Papers on Analysis:
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ON THE CONFORMAL REPRESENTATION OF ALEXANDROV SURFACES

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The concept of a two-dimensional manifold of bounded curvature has been introduced by A.D. Alexandrov already at the end of 40th in continuation of his studies of the intrinsic geometry of convex surfaces. A concise exposition of basic definitions and results of the theory of two-dimensional manifolds of bounded curvature can be found in the publications [1], [2], and [3]. A complete exposition was given in 1962 in the joint book by A.D. Alexandrov and V.A. Zalgaller printed in the series "Trudy MIAN" ([4]).

1. Alexandrov's theory of two-dimensional manifolds or, what is the same, the theory of general surfaces can be treated as an irregular analog of two-dimensional Riemannian geometry or, in other words, of the Gaussian intrinsic geometry of surfaces. Riemannian geometry is determined by a strictly positive quadratic differential form locally representable by the formula $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Using this quadratic form, we can define the length of a curve. Setting $\rho(X, Y)$ to be equal to the greatest lower bound of the lengths of curves joining points X and Y, we get some metric. This metric is referred to as the natural metric of the Riemannian space in question. In Riemannian geometry, the functions E, F and G are usually supposed to be sufficiently smooth.

Now, I formulate some necessary definitions.

I presume the knowledge of the notion of a curve as well as the length of a curve in an arbitrary metric space. Here the length of a curve K will be denoted by $s(K)$. The curve is said to be rectifiable if its length is finite.

A metric space M is said to be pathwise connected if for every pair of points X, $Y \in M$ there exists a curve L joining X and Y, i.e. such a curve that X and Y are its endpoints. We say that M is metrically connected if for every pair of points X, $Y \in M$ there exists a rectifiable curve L joining them.

We say that the metric of M is intrinsic if for every pair of points X, Y in M the distance $\rho(X, Y)$ between these points is equal to the greatest lower bound of the lengths of curves joining X and Y.

A curve L in M with endpoints X and Y is said to be a shortest arc if its length $s(L)$ is equal to the distance between these points, $s(L) = \rho(X, Y)$. A curve L is said to be a geodesic if every sufficiently small arc of this curve is a shortest arc. If a metric space with intrinsic metric is complete, then for every point X in this space and a neighborhood U of X we can find some other neighborhood U' of X such that for every two points belonging to U' there exist a shortest arc joining these points in U.

Suppose that G is an arbitrary subset of a metric space M. We call this set G metrically connected if for every two points X, $Y \in G$ there exists a rectifiable curve in G which joins X and Y. We denote by $\rho_G(X, Y)$ the greatest lower bound of the lengths of curves in G joining X and Y. If a set $G \subset M$ is metrically connected in M, then the function $\rho_G : G \times G \to \mathbb{R}$ is a metric on G. We call ρ_G the induced metric on G.

A topological space M is said to be a two-dimensional manifold if every point of this space has a neighborhood homeomorphic to the closed half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.

A point X of a two-dimensional manifold M is an interior point of M if X has a neighborhood homeomorphic to the open half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. If a point X has no such neighborhood, we say that X is a boundary point of M. The set of all boundary points of a manifold M is denoted by ∂M and called the boundary of M. In the case $\partial M = \varnothing$ we call M a manifold without boundary.

The axiomatic approach to the theory of metric spaces of bounded curvature which was suggested by Alexandrov is based on the following construction. Suppose that X, Y, and Z are three points in a metric space whose metric is assumed intrinsic. Join these points pairwise by shortest arcs. On the ordinary Euclidean plane, consider the triangle $X'Y'Z'$ such that $|X'Y'| = \rho(X, Y)$, $|X'Z'| = \rho(X, Z)$, and $|Y'Z'| = \rho(Y, Z)$. We call the triangle $X'Y'Z'$ the envelope of the triangle XYZ in the metric space under the consideration.

Constructing the envelope of a triangle, we can replace the Euclidean plane with an arbitrary surface of constant curvature, i.e., a hyperbolic plane or an ordinary Euclidean sphere.

Let L and K be two curves in a metric space M with intrinsic metric. Suppose that these curves have a common beginning O. Take arbitrary points $X \in L$ and $Y \in K$ and let $O'X'Y'$ be the envelope of the triangle OXY on the plane. Denote by $\gamma(X, Y)$ the angle of the triangle $O'X'Y'$ at the vertex O'. Set $x = \rho(O, X)$, $y = \rho(O, Y)$, and $z = \rho(X, Y)$. Then $\gamma = \gamma(X, Y)$ can be found from the formula

$$\cos\gamma = \frac{x^2 + y^2 - z^2}{2xy}.$$

The limit (if any) $\lim\limits_{X \to O, Y \to O} \gamma(X, Y)$ as X and Y tend to O along the curves is called the angle between the curves K and L at O.

The requirement of existence of an angle between two curves is very strong. This relates to the fact that in this definition the ratios x/y and y/x are not assumed to be bounded. In this connection, I wish to say that, in Alexandrov's axiomatic approach to the theory of metric spaces with restrictions on curvature, instead of this angle one considers the so called upper angle, i.e., the upper limit

$$\overline{\lim_{X\to O, Y\to O}} \gamma(X,Y).$$

Proving the existence of the angle in the sense of this definition for geodesics in a Riemannian space satisfying the customary regularity requirements is, of course, an exercise, but not a very simple exercise.

We say that a curve K in a metric space M has a definite direction at an endpoint A if the angle that it makes with itself equals 0.

2. A.D. Alexandrov gave two equivalent definitions for two-dimensional manifolds of bounded curvature. These manifolds are considered as metric spaces. The first definition is an axiomatic one. In view of the space limitation, I cannot develop this definition in every detail. We merely say that it relies on the above-described notions of the envelope of a triangle and the upper angle between curves.

From now on, we let M denote an arbitrary two-dimensional manifold without boundary.

The second definition by Alexandrov is based on the approximation of metrics by two-dimensional Riemannian metrics. Given an open set U in $\mathbb{R}^n$ and a metric ρ on U, we say that ρ is an n-dimensional Riemannian metric on U if the metric space (U,ρ) is isometric to some n-dimensional Riemannian space endowed with its natural metric.

