the analysis in [KiSh] is local and the results therein apply to the p-harmonic functions of this paper as well.

In [C, Theorem 4.38] it was shown that if the measure on X is doubling and if X supports a $(1,p)$-Poincaré inequality for some p, then there is a "differentiable" structure on X compatible with the notion of upper gradient. More precisely, the following theorem holds:

Theorem 2.2 (Cheeger). *Let X be a metric measure space equipped with a locally doubling Borel regular measure μ. Assume that X admits a weak $(1,p)$-Poincaré inequality for some $1 < p < \infty$.*

Then there exist measurable sets U_α with $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$ and there exist Lipschitz "coordinate" functions $\mathbf{X}^\alpha = (X_1^\alpha, \dots, X_{k(\alpha)}^\alpha) : X \to \mathbb{R}^{k(\alpha)}$ so that

(i) *for each α, the functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ are all linearly independent on U_α and $1 \leq k(\alpha) \leq N$, where N is a constant depending only on the doubling constant of μ and the constants from the Poincaré inequality,*

(ii) *if* $f : X \to \mathbb{R}$ *is Lipschitz, then there exist unique bounded vector-valued functions* $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ *such that*

$$\lim_{r\to 0+} \sup_{x\in B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0) \cdot (\mathbf{X}^\alpha(x) - \mathbf{X}^\alpha(x_0))|}{r} = 0$$

for μ*-a.e.* $x_0 \in U_\alpha$.

In other words, there exists a finite dimensional L^∞ vector bundle of Banach spaces $T^*X = \{F_x\}_{x\in X}$, the *generalized cotangent bundle*, and a derivation operator D on the algebra of locally Lipschitz functions on X taking values in the sections of T^*X. For x in the "coordinate chart" U_α, the fiber F_x is the dual space to the space $\mathbb{R}^{k(\alpha)}$ mentioned in the theorem. Given a Newtonian function f, the function g_f denotes the *minimal p-weak upper gradient* of f, and is defined by the equation

$$g_f(x) = \inf_g \limsup_{r\to 0+} \fint_{B(x,r)} g\, d\mu,$$

the infimum being taken over all upper gradients g of f (see [Sh2, Corollary 3.7]). With this convention, the norm on the space F_x is given by

$$||\lambda||_{\alpha,x} = g_{\langle\lambda,\mathbf{X}^\alpha\rangle}(x), \tag{3}$$

where, for $\lambda = (\lambda_1, \ldots, \lambda_{k(\alpha)})$ in $\mathbb{R}^{k(\alpha)}$ we set $\langle\lambda, \mathbf{X}^\alpha\rangle$ to be the Lipschitz function given by $\sum_{i=1}^{k(\alpha)} \lambda_i X_i^\alpha$. By linear independence of $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha$ we mean that whenever $g_{\langle\lambda,\mathbf{X}^\alpha\rangle}(x) = 0$ for some $x \in U_\alpha$ we have $\lambda = 0$. Finally, in a given chart U_α, Df is given by $d^\alpha f$.

Note that the generalized cotangent bundle is only a measurable vector bundle, that is, the charts U_α are measurable subsets of X and the change of coordinate functions $\mathbf{X}^\beta \circ (\mathbf{X}^\alpha)^{-1}$ are bimeasurable invertible transformations.

By [Sh1, Theorem 4.10] and [C, Theorem 4.47], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}(X)$-norm of the collection of locally Lipschitz functions on X,[1] and by [FHK, Theorem 10], the derivation operator D can be extended to all of $N^{1,p}(X)$ so that

$$|Df(x)| \approx g_f(x)$$

for all $f \in N^{1,p}(X)$ and μ-a.e. $x \in X$. Here $|\cdot| = ||\cdot||_{\alpha,x}$ denotes the norm in (3).

One easily verifies (see also [C, p. 458]) that the derivation operator Du satisfies the product and chain rules: if v is a bounded Lipschitz function on X, $u \in N^{1,p}(X)$, and $\Phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative, then uv and $\Phi(u)$ belong to $N^{1,p}(X)$ and

$$D(uv) = u\, Dv + v\, Du,$$
$$D(\Phi \circ u) = \Phi'(u)\, Du.$$

We modify the notion of p-harmonicity of [Sh2], [KiSh], and [HoSh] by minimizing $|Du|$ rather than g_u.

[1]This is an abstract analogue of the celebrated "$H = W$" theorem of Meyers-Serrin [MS].

Definition 2.3. Let $\Omega \subset X$ be a domain. A function $u : X \to [-\infty, \infty]$ is said to be *p-harmonic* on Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and for all relatively compact subsets U of Ω and for all functions $\varphi \in N^{1,p}_0(U)$,

$$\int_U |Du|^p \, d\mu \le \int_U |D(u+\varphi)|^p \, d\mu.$$

It must be noted that by the results in [Ch], the p-harmonic functions as defined above are p-quasiminimizers in the sense of [KiSh]. Moreover, p-harmonic functions can be characterized in terms of a weak formulation of the appropriate Euler-Lagrange equation: f is p-harmonic if and only if

$$\int_U |Du|^{p-2} Du \cdot D\phi \, d\mu = 0 \tag{4}$$

for all U and φ as in Definition 2.3. Here we make use of the fact that the norms $|\cdot|$ on the fibers of T^*X can be chosen to be inner product norms; see the discussion in [C, p. 460].

Non-negative Q-harmonic functions on spaces of (locally) Q^--bounded geometry satisfy a (local) Harnack inequality by results from [KiSh]. In particular, for each relatively compact domain Ω, there exists a constant $c_0 > 1$ such that

$$\sup_{B(x,r)} u \le c_0 \inf_{B(x,r)} u \tag{5}$$

whenever u is a non-negative Q-harmonic function in $B(x, 2r) \subset \Omega$. Consequently, for each compact subset $K \subset \Omega$ there exists a constant C_K so that

$$\sup_K u \le C_K \inf_K u \tag{6}$$

whenever u is a non-negative Q-harmonic function in Ω. Finally, as a consequence of the local LLC property, a Harnack inequality on a sphere $S(y, r) = \{x \in X : d(x, y) = r\}$ holds for non-negative Q-harmonic functions on the annulus $B(y, Cr) \setminus B(y, r/C)$ for sufficiently small r; see, e.g., [BMSh, Lemma 5.3].

3. The conformal Martin boundary

In [HoSh] it was shown that every relatively compact domain in a space of locally Q^--bounded geometry supports a Q-singular function which plays a role analogous to the Green function of the Q-Laplacian operator.

