Papers on Analysis:
A volume dedicated to Olli Martio on the occasion of his 60th birthday
Report. Univ. Jyväskylä 83 (2001), pp. 63–72

NECESSARY CONDITIONS FOR POINCARÉ DOMAINS

DAVID E. EDMUNDS AND RITVA HURRI-SYRJÄNEN

ABSTRACT. We give necessary conditions and sufficient conditions for domains in certain classes to support Poincaré-type inequalities. The necessary conditions are sharp on large classes of domains.

1. INTRODUCTION

Let D be a bounded domain in euclidean n-space $\mathbb{R}^n, n \geq 2$. Let $1 \leq p \leq q < \infty$. We say that D supports the (q,p)-Poincaré inequality with weights ν and μ, if there is a constant

$$\mathcal{K} = \mathcal{K}_{q,p}(\nu, \mu, D) < \infty,$$

such that

$$\left(\int_D |u(x) - u_{D,\nu}|^q \,\nu(x)\,dx\right)^{\frac{1}{q}} \leq \mathcal{K}\left(\int_D |\nabla u(x)|^p \,\mu(x)\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

whenever $u \in L^1_p(D, \mu)$ with $-\infty < u_{D,\nu} < \infty$; here

$$u_{D,\nu} = \left(\int_D \nu(x)\,dx\right)^{-1} \int_D u(x)\,\nu(x)\,dx$$

and $L^1_p(D,\mu) = \{u|\int_D |\nabla u(x)|^p \mu(x)\,dx < \infty\}$, $\nu(D) < \infty$. In this case we say D is a (q,p,ν,μ)-Poincaré domain and write $D \in \mathcal{P}(q,p,\nu,\mu)$ or $D \in \mathcal{P}(q,p)$ with ν and μ. When $\nu = \mu = 1$ we recover the classical Poincaré inequality, write $D \in \mathcal{P}(q,p)$ and say D is a (q,p)-Poincaré domain. If $\nu = 1$ and $\mu = \operatorname{dist}(x, \partial D)^{\delta p}$ with $0 \leq \delta \leq 1$, the inequality (1.1) is an improved Poincaré inequality and we write briefly $D \in \mathcal{P}(q,p,\delta)$.

We give a necessary condition for the improved Poincaré inequality to be valid in domains with a shadow property: Theorem 3.1. It is not restrictive: all domains with the quasihyperbolic boundary condition have the shadow property but so do certain other domains, too. Our necessary condition turns out to be sharp at least on all quasihyperbolic boundary conditions domains. Our sufficiency condition, Theorem 4.1, handles doubling weights. Our Corollary 4.3 for the improved Poincaré case is sharp at least on all quasihyperbolic boundary condition domains which are plump.

2000 *Mathematics Subject Classification.* 46E35.

Key words and phrases. Poincaré inequality, Poincaré domain, irregular domain, necessary condition, sufficient condition.

The research of the second author was supported by the Academy of Finland under Grant No. 760095 and Grant No. 42865, by the Väisälä Foundation and the Magnus Ehrnrooth Foundation.

Vl. Maz'ya has established necessary and sufficient conditions with capacity estimates for a (q,p)-Poincaré domain on a very general basis, [12]. We refer also to the book of Maz'ya and S. Poborchi [13]. Using variational and Sobolev capacities D. Herron and P. Koskela characterized the (n,n)-Poincaré inequality in the unweighted case, [5]. However, capacity estimates can be quite difficult to verify for a given domain with an irregular boundary.

The unweighted (p^*,p)-case has been studied by S. Buckley and Koskela; here $p^* = \frac{np}{n-p}$, $1 \leq p < n$. Putting a restriction onto the domain they showed that under their restriction (the so-called separation property) a bounded domain $D \subset \mathbb{R}^n$ is a John domain if and only if D is a (p^*,p)-Poincaré domain, [2]. Note that it is well known that a John domain is always a (p^*,p)-Poincaré domain, [1].