Further I use the following standard notation. Given an arbitrary real number x we set $x^+ = max\{0, x\}$ and $x^- = (-x)^+$. Obviously $x^+ - x^- = x$ and $x^+ + x^- = |x|$. The quantity x^+ is called the positive part of x and x^- is the negative part of x.

Let M be a two-dimensional manifold endowed with a Riemannian metric. Suppose that the metric satisfies regularity conditions customary in differential geometry. Then the Gaussian curvature $K(X)$ is defined at every point $X \in M$. Denote by $\alpha(E)$ the area of a set $E \subset M$. The set function α is defined on all Borel sets $E \subset M$ and is nonnegative. For a Borel set E we define the following integrals

$$\omega(E) = \int\limits_E K(X)\, d\alpha(X), \quad \omega^+(E) = \int\limits_E [K(X)]^+\, d\alpha(X),$$

$$|\omega|(E) = \int\limits_E |K(X)|\, d\alpha(X).$$

We call $\omega(E)$ the integral curvature of E and $|\omega|(E)$ the absolute curvature of E; and finally $\omega^+(E)$, the positive part of the curvature of E.

A two-dimensional space M with a metric ρ is called a two-dimensional space of bounded curvature or, what is the same, a general surface in the Alexandrov sense if and only if for every point $P \in M$ there exists a sequence of Riemannian metrics $(\rho_\nu)_{\nu\in\mathbb{N}}$ which are defined on a neighborhood U of P and are such that the functions ρ_ν converge uniformly to ρ_U on the set $U \times U$ and the sequence

$$(|\omega_\nu|(U)), \quad \nu = 1, 2, \ldots, \tag{$*$}$$

is bounded. (Here $|\omega_\nu|(U)$ denotes the absolute integral curvature, i.e., the integral of the absolute value of the Gaussian curvature of the Riemannian metric ρ_ν over the set U with respect to area.)

This is the primary version of the approximative definition given by Alexandrov in [1].

The boundedness condition for the sequence $(*)$ can be replaced, as shown in [4], by the weaker condition of boundedness of the sequence

$$(\omega_\nu^+(U)), \qquad \nu = 1, 2, \ldots. \tag{$**$}$$

We can say that boundedness of the sequence $(**)$ is a sufficient condition for a two-dimensional manifold with an intrinsic metric to be a manifold of bounded curvature. Boundedness of the sequence $(*)$ is a necessary condition.

So far I considered only two-dimensional manifolds without boundary. How does the concept of a two-dimensional manifold of bounded curvature extend to the case of manifolds with boundary? This question was studied by Yu.F. Borisov [23], [24] and was the subject of his thick (300 pages) PhD dissertation written 50 years ago. Unfortunately, this dissertation remains unpublished as yet.

The basic concepts of the Gaussian theory of surfaces admit some analogs in Alexandrov's theory of general surfaces, but the characteristics related to the points of a surface must be replaced with certain integral characteristics in the general case of Alexandrov surfaces. The Gaussian curvature, regarded as a point function, should be replaced with an additive set function which in the regular case coincides with the integral curvature of sets. In the general case, the integral curvature may be a set function not absolutely continuous with respect to area.

In a similar way, the geodesic curvature of a curve is replaced with the integral curvature of a curve in the theory of general surfaces. For a regular curve on a regular surface, the integral curvature equals the integral of the geodesic curvature with respect to the arclength of the curve. (The integral curvature of a curve is also called the turn or swerve of the curve.)

3. Here I wish to consider a particular instance of manifolds of bounded curvature which can be attributed formally to elementary geometry, but in this example we can see the main pecularieties by which the manifolds of bounded curvature differ from Riemannian spaces.

Let M be a two-dimensional metric space endowed with an intrinsic metric ρ. We say that M is a manifold with polyhedral metric, or shortly that M is a polyhedron, if there exists a triangulation (in the topological sense) of the manifold M which satisfies the following conditions. If we furnish each triangle of the triangulation with the intrinsic metric induced by the metric of the manifold, then each of these triangles becomes isometric to a triangle on the ordinary Euclidean plane $\mathbb{E}^2 = \mathbb{C}$. Speaking of a triangulation of a polyhedron, I will always bear in mind a triangulation that satisfies this condition.

Let M be an arbitrary two-dimensional polyhedron and let $X \in M$ be an interior point of M. We define a certain number $\theta(X)$.

Suppose that we are given a triangulation of the manifold M such that the triangles of this triangulation in the induced metric are isometric to planar triangles. Without loss of generality we can suppose that X is a vertex of this triangulation. (We can satisfy this condition always by subdividing the initial triangulation.) Let $T_1, T_2, \ldots, T_m$ be all triangles of the triangulation which have X as its common vertex and let $\theta_1, \theta_2, \ldots, \theta_m$ be the angles of these triangles at the point X. The sum $\theta(X) = \theta_1 + \cdots + \theta_m$ is independent of the choice of the triangulation of the polyhedron. $\theta(X)$ is called the total angle at the point X of the polyhedron M.

Given an interior point X of M, we call the difference $\omega(X) = 2\pi - \theta(X)$ the curvature of the manifold M at X.

Obviously, $\omega(X) < 2\pi$. If for a point X we have $\omega(X) = 0$, then some neighborhood of X is isometric to a planar disk and the point X corresponds by isometry to the center of this disk.

In the case when X is a boundary point of the polyhedron M, consider the difference $\varkappa(X) = \pi - \theta(X)$. This difference is referred to as the turn of the boundary at X.

Every two-dimensional manifold M with a polyhedral metric is a two-dimensional manifold of bounded curvature. In this case the curvature of a set $E \subset M$ is equal to the sum of the curvatures of the interior vertices of M that belong to E, $\omega(E) = \sum\limits_{X \in E} \omega(X)$. We see that in this case the measure is concentrated on some discrete set. In this case we have $\omega^+(E) = \sum\limits_{X \in E} \omega^+(X)$. Given a set $E \subset \partial M$, put

$$\varkappa(E) = \sum_{X \in E} \varkappa(X).$$

Here I formulate two results about two-dimensional polyhedra.