Definition 3.1. Let Ω be a relatively compact domain in a space X of locally Q^--bounded geometry and let $y \in \Omega$. An extended real-valued function $g = g(\cdot, y)$ on Ω is said to be a *Q-singular function with singularity at y* if it satisfies the following four criteria:

(i) g is Q-harmonic in $\Omega \setminus \{y\}$ and $g > 0$ on Ω;
(ii) $g\big|_{X\setminus\Omega} = 0$ p-q.e. and $g \in N^{1,Q}(X \setminus B(y, r))$ for all $r > 0$;
(iii) y is a singularity, i.e., $\lim_{x\to y} g(x) = \infty$;
(iv) whenever $0 \le a < b < \infty$,

$$\mathrm{Cap}_Q(\Omega^b; \Omega_a) = (b-a)^{1-Q}, \tag{7}$$

where $\Omega^b = \{x \in \Omega : g(x) \ge b\}$, $\Omega_a = \{x \in \Omega : g(x) > a\}$.

Here $\mathrm{Cap}_p(K;\Omega)$ denotes the *relative p-capacity* of a compact set K with respect to an open set $\Omega \supset K$; recall that this is equal to $\inf \int_\Omega |Du|^p \, d\mu$, the infimum being taken over all functions $u \in N^{1,p}(X)$ for which $u|K \geq 1$ and $u|X \setminus \Omega = 0$. If such functions do not exist, we set $\mathrm{Cap}_p(K;\Omega) = \infty$. For more on capacity, see [HeK2], [KiMa], [KaSh], [HKM, Chapter 2], and the references therein.

Since we have fixed the regularity exponent Q of the measure μ in this discussion, we shall simply call such functions singular functions, suppressing the reference to the index. The equality (7) replaces the double inequality in the analogous definition of singular functions of [HoSh], because the use of Cheeger's derivative in the definition of Q-harmonicity gives us access to the Euler-Lagrange equation (4); see [Ho, Lemma 3.8].

By [HeK1, Theorem 5.7], X is *locally Q-Loewner,* that is, there exists a decreasing function $\varphi = \varphi_X : (0,\infty) \to (0,\infty)$ and each point in X has a neighborhood U so that $\mathrm{Mod}_Q(E,F;B) \geq \varphi(t)$ whenever $B \subset U$ is a ball and E and F are two disjoint, nondegenerate continua in B with $\mathrm{dist}(E,F) \leq t \min\{\mathrm{diam}(E), \mathrm{diam}(F)\}$. Here $\mathrm{Mod}_Q(E,F;B)$ is the Q-modulus of the family of all curves in B joining E and F; see [HeK1, Section 2.3]. Hence the proof given in [Ho, Theorem 3.22] regarding the uniqueness of singular functions applies in this setting and we deduce the following:

Theorem 3.2. *Given a relatively compact domain $\Omega \subset X$ and $y \in \Omega$, there exists precisely one singular function on Ω with singularity at y.*

This observation enables us to define a Martin boundary in a manner similar to the classical potential theoretic Martin boundary.

Definition 3.3. Fix $x_0 \in \Omega$. Given a sequence (x_n) of points in Ω, we say that the sequence is *fundamental* (relative to x_0) if the sequence has no accumulation point in Ω and the sequence of normalized singular functions

$$M(x,x_n) := \frac{g(x,x_n)}{g(x_0,x_n)}$$

is locally uniformly convergent. Above we set $M(x,x_0) = 0$ for $x \neq x_0$ and $M(x_0,x_0) = 1$.

Given a fundamental sequence $\xi = (x_n)$, let us denote the limit function

$$M_\xi(x) := \lim_{n\to\infty} M(x,x_n).$$

We say that two fundamental sequences ξ and ζ are equivalent (relative to x_0), $\xi \sim \zeta$, if $M_\xi = M_\zeta$. It is worth noting that M_ξ is a non-negative Q-harmonic function in Ω, with $M_\xi(x_0) = 1$. Hence $M_\xi > 0$ in Ω by local Harnack's inequality. Note that if $\tilde{x}_0$ is another point in Ω, then $g(x,x_n)/g(\tilde{x}_0,x_n) = M(x,x_n)/M(\tilde{x}_0,x_n)$. This observation enables us to conclude that being a fundamental sequence is independent of the particular choice of x_0. Furthermore, fundamental sequences ξ and ζ are equivalent relative to x_0 if and only if they are equivalent relative to any $\tilde{x}_0 \in \Omega$. Thus the following definition is independent of the fixed point x_0.

Definition 3.4. The collection of all equivalence classes of fundamental sequences in Ω is the *conformal Martin boundary* $\partial_{cM}\Omega$ of the domain Ω. This collection is endowed with the local uniform topology: a sequence ξ_n in this boundary is said to converge to a point ξ if the sequence of functions M_{ξ_n} converges locally uniformly to M_ξ.

Because of the uniqueness of singular functions (see Theorem 3.2 above), we can also identify points y in Ω with their corresponding singular functions

$$M(\cdot, y) = g(\cdot, y)/g(x_0, y).$$

By using the uniqueness of singular functions together with results from [Sh3], we see that the topology inherited from the local uniform norm on the collection of such normalized singular functions is compatible with the topology of Ω and that $\Omega \cup \partial_{cM}\Omega$ is compact.

Remark 3.5. From now on we assume without further notice that no fundamental sequence (x_n) contains the point x_0.

Lemma 3.6. *For every $x_\infty \in \partial\Omega$ and a sequence (x_n) in Ω converging to x_∞ there exists a subsequence (x_{n_k}) which is fundamental.*

Proof. Note that $M(x_0, x_n) = 1$ for every $n \in \mathbb{N}$. Therefore the sequence $M(x, x_n)$ is locally bounded by the Harnack inequality; see [KiSh]. By [Sh3, Proposition 4.1] we obtain a locally uniformly convergent subsequence which yields a Q-harmonic limit. $\square$

By Lemma 3.6 we see that given a point in the metric boundary of Ω we can associate with it at least one point on the Martin boundary. Our goal now is to prove Theorem 1.1, that is, to show (under suitable hypotheses) that we can associate to each point in the Martin boundary a unique point on the metric boundary. In what follows, we denote by $l(\gamma)$ the length of a rectifiable curve γ.

Definition 3.7. Let $\Omega \subsetneq X$ be a proper subdomain and let $A \geq 1$. We say that Ω is an *A-uniform* domain if every pair of distinct points $x, y \in \Omega$ can be joined by a rectifiable curve γ lying in Ω for which $l(\gamma) \leq A\, d(x, y)$ and

$$\min\{l(\gamma_{xz}), l(\gamma_{zy})\} \leq A\, \delta(z)$$

for all points z on γ. Here $\delta(z) = \delta_\Omega(z) = \operatorname{dist}(z, X \setminus \Omega)$ denotes the distance from z to the complement of Ω and γ_{ab} denotes the portion of the curve γ which lies between a and b. A curve γ in Ω which satisfies both of these conditions is said to be an *A-uniform curve.* We say that Ω is *uniform* if it is A-uniform for some A.