The basic assumptions on domains in our theorems concerning necessity, Theorem 3.1 and Theorem 3.3, are not related to the condition of Buckley and Koskela. Our basic result on sufficiency, Corollary 4.2, does not impose any restriction on the bounded domain under consideration.

1.1. **Acknowledgement.** The second author would like to thank J. L. Lewis for useful discussions.

2. NOTATION

Throughout the paper the domains $D \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^n$, $n \geq 2$, are assumed to be bounded and it is assumed that $1 \leq p \leq q < \infty$. The Lebesgue n-measure of a set A is denoted by $|A|$. A weight is a non-negative locally integrable function on $\mathbb{R}^n$. The functions ν and μ are weights. We sometimes write $\mu(A)$ instead of $\int_A \mu(x)dx$.

We use the following lemmata frequently:

2.1. **Lemma.** *Let D be a domain and $A \subset D$ a ν-measurable set such that $\nu(A) > 0$. If $u \in L^q(D,\nu)$, then for each $c \in \mathbb{R}$,*

$$\| u - u_{A,\nu} \|_{L^q(D,\nu)} \leq 2 \left(\frac{\nu(D)}{\nu(A)} \right)^{\frac{1}{q}} \| u - c \|_{L^q(D,\nu)} .$$

Proof. Minkowski's inequality and Hölder's inequality. □

2.2. **Lemma.** *Suppose that $D \in \mathcal{P}(q,p,\nu,\mu)$. Let $u \in L^1_p(D,\mu)$ be such that*

$$u_{A,\nu} = \frac{1}{\nu(A)} \int_A u(x)\, d\nu(x) = 0$$

for some set A of positive ν-measure. Then there is a constant $c < \infty$ independent of u such that

$$\| u \|_{L^q(D,\nu)} \leq c \| \nabla u \|_{L^p(D,\mu)} .$$

Proof. Lemma 2.1, and the fact that $D \in \mathcal{P}(q,p,\nu,\mu)$ yield the claim. □

Chains Sets D_i, $i = 0, 1, \cdots, k$, in $\mathbb{R}^n$ are said to form a chain, abbreviated $\mathcal{C}(D_0, D_k) = \mathcal{C}(D_k) = (D_0, D_1, \cdots, D_k)$, if $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The length of a chain $\mathcal{C}(D_k) = (D_0, D_1, \cdots, D_k)$ is defined as $l(\mathcal{C}(D_k)) = k$. The collection of sets in the chain is denoted by $\mathcal{C}(D_k)$ as well.

Decomposition Let W be a family of domains D such that

$$G = \bigcup_{D \in W} D$$

is a bounded domain with the following properties. There exists a constant $c < \infty$ such that

$$\sum_{D \in W} \mathcal{X}_D(x) \leq c$$

for all $x \in \mathbb{R}^n$, and there is a fixed subdomain $D_0 \in W$ such that each $D \in W$ can be joined to D_0 by a chain $\mathcal{C}(D_0, D)$ of subdomains D_j in W with

$$c\nu(D_j \cap D_{j+1}) \geq \max\{\nu(D_j), \nu(D_{j+1})\}$$

for each $j = 0, 1, \ldots, l(D_0, D) - 1$, here $l(D_0, D) \geq 1$. If all subdomains $D \in W$ belong to $\mathcal{P}(q, p, \nu, \mu)$ and there is a constant c such that for all such D,

$$\mathcal{K}_{q,p}(D, \nu, \mu) \leq c < \infty,$$

then we call W a (q, p)-Poincaré decomposition of G with respect to weights ν and μ or simply a (q, p)-Poincaré decomposition of G if $\nu = \mu = 1$.

A similar decomposition without weights can be found in [6, 4.2].

