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A SURVEY OF NEARISOMETRIES

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1. Introduction

1.1. *Notation.* Throughout this article, E and F are real Banach spaces (sometimes Hilbert spaces or just euclidean spaces) of dimension at least one. The norm of a vector x is written as $|x|$. In a Hilbert space we let $x \cdot y$ denote the inner product of vectors x and y.

Open and closed balls with center x and radius r are written as $B(x,r)$ and $\bar{B}(x,r)$, respectively, and we use the abbreviations $B(r) = B(0,r)$ and $\bar{B}(r) = \bar{B}(0,r)$. The open unit ball in the euclidean space R^n is $B^n = B(1)$.

We shall consider maps $f\colon A \to F$ where $A \subset E$. Without further notice we shall always assume that $A \neq \varnothing$. If $g\colon A \to F$ is another map, we set

$$d(f,g) = \sup\{|fx - gx| : x \in A\}$$

with the possibility $d(f,g) = \infty$. We frequently consider the case where $g = T|A$ is the restriction of an isometry $T\colon E \to F$. Then we simply write $d(T,f) = d(T|A,f)$. To simplify expressions we often omit parentheses writing $fx = f(x)$ etc.

1.2. *Nearisometries.* Let $A \subset E$. A map $f\colon A \to F$ is a *nearisometry* if there is a number $\varepsilon \geq 0$ such that

$$|x-y| - \varepsilon \leq |fx - fy| \leq |x-y| + \varepsilon \tag{1.3}$$

for all $x, y \in A$. More precisely, we say that such a map is an ε-*nearisometry.* Observe that f need not be continuous. In the literature, the nearisometries and ε-nearisometries are often called approximate isometries and ε-isometries, respectively.

The condition (1.3) can be regarded as a perturbation of the isometry condition $|fx - fy| = |x-y|$. Another kind of perturbation is given by the multiplicative condition

$$|x-y|/M \leq |fx - fy| \leq M|x-y|. \tag{1.4}$$

A map satisfying (1.4) is called M-*bilipschitz.*

Neither of the conditions (1.3) and (1.4) implies the other. Condition (1.3) is stronger than (1.4) for large distances but weaker for small distances. A bilipschitz map is always continuous, even an embedding.

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However, if A is bounded, then a $(1+t)$-bilipschitz map $f\colon A \to F$ is a $td(A)$-nearisometry, where d denotes diameter. In the other direction, an ε-nearisometry satisfies (1.4) with $M = 1+\varepsilon$ whenever $|x-y| \geq 1+\varepsilon$. The basic question considered in this paper is the following stability problem:

Given an ε-nearisometry $f\colon A \to F$, does there exist an isometry $T\colon E \to F$ such that $d(T,f)$ is small if ε is small? In other words, are the nearisometries near isometries?

It turns out that in many cases we can even find a linear bound $d(T,f) \leq c\varepsilon$. This is true, for example, if A is the whole space and f is surjective. For Hilbert spaces this was proved in 1945 by D.H. Hyers and S.M. Ulam in the famous paper [HU1].

The classical case $A = E$ is considered in Section 2. In Section 3 we consider the case where A is an unbounded subset of E, and Section 4 deals with bounded sets $A \subset E$. In Section 5 we mention some related results.

2. Whole spaces

2.1. *Preliminary results.* We recall the classical Mazur-Ulam theorem (see [BL, 14.1]): Every surjective isometry $T\colon E \to F$ is affine, that is, $Tx = Sx + T(0)$ where S is linear. This is also true for nonsurjective isometries if F is strictly convex, in particular, if F is Hilbert.

In this section we consider ε-nearisometries $f\colon E \to F$ defined in the whole space E. There is an extensive literature dealing with this case starting with the influential paper [HU1] of Hyers and Ulam; see 2.3. To find an isometric approximation $T\colon E \to F$ we may without loss assume that $f(0) = 0$. If T is linear and if $d(T,f) < \infty$, it is easy to see that $Tx = \lim_{t\to\infty} f(tx)/t$ for each $x \in E$. Consequently, if T is surjective or if F is strictly convex, an isometric approximation of f is uniquely determined up to translation.