See [Ma] for more information on uniform domains in the Euclidean setting, and [BHK], [V] for the general metric setting.

In Euclidean space, uniform domains contain no outward pointing cusps. The following lemma provides a version of this statement in general metric spaces. Here, for $x_\infty \in \partial\Omega$ and $r > 0$, $m > 0$, we define

$$\Lambda(x_\infty, r, m) = \{x \in \Omega \cap \bar{B}(x_\infty, r) : \delta(x) \geq mr\}$$

to be the set of points in the (closed) ball of radius r about x_∞ which lie at a distance $\geq mr$ from the complement of Ω.

Lemma 3.8. *Let $\Omega \subset X$ be an A-uniform domain and let $x_\infty \in \partial\Omega$. Then $\Lambda(x_\infty, r, m) \neq \emptyset$ provided $0 < r < \frac{1}{4} \operatorname{diam} \Omega$ and $0 < m \leq \frac{1}{2A}$.*

Proof. It suffices to prove that $\Lambda(x_\infty, r, \frac{1}{2A}) \neq \emptyset$ since $\Lambda(x_\infty, r, m') \subset \Lambda(x_\infty, r, m)$ if $0 < m \leq m'$. Choose $x \in B(x_\infty, \frac{1}{2}r) \cap \Omega$ and $y \in \Omega \setminus B(x_\infty, 2r)$. Let γ be an A-uniform curve joining x to y and let $z \in \gamma \cap \partial B(x_\infty, r)$; such a point clearly exists because γ is a connected set. Moreover, $z \in \Omega$. Thus,

$$\delta(z) \geq \frac{1}{A} \min\{l(\gamma_{xz}), l(\gamma_{zy})\} \geq \frac{1}{A} \min\{d(x,z), d(z,y)\} \geq \frac{1}{2A} r.$$

□

In the proof of Theorem 1.1 we will make use of several results (Lemmas 3.9, 3.11 and 3.12) on the behavior of positive Q-harmonic functions near the boundary of Ω. Analogous results for non-negative harmonic functions in non-tangentially accessible (NTA) domains were proved in [JK, Lemma 4.4] and for positive solutions of uniformly elliptic diffusion operators in [Bau, Lemmas 2.2–2.4]. For the sake of completeness and also because the extension of these results from Euclidean space to general metric spaces involves some slight modifications, we present here detailed proofs of the corresponding lemmata for Q-harmonic functions. However, we wish to acknowledge the debt which our results owe to [Bau] and [JK].

Lemma 3.9. *Let Ω be an A-uniform domain and let $x_\infty \in \partial\Omega$. Let $0 < r < \frac{1}{4} \operatorname{diam} \Omega$ and let u be a positive Q-harmonic function defined on $B(x_\infty, 4r) \cap \Omega$. Then there exist constants A' and C_1 (depending only on A, the diameter of Ω, and the constants involved in the local Q^--bounded geometry of X) so that*

$$u(x) \leq C_1^h u(y) \tag{8}$$

for all natural numbers h and all x and y in $\Lambda(x_\infty, r/A', 2^{-h})$.

The proof of Lemma 3.9 makes use of quasihyperbolic geometry and, in particular, of a characterization of uniform domains in terms of a growth condition on the quasihyperbolic metric. Recall that the *quasihyperbolic metric* k_Ω in a domain $\Omega \subsetneq X$ is defined to be

$$k_\Omega(x,y) := \inf_\gamma \int_\gamma \frac{ds(z)}{\delta(z)}, \qquad x, y \in \Omega, \tag{9}$$

where the infimum is taken over all rectifiable curves γ joining x to y in Ω and the integral denotes the line integral of the weight $\delta(z)^{-1}$ over γ, evaluated by using the arc length parametrization; see [GP]. Any two points in Ω can always be joined by (at least) one *quasihyperbolic geodesic*, i.e., a curve γ which achieves the infimum in (9). See [GO, Lemma 1] or [BHK, Section 2]. For an overview of some useful applications of the quasihyperbolic metric, see [K].

Proof. First we note that a quasihyperbolic geodesic connecting given points in Ω is an A'-uniform curve for some A' depending only on A. See, e.g., [BHK, Theorem 2.10] or [V, Theorem 2.29]. Let $h \in \mathbb{N}$ and let x and y be points in $\Lambda(x_\infty, r/A', 2^{-h})$. Denote by γ a quasihyperbolic geodesic joining x to y. Then γ is an A'-uniform curve. Note that $\gamma \subset B(x_\infty, 2r) \cap \Omega$ since $l(\gamma) \leq A'd(x,y) \leq 2r$ and so $\gamma \subset B(x,r) \cup B(y,r) \subset B(x_\infty, 2r)$.

We now construct a chain of balls B_i, $i = 1, \ldots, N$, covering γ satisfying the following three conditions:

(i) $B_i \cap B_{i+1} \neq \emptyset$ for each i;

(ii) $2B_i \subset B(x_\infty, 4r) \cap \Omega$;
(iii) $N \le 3k_\Omega(x,y)$.

Set $x_1 = x$, $r_1 = \delta(x_1)/2$ and $B_1 = B(x_1, r_1)$. Let x_2 be the terminal point of γ in the ball B_1 (thus $x_2 \in \partial B_1$), $r_2 = \delta(x_2)/2$ and $B_2 = B(x_2, r_2)$. Continuing this process, we construct a sequence of balls $B_i = B(x_i, r_i)$. Since these are open balls and $x_{i+1} \in \partial B_i$, it is clear that (i) holds. Moreover, $2B_i \subset \Omega$ by construction. If $z \in 2B_i$, then

$$d(x_\infty, z) \le d(x_\infty, x_i) + 2r_i = d(x_\infty, x_i) + \delta(x_i) \le 2d(x_\infty, x_i) \le 4r$$

since $\gamma \subset B(x_\infty, 2r)$. This shows that (ii) holds. Finally, let γ_i denote that portion of the geodesic γ contained in B_i which joins x_i to x_{i+1}. These subcurves γ_i are all disjoint. If $z \in \gamma_i$ then $\delta(z) \le \delta(x_i) + r_i = 3r_i$. For each i, we estimate

$$\int_{\gamma_i} \frac{|dz|}{\delta(z)} \ge \frac{1}{3}\frac{l(\gamma_i)}{r_i} \ge \frac{1}{3}.$$

Summing over i, we see that

$$k_\Omega(x,y) = \int_\gamma \frac{|dz|}{\delta(z)} = \sum_i \int_{\gamma_i} \frac{|dz|}{\delta(z)} \ge \frac{1}{3}N$$

which proves (iii) (and incidentally establishes that the number of balls in the chain is finite).