For each $D \in W$ we fix a chain $\mathcal{C}(D_0, D)$ satisfying the above conditions. We write for a fixed $A \in W$

$$A(D_0, W) = \bigcup_{\substack{B \in W \\ A \in \mathcal{C}(D_0, B)}} B.$$

We need also the set

$$Z^*(A) = \bigcup_{\substack{B \in W \\ A \in \mathcal{C}(D_0, B) \\ A \cap B \neq \emptyset}} B.$$

Whitney decomposition If the decomposition of a domain D is a Whitney decomposition W we choose for each Whitney cube Q a chain from a fixed cube Q_0 using a quasihyperbolic geodesic joining the centre points of the cubes $x_0 \in Q$ and $x_k \in Q_k$. Then $l(\mathcal{C}(Q_k))$ is comparable to $k_D(x_0, x_k)$, [6, Proposition 6.1]. Recall that the quasihyperbolic distance between points x_1 and x_2 in D is given by

$$k_D(x_1, x_2) = \inf \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)}$$

where the infimum is taken over all rectifiable curves γ joining x_1 and x_2 in D, [4].

The family

$$\cup \Big\{ \operatorname{int} \frac{9}{8} Q | Q \in W \Big\}$$

is a (q, p)-Poincaré decomposition of D, $1 \leq p \leq q < \infty$.

Let $D \subset \mathbb{R}^n$ be a domain and consider cubes which are near the boundary and far away from Q_0. Let $A \in W$ be such that A is far away from Q_0 and near the boundary. We pick $y_A \in \partial D$ such that $\operatorname{dist}(A, \partial D) = \operatorname{dist}(A, y_A)$. Our main assumption about D is the following: there exists $R > 0$ such that $A(Q_0, W) \subset B(y_A, R)$ (the open ball in $\mathbb{R}^n$ with centre y_A and radius R) and $B(y_A, R) \cap Q_0 = \emptyset$. If D satisfies this condition, then we say that D has a shadow property. Let R_A be the infimum of all such R and define

$$S(A) = B(y_A, R_A) \cap D$$

and

$$Z(A) = \{z \in S(A) \,|\, \operatorname{dist}(z, D \setminus S(A)) \leq |A|^{1/n}\}.$$

3. NECESSARY CONDITIONS

3.1. **Theorem.** *Suppose that $D \subset \mathbb{R}^n$, $n \geq 2$, is a (q,p)-Poincaré domain which has a shadow property. Then there exists a constant $c = c(q,p,n,D) < \infty$ such that*

$$|S(A) \setminus Z(A)| \leq c|Z(A)|^{\frac{q}{p}}|A|^{-\frac{q}{n}}$$

holds for each Whitney cube A which is far away from Q_0.

Proof. We fix a cube $A \in W$. Let D_0 be a subset of $D \setminus B(y_A, R_A)$ containing the centre cube Q_0. We define a Lipschitz function $u_A : D \longrightarrow \mathbb{R}$ such that $u_A(y) = 0$, whenever $y \in D_0$ and otherwise

$$u_A(y) = a_A|A|^{-\frac{1}{p}} \inf_{\gamma \in \Gamma_{x_0,y}} \operatorname{length}(\gamma \cap Z(A)),$$

where $\Gamma_{x_0,y}$ is a set of all rectifiable curves in D joining x_0 and y. We fix the number $a_A \in \mathbb{R}_+$ later. For all $y \in S(A) \setminus Z(A)$,

$$u_A(y) \geq a_A|A|^{\frac{p-n}{np}}.$$

Hence,

$$\int_D |u_A(x)|^q \, dx \geq c(n,q) \int_{S(A)\setminus Z(A)} a_A^q |A|^{\frac{q(p-n)}{np}} \, dx$$
$$= c(n,q)|S(A) \setminus Z(A)| \, a_A^q |A|^{\frac{q(p-n)}{np}}.$$