Theorem 1. *Suppose that M is a compact two-dimensional polyhedron. Then*

$$\varkappa(\partial M) + \omega(M) = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

This theorem is an analog of the Gauss-Bonnet theorem in differential geometry. The theorem is a simple consequence of the definition of the Euler characteristic and the elementary fact that the sum of angles of a planar triangle equals π.

A. D. Alexandrov formulated some extremal problems for two-dimensional manifolds of bounded curvature. He suggested the following method for the solution of these problems: First, the problem under study is considered for polyhedra. In this case we get a problem from elementary geometry. If the problem is solved for polyhedra, then by passing to the limit we can get the solution of the problem in the general case.

Let M be a polyhedron. We say that M is a convex cone if it satisfies the following conditions.

C1) M is homeomorphic to a closed disk on the plane;

C2) For every boundary vertex X of M the turn of the boundary at this point is nonnegative: $\varkappa(X) \geq 0$ for all $X \in \partial M$.

C3) M has at most one interior vertex. If $X \in M$ is an interior vertex of M, then $\omega(X) \geq 0$.

I make some comments on this definition.

In the case when the convex cone M has no interior vertex, M is isometric to a planar convex polygon.

Theorem 2 [11]. *Let M be a polyhedron homeomorphic to a closed disk on the plane. Suppose that $\omega^+(M) < 2\pi$. Then one can construct a convex cone Q for which $\omega(Q) \leq \omega^+(M)$ and there exists a mapping $\varphi : Q \to M$ satisfying the following conditions:*

1) For every pair of points $X,\ Y \in Q$,

$$\rho[\varphi(X), \varphi(Y)] \leq \rho(X, Y).$$

2) $\varphi(\partial Q) = \partial M$.

3) Each arc of the boundary of the cone Q is transformed by the mapping φ into an arc of the boundary of ∂M of the same length.

Theorem 2 can be applied, for instance, to the solutions of isoperimetric problems. The classical example is the following isoperimetric problem:

Among all polyhedra P homemorphic to a closed disk, satisfying the condition $\omega^+(P) \leq \omega_0 < 2\pi$, and having the boundary consisting of straight line segments of given lengths, find a polyhedron with greatest area.

As follows from Theorem 2, the solution must be a convex cone. In this case the problem is reduced to a problem from elementary geometry.

Theorem 2 admits also other applications.

4. In the regular case, it has been already discovered by Gauss that in a neighborhood of an arbitrary point of a surface one can introduce a local coordinate system in which the line element of the surface takes the shape

$$ds^2 = \lambda(x, y)(dx^2 + dy^2). \tag{1}$$

Coordinate systems satisfying this condition are called isothermal coordinate systems. The construction of an isothermal coordinate system in the neighborhood of a point P of a surface is equivalent to the construction of a conformal mapping of this neighborhood into the plane. Isothermal coordinate systems on two-dimensional Riemannian manifolds form an atlas whose transition functions are conformal mappings of plane domains.

Simple calculations shows that if the line element of a surface is given by formula (1), then the Gaussian curvature of the surface at a point with coordinates (x, y) is equal to

$$K(x, y) = -\frac{1}{2\lambda(x, y)} \Delta \log\{\lambda(x, y)\}.$$

Using the familiar formula for a solution to the Poisson equation, from here we can derive a representation for the function λ (further we use the notation $z = x + iy$ and $\zeta = \xi + i\eta$):

$$\log \lambda(z) = \frac{1}{\pi} \int_G \log \frac{1}{|z - \zeta|} K(\zeta)\lambda(\zeta) d\xi d\eta + H(z), \tag{2}$$

where $H(z)$ is a harmonic function and $G \subset \mathbb{C}$ is the domain of the coordinate system under consideration. With the above notations, we can write down (2) as follows

$$\log \lambda(z) = \frac{1}{\pi} \int_G \log \frac{1}{|z - \zeta|} d\omega(\zeta) + H(z), \tag{3}$$

where $H(z)$ is as before a harmonic function.

It is well known that the class of subharmonic functions in a domain G on the plane $\mathbb{C} = \mathbb{R}^2$ coincides with the class of all real functions F defined in this domain and admitting the representation

$$F(z) = H(z) - \int_G \ln \frac{1}{|z - \zeta|} d\mu(\zeta),$$

where $H(z)$ is a harmonic function and μ is a nonnegative measure on the σ-algebra of all Borel subsets of G.

Suppose that a function $F(z)$ can be represented as

$$F(z) = \frac{1}{\pi} \int_G \log \frac{1}{|z - \zeta|} d\omega(\zeta) + H(z),$$

where $H(z)$ is a harmonic function and ω is an arbitrary completely additive set function. Decomposing ω into the difference of two nonnegative measures we get that the function F is the difference of two subharmonic functions.

In the general theory of Alexandrov surfaces the integral curvature is an arbitrary completely additive set function The only restriction on it is as follows. If the set E consists of a single point then we must have $\omega(E) \leq 2\pi$.

In a two-dimensional Riemannian manifold we can also consider some other coordinate systems in which the line element has a special form. However, in the approximation of a manifold of bounded curvature by Riemannian manifolds in accordance with the above definition, the domain in which a special coordinate system can be introduced collapses as a rule to a point as $n \to \infty$. This phenomenon does not take place for isothermal coordinates as it follows from the general principles of complex function theory. The possibility of introducing an isothermal coordinate system in a domain of a two-dimensional Riemannian space depends exclusively on the topological structure of the domain. In particular, such coordinate system can be defined on every domain homeomorphic to an open disk of the plane. This is immediate from the fact that the construction of an isothermal coordinate system is equivalent to finding a conformal mapping of this domain into the plane $\mathbb{C}$.

There is one more type of coordinate systems for which the collapse of the domain does not take place. This is the Chebyshev coordinate system with the line element $ds^2 = dx^2 + \cos\varphi(x,y)dxdy + dy^2$ as it was shown in [21]. But in this case the following condition is necessary. The absolute integral curvature of the domain where the coordinate system should be defined must be at most π.

To prove the existence of an isothermal coordinate system in a domain on a surface without presuming that the size is small, it is necessary, first of all, to show that every point of the surface has a neighborhood where an isothermal coordinate system can be introduced. This neighborhood can be arbitrary small. Then the existence of an isothermal coordinate system in the whole domain can be proven by using the well-known results of complex function theory, namely, the principle of uniformization.