Next, we recall that in an A-uniform domain, the following logarithmic estimate for the quasihyperbolic metric holds true:

$$k_\Omega(a,b) \le 4A^2 \log\left(1 + \frac{d(a,b)}{\min\{\delta(a), \delta(b)\}}\right) \tag{10}$$

for all $a, b \in \Omega$. See [BHK, Lemma 2.13]. Thus

$$\begin{aligned} N \le 3k_\Omega(x,y) &\le 12A^2 \log\left(1 + \frac{d(x,y)}{\min\{\delta(x), \delta(y)\}}\right) \\ &\le 12A^2 \log(1 + 2^{h+1}) \\ &\le 12A^2 \log(2^{3h}) \\ &\le 25A^2 h, \end{aligned} \tag{11}$$

where we have used the fact that $x, y \in \Lambda(x_\infty, r/A', 2^{-h})$.

Since u is a positive Q-harmonic function on $B(x_\infty, 4r) \cap \Omega$, by a repeated use of the Harnack inequality on the balls B_i, we see that

$$u(x) \le c_0^N u(y) \le C_1^h u(y),$$

where $c_0 > 1$ denotes the Harnack constant in (5) and $C_1 = c_0^{25A^2}$. Thus (8) holds. $\square$

Our second lemma states a boundary regularity result for Q-harmonic functions. This result requires a capacity density condition in the complement of the domain, specifically, the so-called uniform Q-fatness condition alluded to in the introduction.

Definition 3.10. We say that Ω has *uniformly Q-fat complement* if there exist constants $c > 0$ and $r_0 > 0$ so that

$$\frac{\mathrm{Cap}_Q(\bar{B}(x,r) \setminus \Omega; B(x,2r))}{\mathrm{Cap}_Q(\bar{B}(x,r); B(x,2r))} \geq c$$

for every $x \in X \setminus \Omega$ and $r \in (0, r_0)$.

Recall that $\mathrm{Cap}_Q(E,U)$ denotes the Q-capacity of E in U, see the remarks following Definition 3.1. For example, a domain obtained by removing a nondegenerate continuum from a ball in $\mathbb{R}^n$ has uniformly n-fat complement. On the other hand, any domain in a metric space of locally Q^--bounded geometry whose boundary contains an isolated point does not have uniformly Q-fat complement, since single points have zero Q-capacity. See [HKM], [Le], [Mi], and [BMSh] for additional information on the uniform fatness condition.

From now on r_0 refers to the constant in the Q-fatness condition of the complement of Ω.

Lemma 3.11. *Let Ω be a bounded domain with uniformly Q-fat complement. Let $x_\infty \in \partial\Omega$. Then there exist constants $C_2 < \infty$ and $\alpha > 0$ so that*

$$\sup_{y \in B(x_\infty, r) \cap \Omega} u(y) \leq C_2 \left(\frac{r}{R}\right)^\alpha \sup_{z \in \Omega \cap \partial B(x_\infty, R)} u(z)$$

whenever $0 < r \leq R < r_0$ and u is a positive Q-harmonic function in $B(x_\infty, R) \cap \Omega$ which vanishes continuously on $\partial\Omega$.

For a proof, see [BMSh, Theorem 5.1]. Observe that the LLC condition is used here.

Our third lemma is a Carleson-type estimate on the values of positive Q-harmonic functions. The proof is modelled on that of Lemma 4.4 in [JK]. See also [Bau], [Ca], and [CFMS].

Lemma 3.12. *Let Ω be a bounded A-uniform domain with uniformly Q-fat complement. Let $x_\infty \in \partial\Omega$, $0 < r < \min\{\frac{1}{4}\operatorname{diam}\Omega, r_0\}$, and let u be a positive Q-harmonic function on $B(x_\infty, 4r) \cap \Omega$ which vanishes continuously on $\partial\Omega$. Then there exists a constant C_3 so that*

$$\sup_{y \in B(x_\infty, r) \cap \Omega} u(y) \leq C_3 \sup_{z \in \Lambda(x_\infty, \frac{r}{A'}, \frac{1}{2A})} u(z). \tag{12}$$

Here $A' = A'(A)$ denotes the constant of Lemma 3.9. Moreover, for all $\beta \leq \frac{1}{2A}$ there exists a constant $C_3'(\beta) \geq 1$ so that

$$\sup_{y \in B(x_\infty, r) \cap \Omega} u(y) \leq C_3'(\beta) u(z) \tag{13}$$

for all $r < r_0/2A'$ and all points z in the set $\Lambda(x_\infty, r, \beta)$.

Note that $\Lambda(x_\infty, r, \beta)$ is nonempty provided r is sufficiently small by Lemma 3.8.

Proof. By a preliminary normalization, we may assume without loss of generality that

$$\sup_{z \in \Lambda(x_\infty, \frac{r}{A'}, \frac{1}{2A})} u(z) = 1.$$

Next, by Lemma 3.11, and by choosing $C_4 \geq \max\{2, (2C_2)^{1/\alpha}\}$ we see that

$$\sup_{y \in B(z_\infty, s/C_4) \cap \Omega} u(y) \leq \frac{1}{2} \sup_{z \in B(z_\infty, s) \cap \Omega} u(z) \tag{14}$$

for all $z_\infty \in B(x_\infty, 4r) \cap \partial\Omega$ and all s such that $B(z_\infty, s) \subset B(x_\infty, 4r)$. By Lemma 3.9, there exists $C_5 > A$ so that if $h \in \mathbb{N}$ with $2^h \geq 2A$, and $w \in B(x_\infty, \frac{r}{A'})$ with $u(w) > C_5^h$, then

$$\delta(w) < C_4^{-h} r. \tag{15}$$

Choose $M \geq 5$ so that $2^M \geq C_5$ and set $C_3 = C_5^M$. Suppose that there exists $y_0 \in B(x_\infty, r) \cap \Omega$ with $u(y_0) > C_3 = C_5^M$. Then $\delta(y_0) < C_4^{-M} r$. Denote by $y_{0,\infty}$ a point of $\partial\Omega$ for which $d(y_{0,\infty}, y_0) = \delta(y_0)$. Then

$$d(y_{0,\infty}, x_\infty) \leq d(y_{0,\infty}, y_0) + d(y_0, x_\infty) \leq C_4^{-M-5} r + r \leq \frac{33}{32} r.$$

Consequently $B(y_{0,\infty}, r) \subset B(x_\infty, 4r)$ and by applying (14) we see that

$$\sup_{y \in B(y_{0,\infty}, r) \cap \Omega} u(y) \geq 2^M \sup_{y \in B(y_{0,\infty}, C_4^{-M} r) \cap \Omega} u(y) \geq 2^M u(y_0) > C_5^{M+1}.$$