On the other hand, since $u_A(y)$ is constant when $y \notin Z(A)$,

$$\int_D |\nabla u_A(x)|^p \, dx = \int_{Z(A)} |\nabla u_A(x)|^p \, dx$$
$$\leq \int_{Z(A)} a_A^p |A|^{-1} \, dx = 1,$$

if we choose $a_A = |A|^{\frac{1}{p}}|Z(A)|^{-\frac{1}{p}}$. Since $D \in \mathcal{P}(q,p)$, we obtain by Lemma 2.1,

$$c(n,q)\, a_A^q |A|^{\frac{q(p-n)}{np}} |S(A) \setminus Z(A)| \leq \int_D |u_A(y)|^q \, dy$$
$$= \int_D |u_A(y) - (u_A)_{Q_0}|^q \, dy$$
$$\leq 2^q |D||Q_0|^{-1} \int_D |u_A(y) - (u_A)_D|^q \, dy$$
$$\leq 2^q |D||Q_0|^{-1} \mathcal{K}_{q,p}^q(D) \left(\int_D |\nabla u_A(y)|^p \, dy \right)^{\frac{q}{p}}$$
$$\leq 2^q |D||Q_0|^{-1} \mathcal{K}_{q,p}^q(D)$$

with $a_A = |A|^{\frac{1}{p}}|Z(A)|^{-\frac{1}{p}}$. Hence,

$$|S(A)\backslash Z(A)| \le c(n,p,q)|D||Q_0|^{-1}\mathcal{K}^q_{q,p}(D)|A|^{-\frac{q}{n}}|Z(A)|^{\frac{q}{p}}.$$

□

The following theorem is proved in a similar manner.

3.2. **Theorem.** *Suppose that $D \subset \mathbb{R}^n$, $n \ge 2$, is a $(q,p,\nu = 1, \mu = \mathrm{dist}(x,\partial D)^{\delta p})$-Poincaré domain with $0 \le \delta \le 1$ and has a shadow property. Then there exists a constant $c = c(q,p,\delta,D) < \infty$ such that*

$$|S(A) \setminus Z(A)| \le c\left(\int_{Z(A)} \mathrm{dist}(x,\partial D)^{\delta p}\,dx\right)^{\frac{q}{p}} |A|^{-\frac{q}{n}}$$

holds for each Whitney cube A which is far away from Q_0.

3.3. **Theorem.** *Let W be a (q,p)-Poincaré decomposition of G into subdomains. Suppose that for each subdomain $A \ne D_0$ the following holds: each path connecting $A(D_0,W)\backslash A$ with D_0 passes through A. If G is a (q,p)-Poincaré domain, then there exists a constant $c = c(q,p,G) < \infty$ such that*

$$|A(D_0,W)\backslash Z^*(A)| \le c|Z^*(A)|^{\frac{q}{p}}|A|^{-\frac{q}{n}}$$

for each subdomain A.

Proof. Let $x_0 \in D_0$. We fix a subdomain $A \in W$. We define a Lipschitz function $u_A : G \longrightarrow \mathbb{R}$ such that $u_A(y) = 0$ whenever $y \in G\backslash A(D_0,W)$, and otherwise

$$u_A(y) = a_A|Z^*(A)|^{-\frac{1}{p}} \inf_{\gamma\in\Gamma_{x_0,y}} \mathrm{length}(\gamma \cap Z^*(A)),$$

where $\Gamma_{x_0,y}$ is the set of all rectifiable curves in G joining x_0 and y. We fix the numbers $a_A \in \mathbb{R}_+$ later. For all $y \in A(D_0,W)\backslash Z^*(A)$,

$$u_A(y) \ge c(n)\, a_A|Z^*(A)|^{-\frac{1}{p}}|A|^{\frac{1}{n}}.$$

The end of the proof is similar to that for Theorem 3.1. □

The following theorem is proved in the same way as the previous one.

3.4. **Theorem.** *Let W be a (q,p)-Poincaré decomposition of G into subdomains with respect to weights $\nu = 1$ and $\mu = \mathrm{dist}(x,\partial G)^{\delta p}$, $0 \le \delta \le 1$. Suppose that for each subdomain $A \ne D_0$ the following holds: each path connecting $A(D_0,W)\backslash A$ with D_0 passes through A . If G is a $(q,p,\nu = 1, \mu = \mathrm{dist}(x,\partial G)^{\delta p})$-Poincaré domain with $0 \le \delta \le 1$, then there exists a constant $c = c(q,p,\delta,G) < \infty$ such that*

$$|A(D_0,W)\backslash Z^*(A)| \le c\left(\int_{Z^*(A)} \mathrm{dist}(x,\partial G)^{\delta p}\,dx\right)^{\frac{q}{p}} |Z^*(A)|^{-\frac{q}{n}}$$

for each subdomain A.