We recall that the *Jung constant* $J(E)$ of E is the infimum of all numbers $r > 0$ such that every set $A \subset E$ of diameter $d(A) \leq 2$ can be covered by a ball of radius r. We have always $1 \leq J(E) \leq 2$, and $J(\mathsf{R}^n) = \sqrt{2n/(n+1)} < \sqrt{2}$ by the classical result proved by H.W.E. Jung [Ju] in 1901; see [Fe, 2.10.45]. Moreover, $J(E) = \sqrt{2}$ for infinite-dimensional Hilbert spaces; see [Da, Th. 2] or [Še1, p. 704]. The upper bound $J(E) = 2$ is obtained, for example, by the Banach spaces c and c_0.

The following result summarizes the work of several authors during 1945–1998.

2.2. **Fundamental theorem.** *Suppose that $f\colon E \to F$ is a surjective ε-nearisometry with $f(0) = 0$. Then there is a surjective linear isometry $T\colon E \to F$ with $d(T,f) \leq 2\varepsilon$. The bound is the best possible.*

Moreover, for each $t > 0$ there is a surjective (hence affine) isometry $S\colon E \to F$ with $d(S,f) \leq J(E)\varepsilon + t$. Also here the bound $d(S,f) \leq 2\varepsilon$ is the best possible.

2.3. *History.* The first part of the Fundamental theorem was proved for Hilbert spaces by Hyers and Ulam [HU1] in 1945 with the bound 10ε. Surprisingly, it took 38 years until J. Gevirtz [Ge] obtained a proof for all Banach spaces (with the bound 5ε) in 1983. Meanwhile, various special cases were considered in [HU2], [Bo1–Bo5] and [Gr]. P.M. Gruber [Gr] got quite close to the solution in 1978. For example, he proved that if there is T with $d(T,f) < \infty$, then $d(T,f) \leq 5\varepsilon$. Moreover, he obtained the result for all finite-dimensional Banach spaces.

M. Omladič and P. Šemrl [OŠ] improved the ideas of Gruber and obtained the bound 2ε in 1995. By a simple example they also proved the sharpness of the bound. The following example, due to S.J. Dilworth [Di], is still simpler.

Let $\varepsilon > 0$ and define $f\colon \mathsf{R} \to \mathsf{R}$ by $fx = x - \varepsilon$ for $x \neq 0, \varepsilon$ and by $fx = -x$ for $x = 0, \varepsilon$. This map is obviously an ε-nearisometry. If $T\colon \mathsf{R} \to \mathsf{R}$ is a linear isometry, then either $T = \mathrm{id}$ or $T = -\mathrm{id}$. Hence $d(T, f)$ is 2ε or ∞.

The second part of the theorem follows rather easily from the first part, as shown by Šemrl [Še1] in 1998. In the same paper he also proved the sharpness of the bound 2ε in the second part for the space $E = c$. A somewhat simpler proof for the space c_0 is given in [HV]. We remark that one can choose $t = 0$ if each set $A \subset E$ with $d(A) \leq 2$ can be covered by a ball of radius $J(E)$. This is true, for example, if E is a Hilbert space.

It is natural to ask whether the bound $d(S, f) \leq J(E)\varepsilon + t$ is sharp for every Banach space E. An affirmative answer for Hilbert spaces was given in [HV] but the general case is open.

One might also think that the result holds with better bounds if only continuous maps $f\colon E \to F$ are considered. However, this is not the case at least for Hilbert spaces, as proved in [HV].

2.4. *Remarks on the proof.* The proof of the Fundamental theorem 2.2 is entirely elementary, and it can be understood (at least if f is bijective) by anyone who knows the definition and some basic properties of Banach spaces. A nice presentation is given in the book [BL, 15.2] of Y. Benyamini and J. Lindenstrauss. The proof consists of three steps.

1. Given $x \in E$, we show that the sequence of the points $y_n = 2^{-n} f(2^n x)$ is Cauchy and hence converges to a limit Tx. We thus obtain a map $T\colon E \to F$, which is clearly an isometry.
2. We show that T is surjective.
3. The inequality $|Tx - fx| \leq 2\varepsilon$ is proved.

If the spaces E and F are Hilbert, the proof is substantially easier than in the general case. For example, the sequence (y_n) converges whenever f is a nearisometry defined in the set $\{2^n x : x \in \mathsf{N}\}$. In particular, the surjectivity of f is not needed in Step 1. A simplified proof for the Hilbert space case is given in [Vä3, 5.1]

However, the surjectivity condition of 2.2 cannot be omitted even in the case $f\colon \mathsf{R} \to \mathsf{R}^2$; see 2.5. It can be omitted if E and F have the same finite dimension (Theorem 2.6). This was proved by R. Bhatia and Šemrl [BŠ] for euclidean spaces and by Dilworth [Di] in the general case.