In light of what was said above, the following conjecture seems very plausible. I came to it already in the spring of 1953. If M is a two-dimensional manifold of bounded curvature, then nearby every point X of this manifold its metric can be defined by a line element $ds^2 = \lambda(z)(dx^2 + dy^2)$ where the function $\lambda(z)$ is such that $\log\lambda(z)$ can be represented as the difference of two subharmonic functions.

I found the proof of this conjecture at the end of 1953. A concise exposition of my results has appeared in Doklady in 1954 [5].

To continue, I introduce some auxiliary notions.

First of all, I must explain what is meant when we say that a metric is defined by the integral of a differential form $\lambda(z)|dz|^2$. If the function λ is defined by formula (3) with an arbitrary set function ω then it can take values ∞ and 0.

As a preliminary, I describe a certain class of plane curves. A curve K is said to be *one-sidedly smooth* if it admits a parameterization $z : [a,b] \to \mathbb{C}$ satisfying the following conditions:

K 1) For every $t \in [a,b)$ the vector-function $z(t)$ has the right derivative

$$z'_r(t) = \lim_{h\to+0} \frac{z(t+h) - z(t)}{h}$$

at t and this derivative differs from 0. For $t \in (a,b]$, $z(t)$ also has the left derivative

$$z'_l(t) = \lim_{h\to-0} \frac{z(t+h) - z(t)}{h},$$

and $z'_l(t) \neq 0$.

K 2) The derivative z_l' is left continuous for $t > a$ in the sense that

$$z_l'(t) = \lim_{u \to t-0} z_l'(u) = \lim_{u \to t-0} z_r'(u),$$

and $z_r'(t)$ is right continuous for $t < b$ in the sense that

$$z_r'(t) = \lim_{u \to t+0} z_l'(u) = \lim_{u \to t+0} z_r'(u).$$

Every one-sidedly smooth curve is rectifiable. The set of points $t \in (a, b)$ where $z_l'(t) \neq z_r'(t)$ is at most countable. If $\zeta(s)$, $s \in [0, L]$, is a parameterization of the curve by arclength, then the vector-function $\zeta(s)$ satisfies the same conditions K 1) and K 2). In this case the derivatives $\xi_l'(s)$ and $\xi_r'(s)$ are unit vectors on the plane.

For every one-sidedly smooth curve K on the plane, we can define a certain number, calling it the *turn of the curve K*.

We suppose that the plane has a definite orientation. For arbitrary nonzero vectors a and b we consider the angle $\angle(a, b) > 0$ if (a, b) is a positively oriented pair of vectors and consider $\angle(a, b) < 0$ if it is a negatively oriented pair.

If a and b are collinear vectors, then the angle between them is defined as follows. We set $\angle(a, b) = 0$ if a and b have the same direction and $\angle(a, b) = \pi$ if these vectors have opposite directions.

In all cases we have

$$-\pi < \angle(a, b) \leq \pi.$$

Let K be an arbitrary one-sidedly smooth curve on the plane $\mathbb{C}$ and let $z(t)$, $t \in [a, b]$, be its parameterization satisfying the above conditions K 1) and K 2). Then we can define a function $\varkappa(t)$, $t \in [a, b]$, so as to satisfy the following conditions:

T 1) $\varkappa(a) = 0$ and for all $t \in (a, b]$ the value $\varkappa(t)$ equals the sum of the angle between $z_r'(a)$ and $z_l'(t)$ and an integer multiple of 2π:

$$\varkappa(t) = \angle(z_r'(a), z_l'(t)) + 2N\pi,$$

where N is an integer.

T 2) The function $\varkappa$ is left continuous on the interval $(a, b]$ and has a finite limit from the right at every point $t < b$, and the jump at every point $t \in (a, b)$ is equal to

$$\varkappa(t + 0) - \varkappa(t) = \angle(z_l'(t), z_r'(t)).$$

The function $\varkappa$ is completely defined by these conditions.

The value $\varkappa(b)$ is independent of the choice of the parameterization $z(t)$, $t \in [a, b]$, of the curve K satisfying the above conditions K 1) and K 2). We set $\varkappa(b) = \varkappa(K)$. We call the value $\varkappa(K)$ the *rotation of the curve K*.

The curve K is said to be a curve with bounded variation of rotation, or shortly a curve *of finite turn*, if $\varkappa(t)$ is a function of bounded variation.

The total variation $\bigvee_a^b \varkappa(t)$ of $\varkappa(t)$ on $[a, b]$ is independent of the choice of the parameterization $z(t)$ of the curve K satisfying conditions K 1) and K 2).

We call this value the *turn of the curve* K and denote it by $|\varkappa|(K)$.

Every polygonal line, i.e., a curve composed of finitely many straight line segments is a curve of finite turn. The image of a curve of finite turn under a conformal mapping of a domain including the curve is again a curve of finite turn. (In fact, this is true for an arbitrary diffeomorphism of the class C^2.)

Let G be an arbitrary plane domain, i.e., let G be a connected open set on the plane $\mathbb{C}$. Suppose that some nonnegative function $\lambda(z)$ is defined in G and that λ is Borel measurable.

Let z_1 and z_2 be two arbitrary points belonging to G and let L be a curve of finite turn in G which joins the points z_1 and z_2.

Let $z(s)$, $0 \leq s \leq l$, be a parameterization of the curve L, where the parameter s is the arclength. (The length s is understood in the sense of the natural Euclidean geometry of the plane.) We set:

$$s_\lambda(L) = \int\limits_0^l \sqrt{\lambda[z(s)]}\, ds.$$

The greatest lower bound of $s_\lambda(L)$ on the set of all curves of finite turn joining the points z_1 and z_2 and lying in the domain G is further denoted by $\rho_\lambda(z_1, z_2)$. So we have:

$$\rho_\lambda(z_1, z_2) = \inf_{L \subset G} s_\lambda(L).$$

The value $\rho_\lambda(z_1, z_2) = \infty$ is not excluded. This happens when for every curve L joining z_1 and z_2 in G we have $s_\lambda(L) = \infty$. We say that the point $z_0 \in G$ is infinitely remote with respect to the line element $ds^2 = \lambda(z)|dz|^2$, shortly λ-infinite, if for all $z \in G$ we have $\rho_\lambda(z_0, z) = \infty$.