4. Sufficient conditions

4.1. **Theorem.** *Let W be a (q,p)-Poincaré decomposition of a domain $G \subset \mathbb{R}^n$, $n \geq 2$, with ν, μ where ν and μ are doubling weights. If there exists a constant $c < \infty$ such that for each $A \in W$,*

$$\sum_{D \in A(W)} l(\mathcal{C}(D_0, D))^{q-1} \nu(D) \leq c \mathcal{K}_{q,p}(A, \nu, \mu)^{-q} \nu(A),$$

then $G \in \mathcal{P}(q,p)$ with ν and μ.

Proof. The proof is a straightforward generalization of the unweighted case with $q = p$, [6, Theorem 4.4], to the (q,p)-case with weights. By Lemma 2.1 it is enough to estimate

$$\int_G |u - u_{D_0}|^q \, d\nu \leq 2^{q-1} \left(\sum_{D \in W} \int_D |u - u_D|^q \, d\nu + \sum_{D \in W} \int_D |u_D - u_{D_0}|^q \, d\nu \right).$$

Since each $D \in \mathcal{P}(q, p, \nu, \mu)$,

$$\sum_{D \in W} \int_D |u - u_D|^q \, d\nu \leq \mathcal{K}_{q,p}(D, \nu, \mu)^q \left(\int_D |\nabla u|^p \, d\mu \right)^{\frac{q}{p}} \leq c \left(\int_G |\nabla u|^p \, d\mu \right)^{\frac{q}{p}}.$$

To estimate the part

$$\sum_{D \in W} \int_D |u_D - u_{D_0}|^q \, d\nu$$

we use the local assumptions and obtain when $D = D_k$,

$$\begin{aligned} |u_D - u_{D_0}|^q &\leq \left(\sum_{j=1}^k |u_{D_j} - u_{D_{j-1}}| \right)^q \\ &\leq k^{q-1} \sum_{j=1}^k |u_{D_j} - u_{D_{j-1}}|^q \\ &= k^{q-1} \sum_{j=1}^k \frac{1}{\nu(D_j \cap D_{j-1})} \int_{D_j \cap D_{j-1}} |u_{D_j} - u_{D_{j-1}}|^q \, d\nu \\ &\leq 2^{\frac{q}{p}} k^{q-1} \sum_{j=1}^k \frac{\mathcal{K}_{q,p}(D_j, \nu, \mu)^q}{c\nu(D_j)} \left(\int_{D_j} |\nabla u|^p d\mu \right)^{\frac{q}{p}} \end{aligned}$$

with $k = l(\mathcal{C}(D_0, D))$. Hence,

$$\begin{aligned}
&\sum_{D\in W}\int_D |u_D - u_{D_0}|^q\,d\nu \\
&\le c\sum_{D\in W}\int_D l(\mathcal{C}(D_0,D))^{q-1}\left(\sum_{A\in\mathcal{C}(D_0,D)}\frac{\mathcal{K}_{q,p}(A,\nu,\mu)^q}{\nu(A)}\left(\int_A|\nabla u|^p\,d\mu\right)^{\frac{q}{p}}\right)d\nu \\
&= c\sum_{A\in W}\sum_{D\in A(W)}\int_D l(\mathcal{C}(D_0,D))^{q-1}\,d\nu\,\frac{\mathcal{K}_{q,p}(A,\nu,\mu)^q}{\nu(A)}\left(\int_A|\nabla u|^p\,d\mu\right)^{\frac{q}{p}} \\
&\le c\left(\int_G|\nabla u|^p\,d\mu\right)^{\frac{q}{p}}.
\end{aligned}$$

The proof is complete. □

Since the length of a chain $l(\mathcal{C}(Q_k))$ is comparable to the quasihyperbolic distance between points $x_0 \in Q_0$ and $x \in Q$, we obtain the following corollary.