Hence we may choose $y_1 \in B(y_{0,\infty}, r) \cap \Omega$ so that $u(y_1) > C_5^{M+1}$ and (by (15)) $\delta(y_1) < C_4^{-M-1} r$. Let $y_{1,\infty}$ be a point of $\partial\Omega$ for which $d(y_{1,\infty}, y_1) = \delta(y_1)$. Continuing in this fashion, we construct two sequences, (y_k) and $(y_{k,\infty})$, with

$$\begin{gathered} y_{k,\infty} \in B(x_\infty, 4r) \cap \partial\Omega \\ y_k \in B(y_{k-1,\infty}, C_4^{-k} r) \cap \Omega \\ \delta(y_k) = d(y_k, y_{k,\infty}) < C_4^{-M-k} r \\ B(y_{k-1,\infty}, C_4^{-k} r) \subset B(x_\infty, 4r) \\ u(y_k) > C_5^{M+k}. \end{gathered}$$

Now the fact that $\delta(y_k) \to 0$ and $u(y_k) \to +\infty$ contradicts the hypothesis that u vanishes continuously on $\partial\Omega$. Thus (12) is verified.

As observed above, $\Lambda(x_\infty, r, \beta)$ is nonempty provided r is sufficiently small and $\beta \leq \frac{1}{2A}$. Moreover, given two points x, y in $\Lambda(x_\infty, r, \beta)$ for some $\beta \in (0,1)$ and $r < r_0/(2A')$, we see that if γ is a quasihyperbolic geodesic connecting x and y in Ω, then $l(\gamma) \leq 2A'r$ and $\delta(z) \geq \beta r/(2A')$ for all points $z \in \gamma$ because $\delta(x), \delta(y) \geq \beta r$ (see Lemma 3.15 below). Therefore, γ can be covered by at most $8(A')^2/\beta$ balls of radii $\beta r/(4A')$. The Harnack inequality can be applied to these balls, and hence inequality (13) follows from (12), where $C_3'(\beta) = (1/c_0)^{8(A')^2/\beta} C_3$. □

We now apply the preceding results to the case of singular functions and in particular to sequences of singular functions giving rise to a Martin boundary function M_χ. The following lemma, which is the key to the proof of Theorem 1.1, establishes the boundary Hölder continuity of Martin boundary functions.

Lemma 3.13. *Let Ω be a uniform domain with uniformly Q-fat complement. Let $x_\infty \in \partial\Omega$ and let $\chi = (x_n)$ be a fundamental sequence with* $\lim_n x_n = x_\infty$. *Then M_χ*

vanishes continuously on $\partial\Omega \setminus \{x_\infty\}$*: for each* $y_\infty \in \partial\Omega \setminus \{x_\infty\}$ *there exists* $r_{y_\infty} > 0$ *so that*

$$\sup_{y \in B(y_\infty, r) \cap \Omega} M_\chi(y) \le C r^s,$$

for every $0 < r \le r_{y_\infty}$*, where* C *and* s *are constants which are independent of* r*.*

Proof. Let $y_\infty \in \partial\Omega \setminus \{x_\infty\}$ and let $\epsilon = \frac{1}{3} d(x_\infty, y_\infty)$. By Lemma 3.11, there exists $s > 0$ so that

$$\sup_{y \in B(y_\infty, r) \cap \Omega} M(y, x_n) \le C \left(\frac{r}{R}\right)^s \sup_{z \in \Omega \cap \partial B(y_\infty, R)} M(z, x_n)$$

whenever $0 < r < R < \min\{r_0, \epsilon\}$ and $x_n \in B(x_\infty, \epsilon)$. Thus it suffices to verify that

$$\sup_n \sup_{z \in \Omega \cap \partial B(y_\infty, R)} M(z, x_n) < \infty$$

for some R as above. By Lemma 3.12, it is enough to show that

$$\sup_n \sup_{z \in \Lambda(y_\infty, \frac{R}{A'}, \frac{1}{2A})} M(z, x_n) < \infty$$

for some R. But observe that

$$\Lambda\left(y_\infty, \frac{R}{A'}, \frac{1}{2A}\right) \subset \left\{x \in \Omega : \delta(x) \ge \frac{R}{2AA'}\right\}$$

which is a compact subset of Ω. Choose, for example, $R = \frac{1}{2}\min\{r_0, \epsilon\}$. Then, letting K be a connected compact subset of Ω containing $\{x \in \Omega : \delta(x) \ge \frac{R}{2AA'}\} \cup \{x_0\}$, we see that

$$M(z, x_n) \le C_K M(x_0, x_n) = C_K$$

for all $z \in K$ and $n \in \mathbb{N}$, where C_K denotes a Harnack constant for the compact set K as in (6). The proof is complete. ☐

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. We begin by showing that whenever $\chi = (x_n)$ and $\xi = (z_n)$ are two fundamental sequences giving rise to the same Martin boundary function $M_\chi = M_\xi$, then $\lim_n x_n$ and $\lim_n z_n$ exist and are equal. Since $\bar{\Omega}$ is compact, it suffices to show that $x_\infty = z_\infty$ provided the limits $\lim_n x_n = x_\infty$ and $\lim_n z_n = z_\infty$ exist.

Suppose that $x_\infty \ne z_\infty$. Then by Lemma 3.13, M_χ vanishes continuously on $\partial\Omega \setminus \{x_\infty\}$ and M_ξ vanishes continuously on $\partial\Omega \setminus \{z_\infty\}$. Thus $M := M_\chi = M_\xi$ vanishes continuously on the entire boundary $\partial\Omega$ of Ω. For $\epsilon \in (0,1)$ let $\Omega_\epsilon := \{x \in \Omega : M(x) > \epsilon\}$. Then the closure of Ω_ϵ is a compact subset of Ω. Let $\Omega_{\epsilon,0}$ be the component of Ω_ϵ containing x_0. Then M is a Q-harmonic function on $\Omega_{\epsilon,0}$ which is in $N^{1,Q}(\bar{\Omega}_{\epsilon,0})$ and $M = \epsilon$ on $\partial\Omega_{\epsilon,0}$. By the maximum principle (see [Sh2] or [KiSh]), M is the constant function $M = \epsilon$ in $\Omega_{\epsilon,0}$. This contradicts the fact that $M(x_0) = 1$ and hence it follows that $x_\infty = z_\infty$.

By appealing to the above discussion, we see that there exists a well-defined surjective mapping from $\partial_{cM}\Omega$ onto $\partial\Omega$. The continuity of this map follows directly from the definition of the topology on the Martin boundary $\partial_{cM}\Omega$. ☐

In general the above map need not be injective. For example, the slit disc in $\mathbb{R}^2$ fails to have this property. Each boundary point on the removed radius corresponds naturally to two distinct Martin boundary points.