The function $\rho_\lambda(z_1, z_2)$ of two variables $z_1,\ z_2 \in G$ satisfies the axioms of a metric with the only exception that the values $\rho_\lambda(z_1, z_2) = \infty$ are admitted.

Now, suppose that the function $\lambda(z)$ in the domain G is given by formula (3), where ω is an arbitrary completely additive set function. In this case we call the metric ρ_λ subharmonic.

If ρ_λ is a subharmonic metric, then for every two different points z_1, z_2 of the domain G we have

$$\rho_\lambda(z_1, z_2) > 0.$$

In this case if the point z_0 is λ-infinite, then $\omega(z_0) \geq 2\pi$. If $\omega(z_0) > 2\pi$ then the converse is true, i.e., in this case z_0 is λ-infinite. If $\omega(z_0) = 2\pi$ then the point can be λ-infinite as well as not be λ-infinite. In particular, we see that in the case when the function $\lambda(z)$ admits the representation (3) the λ-infinite points form some discrete set in G.

The main results of the article under consideration (see [5]) are stated in the following three theorems:

Theorem 3. *Suppose that ρ_λ is a subharmonic metric in a plane domain G, where the function λ is defined by formula* (3)*. The domain G equipped with this metric is a two-dimensional manifold of bounded curvature in the sense of A.D. Alexandrov.*

Theorem 4. *Let U be an open domain in a two-dimensional manifold M of bounded curvature such that its closure is homeomorphic to a closed disk on the plane. Introduce in U the metric that is induced by the metric of the ambient manifold. Then U is isometric to a plane domain equipped with a certain subharmonic metric.*

Theorem 4 means that in a neighborhood of an arbitrary point of a two-dimensional manifold of bounded curvature we can introduce an isothermal coordinate system that defines the metric of the manifold in this neighborhood.

The proofs of Theorems 3 and 4 are based on the following lemma about convergence of metrics.

The Basic Lemma. *Let G be a bounded closed domain on the plane $\mathbb{C}$ such that the boundary of G consists of finitely many closed simple curves of finite turn.*

Let ω_n^1 and ω_n^2, $n = 1, 2, \dots$, be two sequences of nonnegative measures on the Borel subsets of $\mathbb{C}$.

Suppose that all these measures vanish outside G and as $n \to \infty$ the measures ω_n^1 converge weakly to some measure ω^1 and the measures ω_n^2 converge weakly to a measure ω^2. Set $\omega_n = \omega_n^1 - \omega_n^2$, $\omega = \omega^1 - \omega^2$, and let

$$\lambda_n(z) = \exp\left\{ \frac{1}{\pi} \int\limits_G \log \frac{1}{|z-\zeta|}\, d\omega_n(\zeta) \right\},$$

$$\lambda(z) = \exp\left\{ \frac{1}{\pi} \int\limits_G \log \frac{1}{|z-\zeta|}\, d\omega(\zeta) \right\}.$$

Let ρ_{λ_n} and ρ_λ be the subharmonic metrics on G defined by the functions λ_n and λ.

Suppose that for the points z_1, $z_2 \in G$ we have $\omega^1(\{z_1\}) < 2\pi$ and $\omega^2(\{z_1\}) < 2\pi$. Then for arbitrary sequences $z_{1,n}$ and $z_{2,n}$ of points of G such that $z_{1,n} \to z_1$ and $z_{2,n} \to z_2$ as $n \to \infty$ the following relation holds:

$$\rho_\lambda(z_1, z_2) = \lim_{n\to\infty} \rho_{\lambda_n}(z_{1,n}, z_{2,n}).$$

The proofs of Theorems 3 and 4 and the proof of the Basic Lemma were published in [7] and [8].

Each transition function for two different isothermal coordinate systems of a two-dimensional manifold of bounded curvature is conformal, i.e. on each component of the domain of definition it is either a holomorphic function or the conjugate of a holomorphic function as it follows from the following Theorem 5 (see [6] and [13]).

Theorem 5. *Let U and V be some domains on the plane $\mathbb{C}$. Suppose that ρ_λ is a subharmonic metric in U and ρ_μ is a subharmonic metric in V. Suppose that the metric spaces (U, ρ_λ) and (V, ρ_μ) are isometric and $\varphi : U \to V$ is a bijective isometry of these spaces. Then φ is a conformal mapping. In this case for every $z \in U$ we have*

$$\lambda(z)|dz|^2 = \mu[\varphi(z)]|d\varphi(z)|^2. \tag{4}$$

The converse is also true. Namely, suppose that a bijective mapping $\varphi : U \to V$ is conformal and the functions $\lambda(z)$ and $\mu(w)$ defined in U and V are connected by formula (4)*. Then φ is an isometry of the metric spaces (U, ρ_λ) and (V, ρ_μ).*

The proof of the fact that the mapping φ is conformal was sketched in my paper [6]. (This fact was not indicated in my paper [25].) A complete exposition of the proof mentioned in [6] is given in [13]. The formula was established by A. Huber in [17]. The proof of Theorem 5 was given in [17]. My proof of conformality of the mapping φ was based on the well-known criteria of conformality due to D.E. Men'shov. At the end of the paper [17] there is a remark that the conformality of the mapping φ can be proved in this way (with the help of Men'shov's theorem).

Theorems 3 and 4 have a local character. A global version of these theorems was formulated by A. Huber. This global version is based on the above-stated Theorem 5.

By a Riemann surface I mean here a pair $(M, \mathcal{A})$ where M is a two-dimensional manifold and $\mathcal{A}$ is an atlas on M such that for any two overlapping charts $\varphi : U \to \mathbb{C}$ and $\psi : V \to \mathbb{C}$ belonging to $\mathcal{A}$ the transition functions $\theta = \psi \circ \varphi^{-1}$ and $\tau = \varphi \circ \psi^{-1}$ are conformal. For every $z \in \varphi(U \cap V)$ the quantity $|\theta'(z)|$ is well defined. We call the charts in the atlas $\mathcal{A}$ basic.