4.2. **Corollary.** *Let $D \subset \mathbb{R}^n$, $n \ge 2$ be a domain. Let ν and μ be locally doubling weights. If there are a point $x_0 \in D$ and a constant $c < \infty$ such that for each Whitney cube A,*

$$\int_{A(W)} k_D(x_0,x)^{q-1}\,d\nu(x) \le c\mathcal{K}_{q,p}(A,\nu,\mu)^{-q}\,\nu(A),$$

then $D \in \mathcal{P}(q,p)$ with ν and μ.

The following corollary can be proved in the same way as Theorem 4.1 noting the Poincaré constant of a cube.

4.3. **Corollary.** *Let $D \subset \mathbb{R}^n$, $n \ge 2$ be a domain. Let*

$$1 \le p \le q \le \frac{np}{n - p(1-\delta)}$$

where $p(1-\delta) < n$ and $0 \le \delta \le 1$. If there exist a point $x_0 \in D$ and a constant c such that for each Whitney cube A

$$\int_{A(W)} k_D(x_0,x)^{q-1}\,dx \le c|A|^{(\frac{\delta-1}{n}+\frac{1}{p})q},$$

then $D \in \mathcal{P}(q,p,1,\operatorname{dist}(x,\partial D)^{\delta p})$.

The following corollary is a generalization of [6, Theorem 6.6], when $p = q$.

4.4. **Corollary.** *Let D be a domain in $\mathbb{R}^n$, $n \ge 2$. Let*

$$1 \le p \le q \le \frac{np}{n-p}$$

where $p < n$. If there are a point $x_0 \in D$ and a constant c such that for each Whitney cube A

$$\int_{A(W)} k_D(x_0,x)^{q-1}\,dx \le c|A|^{\frac{n-p}{np}q},$$

then $D \in \mathcal{P}(q,p)$.

5. EXAMPLES

5.1. **Definition.** [9] Let $0 < \alpha \leq \beta$. A domain D is a John domain if there is a point $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a curve $\gamma : [0,1] \to D$ parametrized by arc length with total length $l \leq \beta$ and

$$\operatorname{dist}(\gamma(t), \partial D) \geq \frac{\alpha}{l} t$$

for all $t \in [0, l]$. If we write $\frac{\alpha}{\beta} = b$, we call D a b-John domain, too.

5.2. *Note.* Lipschitz domains and bounded uniform domains, [9], (for example, the Koch snow flake) are John domains.

5.3. **Definition.** [3] A domain D satisfies a quasihyperbolic boundary condition, or is a quasihyperbolic boundary condition domain, abbreviated $D \in QHBC$, if there exist a point x_0 and a constant $a \geq 1$ such that

$$k_D(x_0, x) \leq a \log\left(1 + \frac{|x_0 - x|}{\min\{\operatorname{dist}(x_0, \partial D), \operatorname{dist}(x, \partial D)\}}\right)$$

for all $x \in D$. Equivalently if D is bounded, $D \in QHBC$ if there exist a point x_0 and constants $b \leq 1$ and c such that

$$k_D(x_0, x) \leq b^{-1} \log\left(\frac{\operatorname{dist}(x_0, \partial D)}{\operatorname{dist}(x, \partial D)}\right) + c$$

for all $x \in D$.

5.4. *Note.* John domains are quasihyperbolic boundary condition domains, but there are quasihyperbolic boundary condition domains which are not John domains, [3].

5.5. **Definition.** [10] A domain D is α-plump, $0 < \alpha \leq 1$, if there is $\sigma > 0$ such that for every $y \in \partial D$ and for all $t \in (0, \sigma]$ there is $x \in D \cap \overline{B(y,t)}$ with $\operatorname{dist}(x, \partial D) > \alpha t$.

5.6. *Note.* John domains are plump, [11], but there are quasihyperbolic boundary condition domains which are not plump, [6].