2.5. *Example.* Let $\varepsilon > 0$ and define $f\colon \mathsf{R} \to \mathsf{R}^2$ by $fx = (x, \sqrt{2\varepsilon|x|})$. It is easy to see that f is an ε-nearisometry. However $d(T, f) = \infty$ for every isometry $T\colon \mathsf{R} \to \mathsf{R}^2$.

2.6. **Theorem.** *If* $\dim E = \dim F < \infty$*, then the Fundamental theorem holds without the surjectivity condition.*

2.7. *Relaxing the surjectivity condition.* Although the surjectivity condition of the Fundamental theorem 2.2 cannot be removed, it can be weakened. For example, it can be replaced by the condition that $F \setminus fE$ is bounded. A more general result is given in 3.1.

A still weaker condition is given by the property of δ-*ontoness*. This can also be considered relative to a closed linear subspace F_1 of F. Let $\delta \geq 0$. We say that a map $f\colon E \to F$ is δ-*onto* F_1 if the Hausdorff distance $d_H(fE, F_1)$ is at most δ, that is, $d(fx, F_1) \leq \delta$ and $d(y, fE) \leq \delta$ for all $x \in E$ and $y \in F_1$. In particular, a map f is δ-onto F iff $F \setminus fE$ contains no ball of radius larger than δ.

2.8. **Theorem.** *Suppose that* $f\colon E \to F$ *is an* ε*-nearisometry with* $f(0) = 0$ *and that* f *is* δ*-onto* F_1 *where* F_1 *is a closed linear subspace of* F*. Then there is a surjective linear isometry* $T\colon E \to F_1$ *such that* $d(T, f) \leq 2\varepsilon + 4\delta$.

This result was obtained by Dilworth [Di] with a somewhat larger bound. The bound $2\varepsilon + 4\delta$ is from [ŠV]. See also [Ta, Th. 3]. The first term 2ε is the best possible by the Fundamental theorem but the second term 4δ is presumably not. It must be at least δ as is seen from the following example; see [Di, p. 473]:

Let $\delta > 0$, $M > 0$, and let $g\colon \mathsf{R} \to \mathsf{R}$ be an increasing M-Lipschitz function with $\lim_{x\to-\infty} gx = 0$, $\lim_{x\to\infty} gx = 1$. Define $f\colon \mathsf{R} \to \mathsf{R}^2$ by $fx = (x, \delta gx)$. Then f is an ε-nearisometry with $\varepsilon = M\delta/2$. Indeed, if $a \in \mathsf{R}$, $b = a + d > a$, $d' = |fb - fa|$, $t = \delta(gb - ga)$, then ${d'}^2 = d^2 + t^2$ and

$$0 \leq d' - d = t^2/(d + d') \leq Md\delta/2d = M\delta/2.$$

The map f is δ-onto $F_1 = \mathsf{R} \times \{0\}$, and $d(T, f) = \delta$ and $d(T, f) = \infty$ for the two linear isometries $T\colon \mathsf{R} \to F_1$. Since ε is arbitrarily small, Theorem 2.8 is not true if 4δ is replaced by a number less than δ.

2.9. *The case* $F_1 = F$. This case of 2.8 is substantially different from the case $F_1 \neq F$. One can show that the bound $2\varepsilon + 4\delta$ can then be replaced by $2\varepsilon + 2\delta$ but the following conjecture looks plausible:

2.10. *Conjecture.* The Fundamental theorem is true if the surjectivity of $f\colon E \to F$ is replaced by the condition that f is δ-onto F for some δ.

2.11. *Results.* At the time of this writing (August 2001), the conjecture is open. It is known to be true in the following cases:

(1) The norm of E is Fréchet differentiable in a dense set [Di, Th. 2]. This class includes all Asplund spaces (separable subspaces have separable duals) and hence all reflexive spaces.

(2) $E = F$ is l_p or $L_p(X, \mu)$, where $1 \leq p \leq \infty$ and (X, μ) is a measure space [ŠV]. The case $1 < p < \infty$ is included in (1).

(3) $E = F = C(X)$, the space of continuous real-valued functions in a compact space X [ŠV].