Let M be a two-dimensional Riemann surface. Then we say that some quadratic differential σ is given on M if for every basic chart $\varphi : U \to \mathbb{C}$ there is a differential quadratic form $ds^2 = \lambda(z)|dz|^2$ on the set $G = \varphi(U)$ (this form is called the representation of the quadratic differential σ in the local coordinate system φ), and the differential forms $\lambda(z)|dz|^2$ and $\mu(w)|dw|^2$ corresponding to two arbitrary overlapping basic charts $\varphi : U \to \mathbb{C}$ and $\psi : V \to \mathbb{C}$ are connected by the following relation: Let $\theta(z) = \psi[\varphi^{-1}(z)]$ be the transition function for the charts φ and ψ. Then $\lambda(z) = \mu[\theta(z)]|\theta'(z)|^2$.

Theorem 6. *Let M be a two-dimensional manifold of bounded curvature. Then the isothermal coordinate systems of the manifold M define the structure of a Riemann surface on M. The differential quadratic forms $\lambda(z)|dz|^2$ locally defining the metric of the manifold are representations of a quadratic differential σ on M.*

The converse is true. If on a Riemann surface $(M, \mathcal{A})$ a quadratic differential σ is given such that for every basic chart the representation of σ has the form $\lambda(z)|dz|^2$ where the function $\lambda(z)$ admits a representation of the type (1)*, then on this manifold there is an intrinsic metric which is locally defined by the representations of the quadratic differential σ and the manifold M endowed with this metric is a two-dimensional manifold of bounded curvature.*

This theorem can be viewed as a global variant of the theorem about the conformal representation of two-dimensional manifolds of bounded curvature. The fact that the isothermal coordinate systems form an atlas with conformal transition functions was proved independently by A. Huber and Reshetnyak. The formula describing the connection between the quadratic forms $\lambda(z)|dz|^2$ corresponding to overlapping isothermal coordinate systems was established by A. Huber.

5. To illustrate the general theorems formulated above, I consider the case of polyhedra.

Let M be a two-dimensional manifold with an intrinsic metric. We say that M is a manifold of type Σ if M is complete and homeomorphic to a sphere with finitely many points deleted. The deleted points of the sphere correspond to the infinitely remote points of M.

Suppose that M is a manifold of type Σ with a polyhedral metric and the set of its vertices is finite. Then in M there is an isothermal coordinate system which is defined on the whole manifold M except possibly one point. The range of any such coordinate system is the whole plane $\mathbb{C}$ minus a finite set of points, the function $\lambda(z)$ of the line element $\lambda(z)|dz|^2$ has the representation

$$\log\lambda(z) = \sum_{j=1}^{m} \frac{\omega_j}{\pi} \log\frac{1}{|z - z_j|} + C,$$

where C is a constant, and the following inequality is fulfilled:

$$\omega_1 + \omega_2 + \cdots + \omega_m \leq 4\pi.$$

If for some j we have $\omega_j \geq 2\pi$, then z_j in this coordinate system corresponds to an infinitely remote point of M. Otherwise z_j corresponds to a vertex of M and ω_j is equal to the curvature of M at this point.

Consider the particular case in which $m = 1$ and $z_1 = 0$. Then $\lambda(z) = C\left(\frac{1}{|z|}\right)^{\omega/\pi}$. In the case when $\omega \leq 2\pi$, the plane equipped with the line element $\lambda(z)|dz|^2$ is isometric to a cone. If $\omega = 2\pi$ the plane with the line element $\lambda(z)|dz|^2 = \dfrac{C|dz|^2}{|z|^2}$ is isometric to the lateral surface of an infinite cylinder.

6. The basic concepts of the theory of manifolds of bounded curvature can be expressed in terms of isothermal coordinate systems.

The set function ω in equality (3) is the integral curvature of the manifold in the sense of definitions given by Alexandrov.

The difficulties in the investigation of shortest arcs on an arbitrary manifold of bounded curvature are connected with the fact that in the general case we can face situations which are impossible in the case of Riemannian geometry. For instance, unlike in the classical case of Riemannian geometry, in the case of an arbitrary two-dimensional manifold of bounded curvature we cannot assert that a shortest arc joining any two sufficiently close points is unique. Two shortest lines starting from one point can agree up to a certain place and diverge after that place. In the regular case this is impossible.

The functional whose extremals are shortest arcs is as follows:

$$\int_a^b \sqrt{\lambda[z(t)]\{[x'(t)]^2 + [y'(t)]^2\}}dt.$$

The function $\lambda(z)$ can be equal to 0 or ∞ at some points of its domain. We can only assert that $\log\{\lambda(z)\}$ has weak first derivatives (generalized derivatives in the sense of S.L. Sobolev) integrable to the power p for all p in the interval $[1, 2)$.

So we see that the study of shortest arcs in a manifold of bounded curvature leads to a variational problem which fails to satisfy the traditional regularity requirements of variational calculus.

Suppose that $z(t)$, $a \leq t \leq b$, is an arbitrary parameterized curve on the complex plane $\mathbb{C}$. It means that z is a continuous mapping of the interval $[a, b]$ into $\mathbb{C}$. Suppose that $z_0 \neq z(t)$ for all $t \in [a, b]$. Then on the interval we can define a continuous function $\varphi_K(t)$ such that $\varphi_K(a) = 0$ and

$$\varphi_K(t) = \arg\left(\frac{z(t) - z_0}{z(a) - z_0}\right) + 2\pi m,$$

where m is an integer. We put $\varphi(K, z_0) = \varphi_K(b)$. The following inequality is true

$$\bigvee_a^b \varphi_K(t) \leq |\varkappa|(K) + \pi.$$

This inequality was first proved by Radon in his investigations of potential theory on the plane. A refined version of this inequality for curves in $\mathbb{R}^n$ was established by Khovansky.

Assume that in a flat domain G we have a Riemannian metric with the line element $ds^2 = \lambda(z)|dz|^2$. Suppose that $\lambda(z) > 0$ for all $z \in G$ and that the function $\lambda(x, y)$ has all partial derivatives of the first and second order and these derivatives are continuous. As it was said above, the function $\lambda(x, y) \equiv \lambda(z)$ then has the representation

$$\log \lambda(z) = \frac{1}{\pi}\int\int_G \log \frac{1}{|z - \zeta|} d\omega(\zeta) + H(z), \tag{5}$$

where the set function $\omega(E)$ is the integral curvature and $H(z)$ is a harmonic function.