5.7. **Example.** Let $D \subset \mathbb{R}^n$ be a b-John domain and let $0 \leq \delta \leq 1$. Then D satisfies the shadow property with a constant $R_A = c(n,b)|A|^{1/n}$, [6, Lemma 8.3]. The necessary condition gives

$$(1-\delta)\frac{1}{n} + \frac{1}{q} - \frac{1}{p} \geq 0$$

with $p(1-\delta) < n$. It is known that $D \in \mathcal{P}(q,p,\delta)$, whenever

$$(1-\delta)\frac{1}{n} + \frac{1}{q} - \frac{1}{p} \geq 0$$

with $p(1-\delta) < n$, [7, Theorem 1.3] Hence, the results are sharp.

5.8. **Example.** Let $D \subset \mathbb{R}^n$ be a b-quasihyperbolic boundary condition domain and let $0 \leq \delta < 1$. The necessary condition for an arbitrary domain does not give good bounds. The triangle domain (a triangle to which a decreasing sequence of small triangles are attached) shows that the upper bound for $|S(A) \setminus Z(A)|$ is attained and is $c|A|^{1/b}$; the lower bound for $|Z(A)|$ is attained at $c|A|$. The construction of the triangle domain is as follows [3]: Let G_0 be the open rectangle bounded by the lines

$x_1 = 0$, $x_2 = 0$, $x_1 = 1$, and $x_2 = -1$. For $j = 1, 2, \cdots$ let G_j be the open triangle bounded by the lines $x_1 = 2^{-2j}$, $x_2 = 2^{-2j} - 2^{-2\eta j}$, $x_1 + x_2 = 2^{-2j} - 2^{-2\eta j}$ where $\eta \geq 2$ is a constant. We set $G = \cup_{j=0}^{\infty} G_j$. Then G satisfies a quasihyperbolic boundary condition with $a = 36\eta$.

A quasihyperbolic boundary condition domain has a shadow property with $R = c(n,b)|A|^{1/bn}$, [6, Lemma 7.25]. (We remark that the exponent can be improved there to be as given here.) The necessary condition gives

$$\frac{n}{1-\delta}\left(1 - \frac{1}{b}\right) \leq p \leq q \leq \frac{np}{b(n - p(1-\delta))}$$

and $p(1-\delta) < n$. The sufficient condition implies that a plump quasihyperbolic boundary condition domain $D \in \mathcal{P}(q, p, \delta)$, whenever

$$\frac{n}{1-\delta}\left(1 - \frac{1}{b}\right) < p \leq q < \frac{np}{b(n - p(1-\delta))}$$

with $p(1-\delta) < n$, [6, Lemma 7.30]. This is known for $\delta = 0$ by [8] for all domains satisfying a quasihyperbolic boundary condition with a constant $b \leq 1$. An upper bound for q is $\frac{(n-\lambda)p}{b(n-p(1-\delta))}$ where $\lambda \in [n-1, n)$ a Whitney cube #-constant, [7].

5.9. *Remark.* A ball with a slit removed shows that even in the case of a John domain one must require $\delta \leq 1$ in order to have a (q,p)-Poincaré inequality with $\nu = 1$ and $\mu = \operatorname{dist}(x, \partial D)^{\delta p}$. A ball to which small triangles are attached gives an example of a quasihyperbolic boundary condition domain for which one must have $\delta < 1$ in order to have a weighted Poincaré inequality, [7, Remarks 3.11].

We consider an example with a non-Whitney decomposition.

5.10. **Example.** Let

$$G_1 = \bigcup_{i=1}^{\infty} (D_{2i-1} \cup P_{2i})$$

where the sets D_{2i-1} and P_{2i}, $i = 1, 2, \cdots$, are defined as follows: Let (h_{2i}) and (η_{2i}) be sequences, where $h_i = M^{-i}$, $M > 1$, and $\eta_{2i} = bM^{-2ai}$, $b > 0, a > 1$. Write $\sum_{i=1}^{k} h_i = d_k$, $k = 1, 2, \cdots$. Define

$$D_{2i-1} = (d_{2i-1} - h_{2i-1}, d_{2i-1}) \times \left(-\frac{1}{2}h_{2i-1}, \frac{1}{2}h_{2i-1}\right)^{n-1},$$

$$P_{2i} = [d_{2i-1}, d_{2i-1} + h_{2i}] \times \left(-\frac{1}{2}\eta_{2i}, \frac{1}{2}\eta_{2i}\right)^{n-1},$$