Moreover, the conjecture is true for all spaces with the bound 2ε replaced by 3ε [ŠV].

2.12. *The number* $\tau(Q)$. We next try to replace the δ-ontoness in 2.10 by a still weaker condition. Let Q be a nonempty set in a Banach space and let u be a unit vector. We set

$$\varrho(u, Q) = \liminf_{|t|\to\infty} d(tu, Q)/|t|, \quad \tau(Q) = \sup_{|u|=1} \varrho(u, Q).$$

Then

$$\varrho(-u, Q) = \varrho(u, Q), \quad 0 \leq \varrho(u, Q) \leq 1, \quad 0 \leq \tau(Q) \leq 1.$$

Moreover, these numbers are translation invariants: $\varrho(u, Q+z) = \varrho(u, Q)$, $\tau(Q+z) = \tau(Q)$.

If Q is bounded or contained in a hyperplane, then $\tau(Q) = 1$. If Q contains a half space, then $\tau(Q) = 0$. If the function $y \mapsto d(y, Q)$ is bounded, then $\tau(Q) = 0$. Hence $\tau(fE) = 0$ if a map $f\colon E \to F$ is δ-onto F for some δ.

In view of Conjecture 2.10, it is reasonable to ask whether an ε-nearisometry $f\colon E \to F$ with $\tau(fE) = 0$ or maybe with $\tau(fE) < 1$ can be approximated by a surjective isometry. An answer for Hilbert spaces is given in 2.13. Observe that $\tau(f\mathsf{R}) = 1$ for the map $f\colon \mathsf{R} \to \mathsf{R}^2$ considered in 2.5.

2.13. **Theorem.** [Vä3, 5.4] *Suppose that E and F are Hilbert spaces and that $f\colon E \to F$ is an ε-nearisometry with $f(0) = 0$ and $\tau(fE) < 1$. Then there is a surjective linear isometry $T\colon E \to F$ with $d(T, f) \le 2\varepsilon$.*

Observe that the condition $d(T, f) \le 2\varepsilon$ implies that $\tau(fE) = 0$. It follows that $\tau(fE) \in \{0, 1\}$ for each nearisometry between Hilbert spaces.

2.14. *Inverse results.* An isometry $f\colon E \to F$ between Banach spaces with $f(0) = 0$ need not be linear, but T. Figiel ([Fi], [BL, 14.2]) and W. Holsztyński [Ho] proved in 1968 that there is a unique linear map $T\colon \overline{\operatorname{span}}\, fE \to E$ such that $|T| = 1$ and $Tf = \mathrm{id}$. In view of this result, it is reasonable to conjecture that if $f\colon E \to F$ is an ε-nearisometry with $f(0) = 0$, then there is a linear map $T\colon \overline{\operatorname{span}}\, fE \to E$ such that $|T| = 1$ and $d(Tf, \mathrm{id}) \le c\varepsilon$ for some constant c. This conjecture was disproved by S. Qian [Qi, Ex. 1] in 1995, but Qian proved that the conjecture holds (with $c = 6$) in certain cases, for example, if F is Hilbert or if E and F are L^p spaces with $1 < p < \infty$. Further results in this direction are given in [ŠV].

3. Unbounded subsets

In this section we consider ε-nearisometries $f\colon A \to F$ where A is an unbounded subset of E. We look for a surjective isometry $T\colon E \to F$ such that $d(T, f)$ is finite and hopefully bounded by $c\varepsilon$ for some constant c, possibly depending on A. As in Section 2, the proofs are based on the behavior of f near the point at infinity.

The results in this section are due to the author [Vä3]. Most of them deal with Hilbert spaces or just with euclidean spaces. However, the following improvement of the Fundamental theorem 2.2 is valid for all Banach spaces. It shows that the Fundamental theorem is not, after all, a truly global result but a local property of maps at the point ∞. The result was suggested to the author by O. Martio.