Let K be an arbitrary curve in the domain G and let $z(t)$, $0 \leq t \leq l$, be a parameterization of this curve. Suppose that the function $z(t)$ has a continuous second order derivative and $z'(t) \neq 0$ for all $t \in [a, b]$. Then the geodesic curvature of the curve K is defined at every point of the curve K. From the well-known formulas of differential geometry it follows that the geodesic curvature $\kappa_g(t)$ of the curve K at the point $z(t)$ is given by the formula

$$\kappa_g(t) = \frac{\kappa(t)}{\sqrt{\lambda[z(t)]}} + \frac{1}{|z'(t)|\sqrt{\lambda[z(t)]}}[-(\log\lambda)_y x'(t) + (\log\lambda)_x y'(t)]. \tag{6}$$

Here $\kappa(t)$ is the usual (Euclidean) curvature of the curve K at the point $z(t)$. We multiply both sides of equality (6) by $\sqrt{\lambda[z(t)]}|z'(t)|dt$ and then integrate the result over the interval $[a,b]$. We get a representation of the integral geodesic curvature or, which is the same, of the geodesic turn of the curve K:

$$\varkappa_g(K) = \varkappa(K) - \frac{1}{2}\int\limits_a^b \frac{\partial \log\lambda(z)}{\partial\nu}|z'(t)|dt.$$

Here $\nu = \nu(t)$ denotes the unit normal vector of the curve K at the point $z(t) = x(t) + iy(t)$. We have $\nu(s) = -\dfrac{y'(t)}{|z'(t)|} + i\dfrac{x'(s)}{|z'(t)|}$. Using formulas (5) and (6), we get the following expression for the geodesic turn of a regular curve in an isothermal coordinate system:

$$\varkappa_g(K) = \tag{7}$$

$$\varkappa(K) - \frac{1}{2\pi}\int\limits_G \left\{\int\limits_a^b \frac{\partial}{\partial\nu}\left(\log\frac{1}{|z(t)-\zeta|}\right)dt\right\} d\omega(\zeta) - -\frac{1}{2}\int\limits_a^b \frac{\partial h}{\partial\nu}[z(t)]|z'(t)|dt.$$

The last integral here is the integral along K of the differential form $\eta(z) = -\frac{\partial h}{\partial y}(z)dx + \frac{\partial h}{\partial x}(z)dy$. Since $h(z)$ is a harmonic function, this form is closed. This allows us to conclude that the value of the integral

$$\int\limits_a^b \frac{\partial h}{\partial\nu}[z(s)]ds$$

depends only on the points $z(a)$ and $z(b)$ and does not change if we replace K by any other curve with the same endpoints which is homotopic to K.

Furthermore, we have the equality

$$\int\limits_a^b \frac{\partial}{\partial\nu}\left(\log\frac{1}{|z(t)-\zeta|}\right)|z'(t)|dt = \varphi(K,\zeta).$$

It follows from the fact that the function $\arg z$ is conjugate to the harmonic function $\log|z|$ on the Riemann surface of the function $\log z$. Finally we get the following representation for the geodesic turn of a curve in the isothermal coordinate system with the line element $ds^2 = \lambda(z)|dz|^2$, where $\lambda(z)$ admits the representation (1). We get

$$\varkappa_g(K) = \varkappa(K) - \frac{1}{2\pi}\int\limits_G \varphi(K,\zeta)d\omega(\zeta).$$

Now, we go on to the general case of two-dimensional manifolds of bounded curvature. Suppose that K is a one-sidedly smooth simple arc in a plane domain G. The function $\varphi(K,z)$ of the variable z is continuous on the set $G \setminus K$. Let p and q be the endpoints of the curve K. The curve K has tangents at this points. We define the value $\varphi(K,p)$ as follows: Let w be an arbitrary interior point of the curve K. It divides the curve K into two segments: $K'_w = [w,q]$ and $K''_w = [p,w]$. The fact that the curve K has a tangent at each of these points implies existence of the final limits

$$\lim_{w\to p}\varphi(K'_w,p), \qquad \lim_{w\to q}\varphi(K''_w,q).$$

We set $\varphi(K,p)$ to be equal to the first of these two limits. The second limit equals $\varphi(K,q)$. The limit $\lim_{z\to z_0}\varphi(K,z)$ does not necessarily exists for every interior point z_0 of the curve K. But in this case the value $\varphi(K,z)$ converges to a definite limit as z tends to z_0 being on one side of the curve K. The limit of $\varphi(K,z)$ as z tends to the point z_0 from the left side of the curve K is denoted by $\varphi_l(K,z_0)$ and the limit of $\varphi(K,z)$ as z tends to z_0 from the right is denoted by $\varphi_r(K,z_0)$. It is not difficult to see that $\varphi_l(K,z_0) - \varphi_r(K,z_0) = 2\pi$.

Suppose that $ds^2 = \lambda(z)|dz|^2$ is a differential quadratic form in a plane domain G, where $\lambda(z)$ admits the representation of the type (5) with a countably additive set function ω satisfying the following condition. The metric defined by the line element ds^2 has no infinite points, i.e., $\rho_\lambda(z_1,z_2)$ is finite for every pair (z_1,z_2) of points of the domain G. Suppose that K is a one-sidedly smooth simple arc in the domain G. Then we put

$$\varkappa_r(K) = \varkappa(K) - \frac{1}{2\pi}\int\limits_G \varphi_r(K,\zeta)\,d\omega(\zeta) - \frac{1}{2}\int\limits_K \frac{\partial h}{\partial\nu}ds,$$

$$\varkappa_l(K) = -\varkappa(K) + \frac{1}{2\pi}\int\limits_G \varphi_l(K,\zeta)\,d\omega(\zeta) + \frac{1}{2}\int\limits_K \frac{\partial h}{\partial\nu}ds.$$

We say that $\varkappa_r(K)$ is the geodesic turn from the right of the curve K in the metric ρ_λ and $\varkappa_l(K)$ is the geodesic turn of this curve from the left. (In the notations we of course should place λ somewhere, but for the sake of simplicity I omit it.) It is easy to see that for every one-sidedly smooth curve K we have the equality

$$\varkappa_l(K) + \varkappa_r(K) = \omega(K^\circ).$$

Here K° denotes the arc K without its endpoints.