Question 1: When is the map defined in Theorem 1.1 injective and hence a homeomorphism?

Question 1′: In the linear setting of usual harmonic functions, Question 1 is closely related to the validity of so-called boundary Harnack principle; see [Ai], [An1], [An2], [AS], [JK], and [Wu]. It would be very interesting to know for which domains Ω the boundary Harnack principle holds for Q-harmonic functions and whether this would be useful in showing that $\partial_{cM}\Omega = \partial\Omega$.

Question 2: It is easy to see that the topology on the conformal Martin compactification $\Omega \cup \partial_{cM}\Omega$ is metrizable. One possible candidate metric is

$$d_Q(x,y) = \int_\Omega \frac{|M(w,x) - M(w,y)|}{1 + |M(w,x) - M(w,y)|}\, d\mu(w), \tag{16}$$

and the topology on Ω induced by this metric is compatible with the underlying topology. Is this metric locally biLipschitz or locally Hölder equivalent with the underlying metric on Ω? It would also be interesting to know how this metric is distorted by quasiconformal maps.

Question 3: In the Euclidean setting and for the classical Martin boundary, Aikawa [Ai] has shown the equivalence of the Martin boundary and the metric boundary assuming only the uniformity, which is an interior condition on Ω. Is the exterior condition of Q-fatness on the complement of Ω needed in Theorem 1.1, or can it be replaced with the milder condition of regularity of the boundary of Ω?

Question 3′: It is easy to see that the estimate of Lemma 3.11 fails if Ω does not satisfy the exterior condition of Q-fatness; consider, for example, the domain $\Omega = B(0,1) \setminus \{(\frac{1}{2},0),(\frac{-1}{2},0)\} \subset \mathbb{R}^n$. This domain does not violate the conclusions of Theorem 1.1. However, if we know that for every point $x^* \in \partial\Omega$, each Martin function M_χ associated with a fundamental sequence $\chi = (x_n)_n$ tending to x^* is unbounded in every neighborhood of x^*, then a modified version of Lemma 3.13 would help us prove Theorem 1.1. Simply connected proper subdomains of $\mathbb{R}^2$ satisfy this condition. Do all inner uniform domains satisfy this condition?

Our next proposition can be informally stated as follows: the values of a singular function can decrease by at most an absolute multiplicative constant under the operation of taking the quasihyperbolic convex hull. While this result plays no role in the proof of Theorem 1.1, we feel that it is of some independent interest and illustrates some different conclusions that can be drawn from Lemma 3.12.

Proposition 3.14. *Let Ω be a uniform domain whose complement is uniformly Q-fat. Let g be a singular function on Ω with singularity at $w \in \Omega$. For $\tau \in (0,\infty]$, set*

$$E_\tau := \{z \in \Omega \,:\, g(z) \geq \tau\}.$$

Then every quasihyperbolic geodesic γ connecting two points $x, y \in E_\tau$ lies entirely in the set $E_{c\tau}$, where c is a positive constant which is independent of x, y, τ and g.

The proof of this proposition needs the following two lemmata.

Lemma 3.15. *Let x, y be two points in Ω so that $\min\{\delta(x), \delta(y)\} \geq r$. Then whenever γ is a quasihyperbolic geodesic in Ω connecting x and y, every point $z \in \gamma$ has the property $\delta(z) \geq \frac{r}{2A'}$.*

Proof. Suppose there is a point $z \in \gamma$ so that $\delta(z) < \frac{r}{2A'}$. Without loss of generality, we can assume that $l(\gamma_{xz}) \leq l(\gamma_{zy})$. Then, as γ is A'-uniform, we see that $l(\gamma_{xz}) \leq r/2$. Therefore, $d(x, z) \leq r/2$. However, $\delta(z) \geq \delta(x) - d(z, x)$. Therefore, we have $\delta(z) \geq r - \frac{r}{2} = r/2 \geq \frac{r}{2A'}$, leading to a contradiction. Therefore, no such point $z \in \gamma$ exists, and the proof is now complete. □

Lemma 3.16. *Let $x \in \Omega$ and γ be a quasihyperbolic geodesic from x to another point y in Ω so that $\delta(y) \geq \delta(x)$. Let Q_x denote a point on $\partial\Omega$ so that $\delta(x) = d(Q_x, x)$. Then if $z \in \gamma$ so that $l(\gamma_{xz}) \leq l(\gamma_{zy})$, one of the following two conditions hold true for z:*

1. *$d(Q_x, z) \leq 3\delta(x)$, and $\delta(z) \geq \frac{\delta(x)}{2A'}$,*
2. *$d(Q_x, z) \geq 3\delta(x)$, and $\delta(z) \geq \frac{2}{3A'} d(z, Q_x)$.*

The proof of Lemma 3.16 is an easy consequence of the uniformity of Ω together with Lemma 3.15, provided we observe in the second case of Lemma 3.16 that $l(\gamma_{xz}) \geq d(x, z) \geq \frac{2}{3} d(Q_x, z)$. We therefore leave the proof of this result to the reader.

Proof of Proposition 3.14. Let x, y be two points in E_τ. Let γ be a quasihyperbolic geodesic connecting x and y in Ω. We consider four cases. In what follows, $r_1 = r_0/(4A')^2$, where r_0 is the bound on radii associated with the uniform Q-fatness of the complement of Ω.

Case 1: Both x and y are at least a distance $\frac{r_1}{2A'}$ from the boundary of Ω; that is, $\min\{\delta(x), \delta(y)\} \geq \frac{r_1}{2A'}$. Then, by Lemma 3.15, $\gamma \subset \{z \in \Omega \colon \delta(z) \geq \frac{r_1}{(2A')^2}\}$ which is a compact subset of Ω. By the Harnack inequality (6), we see that for all points z in γ,

$$g(z) \geq \frac{1}{C_1}\tau. \tag{17}$$

Case 2: $\delta(x) = \min\{\delta(x), \delta(y)\} < \frac{r_1}{2A'}$, and $d(x, y) \leq 3\delta(x)$.

Then, by the fact that γ is an A'-uniform curve, $l(\gamma) \leq A' d(x, y) \leq 3A'\delta(x) < r_0/2$, and hence $\gamma \subset B(Q_x, 4A'\delta(x))$, and by Lemma 3.15, for all $z \in \gamma$ we have $\delta(z) \geq \delta(x)/(2A')$. Therefore, applying the second conclusion of Lemma 3.12 with $r = 4(A')^2\delta(x)$ and $x_\infty = Q_x$, we see that all $z \in \gamma$ satisfy

$$g(z) \geq \frac{1}{C_3}\tau. \tag{18}$$

Case 3: $\delta(x) = \min\{\delta(x), \delta(y)\} < \frac{r_1}{2A'}$, and $3\delta(x) < d(x, y) \leq 3A'\delta(x)$.