3.1. **Theorem.** *Suppose that $A \subset E$ and that $f\colon A \to F$ is an ε-nearisometry such that the sets $E \setminus A$ and $F \setminus fA$ are bounded. Then there is a surjective isometry $T\colon E \to F$ with $d(T, f) \le 2\varepsilon$. For each $x_0 \in A$ we can choose T so that $Tx_0 = fx_0$.*

Proof. We may assume that $x_0 = 0$, $fx_0 = 0$. Choose a number $R > \varepsilon$ such that $E \setminus A \subset B(R)$ and $F \setminus fA \subset B(2R)$. Define $f_1\colon E \to F$ by $f_1 x = 2x$ for $|x| \le R$ and by $f_1 x = fx$ for $|x| > R$. Then f_1 is a surjective nearisometry. By the Fundamental theorem 2.2, there is a linear surjective isometry $T\colon E \to F$ with $d(T, f_1) < \infty$. Since $d(T, f) < \infty$, an easy modification of the proof of 2.2 (see [BL, p. 362]) shows that $d(T, f) \le 2\varepsilon$. $\square$

3.2. *Hilbert spaces.* In the rest of this section we assume that E and F are Hilbert spaces. As mentioned in 2.4, the sequence of points $y_n = 2^{-n} f(2^n x)$ converges as soon as the nearisometry f is defined in the set $\{2^n x : n \in \mathsf{N}\}$. A more general result is given in 3.3. These results are not valid in general Banach spaces, which makes the theory in Banach spaces considerably more difficult.

3.3. **Lemma.** *Suppose that E and F are Hilbert spaces and that (x_j) is a sequence in E such that $|x_j| \to \infty$ and $x_j/|x_j|$ converges to a limit as $j \to \infty$. Suppose also that $f\colon \{x_j : j \in \mathsf{N}\} \to F$ is a nearisometry. Then the sequence of the points $fx_j/|x_j|$ is convergent.*

3.4. *Terminology.* Suppose that A is an unbounded subset of a Hilbert space E. We say that a unit vector $u \in E$ is a *cluster direction* of A if there is a sequence (x_j) in A such that $|x_j| \to \infty$ and $x_j/|x_j| \to u$. We let $\operatorname{cd} A$ denote the set of all cluster directions of A.

The theory is easier in the case $\dim E < \infty$, because then each sequence (x_j) with $|x_j| \to \infty$ has a subsequence (y_j) such that the sequence $(y_j/|y_j|)$ is convergent. In particular, $\operatorname{cd} A$ is then compact and nonempty. In a general Hilbert space, an unbounded set need not have any cluster directions.

For an unbounded set $A \subset E$ we define a number $\mu(A) \in [0,1]$ as follows. First, for each unit vector $e \in E$ we set $\sigma(e, A) = \sup\{|u \cdot e| : u \in \operatorname{cd} A\}$ and then

$$\mu(A) = \inf\{\sigma(e, A) : |e| = 1\}.$$

The number $\mu(A)$ is small iff $\operatorname{cd} A$ lies in a narrow neighborhood of a hyperplane through the origin.

For example, $\mu(A) = 1$ if A contains a half space or if $d(x, A)$ is bounded for $x \in E$. Furthermore, $\mu(C) = \sin\alpha$ for the cone $C = \{x : |x \cdot e| \geq |x| \cos\alpha\}$ where $0 < \alpha \leq \pi/2$, $|e| = 1$.

If $\dim E < \infty$, then $\mu(a)$ is the infimum of all $t \in [0,1]$ such that the double cone $\{x \in E : |x \cdot e| > t|x|\}$ meets A in a bounded set for some unit vector e. Moreover, in this case $\mu(A) = \sqrt{1 - \tau(A)^2}$, where τ is defined in 2.12.

Let $c > 0$. We say that a set $A \subset \mathsf{R}^n$ has the *c-isometric approximation property*, abbreviated c-IAP, if for every $\varepsilon \geq 0$ and for every ε-nearisometry $f\colon A \to \mathsf{R}^n$ there is an isometry $T\colon \mathsf{R}^n \to \mathsf{R}^n$ with $d(T, f) \leq c\varepsilon$. If A has the c-IAP and contains at least two points, then clearly $c \geq 1/2$.

For example, the whole space R^n has the $\sqrt{2}$-IAP by 2.6. From Example 2.5 it follows that a line in R^n, $n \geq 2$, does not have the c-IAP for any c. The following result [Vä3, 2.3] gives a quantitative geometric characterization for unbounded subsets of R^n with the c-IAP. The corresponding question for bounded sets will be considered in Section 4.

3.5. **Theorem.** *For an unbounded set $A \subset \mathsf{R}^n$, the following conditions are quantitatively equivalent.*

(1) *A has the c-IAP,*

(2) $\mu(A) \geq 1/c' > 0$.