Theorem 7. *Let G be a plane domain and let ρ_λ be a subharmonic metric in G. Suppose that a curve $K \subset G$ is a shortest arc in this metric and for its endpoints p and q we have $\omega(\{p\}) < 2\pi$ and $\omega(\{q\}) < 2\pi$. Then K is a plane curve of finite turn and for each arc L of this curve we have the inequalities*

$$\varkappa_l(L) \le 0, \qquad \varkappa_r(L) \le 0. \tag{8}$$

Inequalities (8) are certain analogs of the Euler equation for geodesics, and in the regular case these inequalities are equivalent to the Euler equation.

The degree of regularity of a shortest arc which is established in this theorem is the best possible that can be achieved in the general case. It should be said that if for a point X of a shortest arc K we have $\omega(X) = 2\pi$ then X is an endpoint of K.

One can construct examples of points in a two-dimensional manifold of bounded curvature with $\omega(X) = 2\pi$ with the following property: There exist two shortest arcs starting at a point X such that for every topological mapping of a neighborhood of X in the plane the image of at least of one of these shortest arcs will have no tangent at the starting point. It means that the image of one of these shortest arcs is not a curve of finite turn.

The values $\varkappa_l(K)$ and $\varkappa_r(K)$ can be defined in a purely geometric way without using isothermal coordinates. I will not speak of this topic.

The theory discussed here admits some applications to the theory of complex functions. I was able to find only some trivial applications. Deeper results were established not long ago by Bonk and Eremenko [18].

References

1. Alexandrov, A. D., Foundations of the inner geometry of surfaces. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **60** (1948), 1483–1486.
2. Alexandrov, A. D., Curves on manifolds of bounded curvature. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **63** (1948), 349–352.
3. Alexandrov, A. D., Quasigeodesics. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **69** (1949), 717–720.
4. Alexandrov, A. D. and V. A. Zalgaller, Two-dimensional manifolds of bounded curvature, Moscow "Nauka", 1962.
5. Reshetnyak, Yu. G., Isothermal coordinates in manifolds of bounded curvature. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 631–633.
6. Reshetnyak, Yu. G., Investigation of manifolds of bounded curvature with the help of isothermal coordinates, Izvestiya of Siberian Branch of Academy of Sciences USSR no. **10** (1959), 15–28.
7. Reshetnyak, Yu. G., Isothermal coordinates on manifolds of bounded curvature I, (Russian) Sibirsk. Mat. Zh. **1** (1960), no. 1, 88–116.
8. Reshetnyak, Yu. G., Isothermal coordinates on manifolds of bounded curvature II, (Russian) Sibirsk. Mat. Zh. **1** (1960), no. 2, 248–276.
9. Reshetnyak, Yu. G., An isoperimetric property of two-dimensional manifolds with curvature not greater k. (Russian) Vestnik Leningrad. Univ. **16** (1961), no. 19, 58–76.
10. Reshetnyak, Yu. G., A special mapping of a cone onto a polyhedron. (Russian) Mat. Sb. (N.S.) **53** (1961), 39–52.
11. Reshetnyak, Yu. G., A special mapping of a cone into a manifold of bounded curvature. (Russian) Sibirsk. Mat. Zh. **3** (1962), no. 2, 256–272.
12. Reshetnyak, Yu. G., Arc length in a manifold of bounded curvature with an isothermal linear element. (Russian) Sibirsk. Mat. Zh. **4** (1963), no. 1, 212–226.

13. Reshetnyak, Yu. G., A rotation of a curve in a manifold of bounded curvature with an isometric linear element. (Russian) Sibirsk. Mat. Zh. **4** (1963), no. 4, 870–911.
14. Reshetnyak, Yu. G., Segments on Ljapunov surfaces with a metric of bounded curvature. (Russian) Sibirsk. Mat. Zh. **5** (1964), no. 2, 477–479.
15. Reshetnyak, Yu. G., Non-expansive maps in a space of curvature no greater than K. (Russian) Sibirsk. Mat. Zh. **9** (1968), no. 4, 918–927.
16. Beurling, A., Sur la géométrie métrique des surfaces à courbure totale < 0. Comm. Sém. Math. Univ. Lund (Medd. Lunds Univ. Mat. Sem.) (1952), Tome Supplementaire, 7–11.
17. Huber, A., Zum potentialtheoretischen Aspekt der Alexandrovschen Flächentheorie, Comment. Math. Helv. **34** (1960), no. 2, 99–126.
18. Bonk, M. and A. Eremenko, Covering properties of meromorphic functions, negative curvature and spherical geometry.
19. Müller, S. and V. Šverak, On surfaces of finite total curvature. J. Differential Geom. **42** (1995), no. 2, 229–258.
20. Ossermann, R., Isoperimetric and related inequalities. Proc. AMS Symp. in Pure and Applied Math. **XXVII**, Part 1 (1975), pp. 207–215.
21. Alexandrov, A. D. and V. V. Strel'tsov, The isoperimetric problem and estimates of the length of a curve on a surface. (Russian) Trudy Mat. Inst. Steklov **76** (1965), 67–80.
22. Bakel'man, I. Ya., Chebyshev's nets in manifolds of bounded curvature, (Russian) Trudy Mat. Inst. Steklov **76** (1965), 124–129.
23. Borisov, Yu. F, Curves on complete two-dimensional manifolds with boundaries. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **64** (1949), 9–12.
24. Borisov, Yu. F, Manifolds of bounded curvature with a boundary. (Russian) Dokl. Akad Nauk SSSR (N.S.) **74** (1950), 877–880.
25. Reshetnyak, Yu. G., Two-dimensional manifolds of bounded curvature, in Geometry IV: Non-Regular Riemannian Geometry (Encyclopaedia of Mathematical Sciences vol. 70), Springer-Verlag, 1993, pp. 3–163.

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