Now, we can break γ up into subarcs γ_i, $i = 1, \ldots, N$ with $N \leq 6(A')^2$, so that $\frac{1}{2A'}\delta(x) \leq l(\gamma_i) \leq \delta(x)/A'$. Let x_i, y_i denote the endpoints of γ_i. If both $\delta(x_i)$ and $\delta(y_i)$ are not smaller than $\frac{r_1}{2A'}$, then the argument of Case 1 covers the points on γ_i. If on the other hand $\min\{\delta(x_i), \delta(y_i)\} \leq \frac{r_1}{2A'}$, then as $d(x_i, y_i) \leq 2\min\{\delta(x_i), \delta(y_i)\}$, the points in γ_i are covered by the argument of Case 2. Thus we see that for every point $z \in \gamma$,

$$g(z) \geq \left(\min\{1/C_1, 1/C_3\}\right)^{6(A')^2} \tau. \tag{19}$$

Case 4: $\delta(x) = \min\{\delta(x), \delta(y)\} < \frac{r_1}{2A'}$, and $d(x, y) > 3A'\delta(x)$.

In this case, let z_0 denote the midpoint of γ, and consider the ball $B(Q_x, r_1)$. Let x' denote the point on γ at which γ first exits the ball $B(Q_x, r_1)$. We consider two subcases:

Case 4a: $z_0 \notin \gamma_{xx'}$. Then for every point z in $\gamma_{xx'}$, $l(\gamma_{xz}) \le l(\gamma_{zy})$. Now we are in the situation of Lemma 3.16. Consider $z \in \gamma_{xx'}$, and let $r = 2\delta(x)$ in the first case considered in this lemma, and let $r = \frac{3}{2}d(Q_x, z)$ in the second case. Then $x \in B(Q_x, r)$ and $z \in \Lambda(Q_x, r, \frac{1}{2A'})$. Now, by Lemma 3.12, we have

$$g(z) \ge \frac{1}{C_3}\tau. \tag{20}$$

Case 4b: $z_0 \in \gamma_{xx'}$ or x' does not exist. Then, we repeat the argument of Case 4a for points in γ_{xz_0}, to obtain inequality (20) for points in γ_{xz_0}.

Together, Cases 4a, 4b, and Case 1 yield inequality (20) for all points z in the segment γ_{xz_0}. We repeat Case 4 for the segment γ_{z_0y} by replacing the role of x with the role of y in the case that $\delta(y) \le \frac{r_1}{2A'}$, and we repeat the argument of Case 1 for the segment γ_{z_0y} otherwise, with x' playing the role of x.

Together, the above four cases yield the desired result. □

4. The conformal Martin boundary and conformal mappings

It was shown in [HKShT] that if $Y_1 = (Y_1, d_1, \mu_1)$ and $Y_2 = (Y_2, d_2, \mu_2)$ are two metric measure spaces of locally Q-bounded geometry, then a homeomorphism $f : Y_1 \to Y_2$ is quasiconformal if and only if $f \in N^{1,Q}_{\mathrm{loc}}(Y_1; Y_2)$ and there exists a constant $K \ge 1$ so that

$$\operatorname{Lip} f(x)^Q \le K\, J_f(x)$$

for μ-almost every $x \in Y_1$. See [HKShT, Theorem 9.8]. Here

$$\operatorname{Lip} f(x) = \limsup_{r\to 0} \left(\operatorname{ess\,sup}_{d_1(x,y)\le r} \frac{d_2(f(x), f(y))}{r} \right)$$

denotes the *maximal stretching* of f at x and

$$J_f(x) = \limsup_{r\to 0} \frac{\mu_2(fB(x, r))}{\mu_1(B(x, r))}$$

denotes the *infinitesimal volume distortion* of f at x. For the definition of the metric space-valued Sobolev space $N^{1,Q}_{\mathrm{loc}}(Y_1; Y_2)$; see [HKShT, Section 3].

Under our standing assumptions on X (see section 2), if $\Omega \subset X$ is a relatively compact domain, then Ω is of locally Q-bounded geometry. Thus if $\Omega \subset X$ and $\Omega' \subset Y$ are relatively compact subdomains of metric measure spaces of locally Q-bounded geometry, then the results of [HKShT, Section 9] apply to quasiconformal maps from Ω to Ω'. Let $f : \Omega \to \Omega'$ be such a map. By the discussion in [HKShT, Section 10], there exists a matrix-valued map df, the *transposed Jacobian,* on Ω so that for every Lipschitz function φ on Ω',

$$D(\varphi \circ f)(x) = df(x)\, D\varphi(f(x))$$

for μ-a.e. $x \in \Omega$.

The following lemma is an immediate consequence of the discussion in [C, p. 460] together with [C, Lemma 4.35] which proves that $|D\varphi| \approx \mathrm{Lip}\,\varphi$ for Lipschitz functions φ. Here we denote by $\|df(x)\|$ the operator norm of $df(x)$ as a map from the fiber $F'_{\varphi(x)}$ in the generalized cotangent bundle T^*Y over $\varphi(x)$ to the fiber F_x in T^*X over x.

Lemma 4.1. *For μ_X-a.e. $x \in \Omega$, $\|df(x)\| \le C\ \mathrm{Lip}\, f(x)$.*

Definition 4.2. We say that a homeomorphism $f : \Omega \to \Omega'$ is a *conformal mapping* if it is a quasiconformal mapping and for μ_X-a.e. $x \in \Omega$,

$$\|df(x)\|^Q \le J_f(x). \tag{21}$$

Note by the above lemma that if $f : \Omega \to \Omega'$ is quasiconformal, then there exists $K_0 \ge 1$ so that $\|df(x)\|^Q \le K_0\, J_f(x)$ for μ_X-a.e. $x \in \Omega$. Thus f is conformal if $K_0 = 1$.

Remark 4.3. Observe that the Cheeger derivative structures on X and Y are not unique. Thus, unlike the definition of quasiconformality, the definition of conformality given above depends heavily on the choice of the derivative structures of both X and Y. Indeed, the identity map $f(x) = x$ from $X = \mathbb{R}^n$ to $Y = \mathbb{R}^n$ is not conformal if Y is equipped with the standard Euclidean differentiation structure and X is equipped with a nonstandard Cheeger differentiation structure, say, the structure in which $D\varphi(x) = 2\nabla\varphi(x)$ if the first component of the vector x is negative and $D\varphi(x) = \nabla\varphi(x)$ if the first component of x is non-negative. Recall that the coordinate charts U_α in the Cheeger differentiation structure need only be measurable sets.