More precisely, (1) *implies* (2) *with $c' = 17c$, and* (2) *implies* (1) *with $c = \sqrt{2}c'$. The constant $\sqrt{2}$ is the best possible.*

3.6. *Remarks.* 1. It follows from 3.5 that $\mu(A) = 0$ iff A does not have the c-IAP for any c. This happens, for example, if A is a linear subspace of R^n with $\dim A < n$. This can be directly seen as in Example 2.5. In fact, the proof for the part (1) $\Rightarrow$ (2) of 3.5 is based on an elaboration of this example.

2. We sketch the proof for the part (2) $\Rightarrow$ (1). Suppose that $\mu(A) \geq 1/c'$ and that $f\colon A \to \mathsf{R}^n$ is an ε-nearisometry. We may assume that $0 \in A$ and that $f(0) = 0$. We first define a map $\varphi\colon \operatorname{cd} A \to \mathsf{R}^n$ as follows. Let $u \in \operatorname{cd} A$ and choose a sequence (x_j) in A such that $|x_j| \to \infty$ and $x_j/|x_j| \to u$. By 3.3, the limit $\varphi u = \lim_{j\to\infty} f x_j/|x_j|$ exists. One can show that the limit is independent of the choice of the sequence and that the map $\varphi\colon \operatorname{cd} A \to \mathsf{R}^n$ is an isometry, which extends to a linear isometry $T\colon \mathsf{R}^n \to \mathsf{R}^n$ with $d(T, f) \leq 2\varepsilon$. The bound $\sqrt{2}\varepsilon$ is obtained by composing T with a suitable translation.

Most of these arguments can be carried out in an arbitrary Hilbert space, and we obtain the following variation of the part (2) $\Rightarrow$ (1) of 3.5:

3.7. **Theorem.** *Suppose that E and F are Hilbert spaces and that $A \subset E$ is an unbounded set with $\mu(A) \geq 1/c'$. Suppose also that $f\colon A \to F$ is an ε-nearisometry. Then there is an isometry $T\colon E \to F$ such that $d(T, Pf) \leq \sqrt{2}\varepsilon$, where $P\colon F \to TE$ is the orthogonal projection onto the affine subspace TE of F.*

4. Bounded subsets

In this section we consider nearisometries of a bounded set $A \subset \mathsf{R}^n$. The target space will be R^n (with the same n) except in the first result 4.1. Most of the results are due to P. Alestalo, D.A. Trotsenko and the author.

The proofs are very much different from those in Sections 2 and 3, because we cannot use a limiting process where points tend to ∞. The basic tool is the simple formula

$$2a \cdot b = |a|^2 + |b|^2 - |a - b|^2,$$

and the proofs are elementary but rather long except if the set A is sufficiently regular (for example a ball), in which case one can make use of the ideas of F. John [Jo]; see 4.3.

We identify R^n in the natural way with a subspace of the Hilbert space l_2.

4.1. **Theorem.** [ATV1, 2.2] *Suppose that $A \subset \mathsf{R}^n$ is bounded and that $f\colon A \to l_2$ is an $\varepsilon d(A)$-nearisometry with $\varepsilon \leq 1$. Then there is a surjective isometry $T\colon l_2 \to l_2$ with $d(T, f) \leq c(n)\sqrt{\varepsilon}d(A)$. If $fA \subset \mathsf{R}^n$, we can choose T so that $T\mathsf{R}^n \subset \mathsf{R}^n$.*

4.2. *Example.* Let A be the interval $[-1, 1]$, let $\varepsilon > 0$, and let $f\colon A \to \mathsf{R}^2$ be the map $fx = (x, |x|\sqrt{\varepsilon})$. Then f is an ε-nearisometry, and $d(T, f) \geq \sqrt{\varepsilon}/2$ for each isometry $T\colon A \to \mathsf{R}^2$. Hence the bound in 4.1 has the correct order of magnitude.

4.3. *John's method.* F. John [Jo] considered in 1961 isometric approximation of locally $(1+\varepsilon)$-bilipschitz maps $f\colon G \to \mathsf{R}^n$ where $G \subset \mathsf{R}^n$ is a ball or, more generally, of a class later called John domains. His method is elegant compared with the proofs of the other results in this section. It can easily be modified so as to prove the IAP (see 3.4) of sufficiently regular bounded sets. We give the result for balls; a more general result is given in [ATV1, 3.12].