$i = 1, 2, \cdots$. We define $G = G_1 \cup G_2 \cup G_3$, where G_2 is the reflection of G_1 with the hyperplane $x_1 = 0$ and $G_3 = (-h_1/2, h_1/2)^n$. The necessary condition is that

$$p \geq \frac{(n-1)(a-1)}{1 - \delta a}$$

whenever $0 \leq \delta < 1/a$. On the other hand, it has been known for $\delta = 0$ that $G \in \mathcal{P}(p,p)$ if and only if $p \geq (n-1)(a-1)$, [6, Remark 5.9].

5.11. *Remark.* We can conclude that our necessary condition in Theorem 3.2 is sharp in a large class of domains for an improved Poincaré inequality where the weight on the right hand side is the distance function to a positive power. On the other hand, our Theorem 4.1 on sufficiency yields a sufficient condition with more general weights. Its version for the improved Poincaré inequality case, Corollary 4.3, is sharp on a large class of domains with the distance function as a weight on the right hand side.

REFERENCES

[1] BOJARSKI, B. Remarks on Sobolev imbedding inequalities. In *Proceedings of the Conference Complex Analysis* (Joensuu, 1987), Lecture Notes in Mathematics 1351, Springer-Verlag, pp. 1–106.

[2] BUCKLEY, S., AND KOSKELA, P. Sobolev–Poincaré implies John. *Math. Res. Lett. 2 5* (1995), 577–593.

[3] GEHRING, F., AND MARTIO, O. Lipschitz classes and quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math. 10* (1985), 203–219.

[4] GEHRING, F., AND PALKA, B. Quasiconformally homogeneous domains. *J. Analyse Math. 30* (1976), 172–199.

[5] HERRON, D., AND KOSKELA, P. Poincaré domains and quasiconformal mappings (Russian). *Sibirsk. Mat. Zh 36, 6* (1995), 1416–1434 iv. Translation in Siberian Math. J. 36 no 6, 1232–1246.

[6] HURRI, R. Poincaré domains in R^n. *Ann. Acad. Sci. Fenn. Ser. I A, Dissertationes 71* (1988), 1–42.

[7] HURRI-SYRJÄNEN, R. An improved Poincaré inequality. *Proc. Amer. Math. Soc. 120* (1994), 213–222.

[8] KOSKELA, P., ONNINEN, J., AND TYSON, J. Quasihyperbolic boundary conditions and capacity: Poincaré domains. *University of Jyväskylä preprint 222* (2000).

[9] MARTIO, O., AND SARVAS, J. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math. 4* (1978-1979), 383–401.

[10] MARTIO, O., AND VÄISÄLÄ, J. Global l^p-integrability of the derivative of a quasiconformal mapping. *Complex Variables Theory Appl. 9* (1988), 309–319.

[11] MARTIO, O., AND VUORINEN, M. Whitney cubes, p-capacity and Minkowski content. *Expo. Math. 5* (1987), 17–40.

[12] MAZ'YA, V. *Sobolev Spaces.* Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985. Translated from the Russian by T.O.Shaposhnikova, Springer Series in Soviet Mathematics.

[13] MAZ'YA, V., AND POBORCHI, S. *Differentiable Functions in Bad Domains.* World Scientific Publishing Co.Inc., River Edge, NJ, 1997.

CENTRE FOR MATHEMATICAL ANALYSIS AND ITS APPLICATIONS, UNIVERSITY OF SUSSEX, FALMER, BRIGHTON, BN1 9QH, U. K.

DEPARTMENT OF MATHEMATICS, P.O. BOX 4, FIN-00014 UNIVERSITY OF HELSINKI, FINLAND
E-mail address: ritva.hurri-syrjanen@helsinki.fi