Lemma 4.4. *If $f : \Omega \to \Omega'$ is a conformal map and $u : \Omega \to \mathbb{R}$ is Q-harmonic on a domain $V \subset \Omega$, then $u' := u \circ f^{-1}$ is Q-harmonic on $f(V) \subset \Omega'$.*

Proof. First, $u' \in N^{1,Q}_{\mathrm{loc}}(f(V))$ by [HKShT, Theorem 9.10]. Next, let U' be a compact subdomain of $f(V)$ and $\varphi' \in N^{1,Q}_0(U')$. By the Q-harmonicity of u,

$$\int_U |Du(x)|^Q\, dx \le \int_U |D(u+\varphi)(x)|^Q\, dx,$$

where $U = f^{-1}(U')$ and $\varphi = (u' + \varphi') \circ f - u$. By the absolute continuity of quasiconformal mappings (see [HKShT, Section 9]) and by the definition of conformality,

$$\int_{U'} |Du'(x')|^Q\, dx' \le \int_{U'} |D(u'+\varphi')(x')|^Q\, dx',$$

and so u' is Q-harmonic. □

Corollary 4.5. *If $f : \Omega \to \Omega'$ is a conformal map and g is a (Q-)singular function on Ω, then $g \circ f^{-1}$ is a singular function on Ω'.*

The only additional point worth noting for the proof of Corollary 4.5 is the invariance of capacity under conformal maps: if K is a compact subset of Ω and $f : \Omega \to \Omega'$ is conformal, then

$$\mathrm{Cap}_Q(f(K), \Omega') = \mathrm{Cap}_Q(K, \Omega).$$

Theorem 1.2 clearly follows directly from Corollary 4.5. Indeed, the image $\xi' = (f(x_n))_n$ of a fundamental sequence $\xi = (x_n)_n$ in Ω is again a fundamental sequence in Ω'.

We use Theorem 1.2 to give an example of a domain Ω for which the conclusion of Theorem 1.1 is false: a nontrivial portion of the boundary $\partial\Omega$ is collapsed onto a single point in the conformal Martin boundary $\partial_{cM}\Omega$.

Example 4.6. Consider the simply connected domain $\Omega \subset \mathbb{C}$ constructed in the following manner. Let $\{\theta_n : n \in \mathbb{N}\}$ be an enumeration of all the rational numbers between 0 and 2π and let $r_n = 1 - 2^{-n}$. For $n \in \mathbb{N}$, set

$$C_n := \{r_n e^{i\theta} : |\theta - \theta_n| \geq 2^{-n-2} \bmod 2\pi\} \cup \{-re^{i\theta_n} : r_n \leq r \leq r_{n+1}\}$$

and let $\Omega = \mathbb{D} \setminus \bigcup_{n=1}^{\infty} C_n$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. See Figure 1.

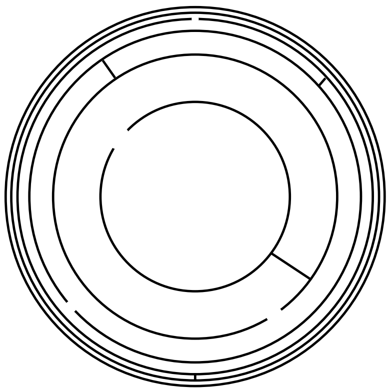

FIGURE 1. A "maze-like" domain Ω

Let f be a conformal map from $\mathbb{D}$ onto Ω. By Theorem 1.2, f extends as a homeomorphism between the respective Martin boundaries in a natural manner. Now the Martin boundary of $\mathbb{D}$ is equivalent with the metric boundary $S(0,1)$ of $\mathbb{D}$. By the theory of prime ends (see [N]), $S(0,1)$ is naturally equivalent to the set of prime ends of $\mathbb{D}$, and f extends as a homeomorphism between the sets of prime ends of $\mathbb{D}$ and Ω in a natural manner. Observe that there is only one prime end corresponding to the outer boundary $S(0,1)$ of Ω. Hence each point on $S(0,1) \subset \partial\Omega$ corresponds to the same Martin kernel function; $S(0,1)$ collapses to a single point on the Martin boundary.

Question 4: Example 4.6 raises the question whether, for a domain Ω, for which the conclusion of Theorem 1.1 is false, the part of the boundary $\partial\Omega$ that collapses onto a single point has Q-harmonic measure zero. See [HKM] for the definition of Q-harmonic measure.

Now let $f : \Omega \to \Omega'$ be a quasiconformal map. We introduce a new variational kernel, which is a deformation of the classical kernel $K_0(x,\xi) = |\xi|^Q$ by the quasiconformal map. For $x \in \Omega$ and $\xi \in F_x$ (the fiber over x in the Cheeger generalized cotangent bundle), let

$$K_f(x,\xi) = J_f(x)|df(x)^{-1*}\xi|^Q,$$

where A^* denotes the transpose of the matrix A. As in the undeformed case, we may now define *K_f-harmonic functions* to be minimizers of the energy

$$\int_U K_f(x, Du(x))\, dx, \qquad U \subset \Omega.$$

The usual variational argument combined with the change of variables formulas from [C, Section 4] and [HKShT, Section 10] shows that each K_f-harmonic function g is a solution to the weak PDE

$$\int_\Omega |Dg|^{Q-2}\, Dg \cdot D\varphi \, \frac{\|df\|^Q}{J_f} = 0$$

for every Lipschitz function φ with support in $\Omega\setminus\{y\}$. The existence and uniqueness of K_f-singular functions follows by the arguments of [HoSh] and [Ho] and we may define a K_f-*Martin boundary* $\partial_M^{K_f}\Omega$ by the procedure outlined in section 3. By exactly the same argument as above, we find that f extends to a homeomorphism of $\partial_M^{K_f}\Omega$ onto $\partial_{cM}\Omega'$.

More generally, starting from any kernel $K(x',\xi)$ defined on Ω' which satisfies the structure conditions of [HKM, §5.1], we may define K-harmonic functions and a K-Martin boundary $\partial_M^K\Omega'$. Then f extends to a homeomorphism of $\partial_M^{f^\# K}\Omega$ onto $\partial_M^K\Omega'$, where

$$f^\# K(x,\xi) = J_f(x)K(f(x), df(x)^{-1*}\xi).$$

Question 5: In the above setting, when does f extend as a homeomorphism between the usual conformal Martin boundaries $\partial_{cM}\Omega$ and $\partial_{cM}\Omega'$? See [Seg] and [SegT] for examples of planar domains for which such a homeomorphic extension does not exist.

The above question can be partially answered by answering the following question.
Question 6: Given a kernel K satisfying the structure conditions of [HKM, §5.1], when is $\partial_M^K\Omega$ homeomorphic to $\partial_{cM}\Omega$?

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