4.4. **Theorem.** *A ball in R^n has the c-IAP with $c = 10n^{3/2}$.*

4.5. *Remark.* The constant c in 4.4 must depend on n. This was recently proved by E. Matoušková [Ma], who showed that for each $t > 0$ there is an integer n and a $(1+t)$-bilipschitz map $f\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $d(T|B^n, f|B^n) \ge 1/\sqrt{2}$ for each isometry $T\colon \mathsf{R}^n \to \mathsf{R}^n$. Since $f|B^n$ is a $2t$-nearisometry, the unit ball B^n does not have the c-IAP for $c < (2t\sqrt{2})^{-1}$.

The map f is defined as follows. Let $h\colon \mathsf{R}^2 \to \mathsf{R}^2$ be the $(1+t)$-bilipschitz spiral map defined by $h(r,\varphi) = (r, \varphi + \pi/2 + t\log r)$ in polar coordinates. Choose an integer $N > \pi/t\log 2$ and set $m = 2^N$, $n = 2m$. Then $\mathsf{R}^n = E_1 \oplus \cdots \oplus E_m$ where $E_j = \operatorname{span}(e_j, e_{m+j})$. The map h induces in a natural way maps $h_j\colon E_j \to E_j$, and we set $h = h_1 \oplus \cdots \oplus h_m\colon \mathsf{R}^n \to \mathsf{R}^n$. Then f is $(1+t)$-bilipschitz with $|fx| = |x|$ and $f(-x) = fx$ for all $x \in \mathsf{R}^n$. One can show that the image of $\mathsf{R}^m = \operatorname{span}(e_1, \dots, e_m)$ contains an orthonormal basis $\bar{u}$ of R^n. If $T\colon \mathsf{R}^n \to \mathsf{R}^n$ is an isometry, then $L = T\mathsf{R}^m$ is an affine subspace of R^n, and one can show that there is a member u of $\bar{u}$ such that $d(u,L) \vee d(-u,L) \ge 1/\sqrt{2}$. Hence f is the desired map.

4.6. *Thickness.* For a unit vector $u \in \mathsf{R}^n$ we define the projection $\pi_u\colon \mathsf{R}^n \to \mathsf{R}$ by $\pi_u x = x \cdot u$. The *thickness* of a bounded set $A \subset \mathsf{R}^n$ is the number

$$\theta(A) = \inf_{|u|=1} d(\pi_u A).$$

We have always $\theta(A) \le d(A)$, and $\theta(A) = 0$ if and only if A is contained in a hyperplane.

It follows from Example 4.2 that a line segment $J \subset \mathsf{R}^2$ does not have the IAP. In this case we have $\theta(A) = 0$. One can show that a bounded set $A \subset \mathsf{R}^n$ containing at least $n+1$ points has the IAP if and only if $\theta(A) > 0$. If A contains no isolated points, this holds in the following quantitative form:

4.7. **Theorem.** *Suppose that $A \subset \mathsf{R}^n$ is a bounded set without isolated points. Then the following properties are quantitatively equivalent:*
(1) *A has the c-IAP,*
(2) $\theta(A) \ge d(A)/c'$.
More precisely, (1) *implies* (2) *with a constant $c' = c'(c,n)$ and vice versa.*

Part (2) $\Rightarrow$ (1) was proved in [ATV1, 3.3], and it is true for all bounded sets. Part (1) $\Rightarrow$ (2) follows from 4.10 below.

4.8. *Sets with isolated points.* If $A \subset \mathsf{R}^n$ is a bounded sets containing isolated points, the part (1) $\Rightarrow$ (2) of 4.7 does not hold quantitatively. This is seen from the following example due to Trotsenko.

Let $0 < t \le 1$ and let $A \subset \mathsf{R}^2$ be the three-point set $\{0, e_1, te_2\}$. Then $d(A)/\theta(A) \ge t + 1/t$ is arbitrarily large. However, a direct proof shows that A has the 8-IAP for all t.

As another example we consider the set $A' = \{0, e_1, te_2, e_1 + te_2\} \subset \mathsf{R}^2$. It turns out that A' has the c-IAP with some $c = c(t)$ but $c(t) \to \infty$ as $t \to 0$. This is seen by considering the map $f\colon A' \to \mathsf{R}^2$ with $f(e_1 + te_2) = e_1 - te_2$ and $fx = x$ for the other three points $x \in A'$.

To get a quantitative geometric characterization for all bounded sets with the c-IAP we introduce the following concept.

4.9. *Definition.* Let $c \geq 1$. We say that a bounded set $A \subset \mathsf{R}^n$ is a *c-solar system* if there is a finite set $H = \{u_0, \dots, u_n\} \subset A$ such that

(1) $|u_k - u_0| \leq cd(u_k, \operatorname{aff}(H \setminus \{u_k\})$ for all $1 \leq k \leq n$,

(2) $A \setminus H \subset \bar{B}(u_0, c\min\{|u_k - u_0| : 1 \leq k \leq n\})$.

Here aff S denotes the affine subspace spanned by a set $S \subset \mathsf{R}^n$.

The point u_0 plays a special role, and it is called the center of the system. The points $u_1, \dots, u_n$ are the planets and the set $A \setminus H$ is the sun. The system may degenerate to $\{u_0, \dots, u_i\}$ with $0 \leq i < n$; then we assume that $u_{i+1} = \cdots = u_n = u_0$. If $n \geq 3$ and if the system is nondegenerate, the planets do not lie in a plane (as in the real solar system). Observe that there are no restrictions for the distances $|u_k - u_0|$.

The three-point set A of 4.9 is a 1-solar system but the set A' is a c-solar system only for $c \geq \sqrt{1 + 1/t^2}$.

4.10. **Theorem.** [Vä2, 2.5] *For a bounded set $A \subset \mathsf{R}^n$, the following conditions are quantitatively equivalent:*

(1) *A has the c-IAP,*

(2) *A is a c'-solar system.*

5. Related results

5.1. *Weak nearisometries.* Let $\varphi\colon\ [0, \infty[\to \mathsf{R}$ be an increasing function. We consider maps $f\colon E \to F$ between Banach spaces satisfying the condition

$$\big||fx - fy| - |x - y|\big| \leq \varphi(|x - y|)$$

for all $x, y \in E$. If φ is the constant function $\varphi(t) = \varepsilon$, this means that f is an ε-nearisometry. J. Lindenstrauss and A. Szankowski ([LS], [BL, 15.4]) proved that if $f(0) = 0$ and if φ increases so slowly that

$$\int_1^\infty \frac{\varphi(t)}{t^2} dt < \infty,$$

then there is a surjective linear isometry $T\colon E \to F$ such that $|Tx - fx|/|x[\to 0$ as $|x| \to \infty$.

5.2. *Stability.* The theory considered in this article is an example of stability (see [Ul, p. 63]). We consider a class C of maps (isometries) $f\colon X \to Y$. Then we relax the definition and get a larger class C^* of maps (ε-nearisometries) involving a parameter ε. Then we ask how well we can approximate a member f of C^* by members T of C. Instead of estimating the distance $d(T, f)$ it is sometimes more convenient to consider maps $T\colon Y \to X$ and the distance $d(Tf, \mathrm{id})$.

Various stability theories are considered in the survey articles of D.H. Hyers [Hy] and G.L. Forti [Fo]. We mention some examples.

1. C = similarities, C^* = quasisymmetric maps [ATV1, 4.6].
2. C = Möbius maps, C^* = quasiregular maps [Re, II.12.5].
3. C = additive maps, C^* = almost additive maps [BL, 15.1].
4. C = convex functions, C^* = almost convex functions [HU3].

5.3. *Applications.* The fundamental theorem 2.2 is beautiful, but the author does not know of any applications of this result. The IAP of thick sets (Th. 4.7) can be applied to prove the following result on bilipschitz extensions [ATV2]. Its proof follows the ideas in [Vä1] and [Tr].

5.4. **Theorem.** *For each positive integer n and for each $c \geq 1$ there are positive numbers $\varepsilon_0 = \varepsilon_0(c, n)$ and $c' = c'(c, n)$ such that the following holds.*

Suppose that A is a subset of R^n such that $\theta(A \cap B(x, r)) \geq r/c$ whenever $x \in A$ and $A \setminus B(x, r) \neq \varnothing$. Then every $(1 + \varepsilon)$-bilipschitz map $f\colon A \to F$ with $\varepsilon \leq \varepsilon_0$ can be extended to a $(1 + c'\varepsilon)$-bilipschitz map $g\colon \mathsf{R}^n \to \mathsf{R}^n$.

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