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ON DENSITY PROPERTIES OF CAPACITIES ASSOCIATED TO GENERAL KERNELS

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In [MP] we investigated the density properties of the Riesz capacities C_s in $\mathbf{R}^N$ and the analytic capacity γ_+ in $\mathbf{R}^2$. The Riesz capacity C_s, $0 < s < N$, is associated to the kernel K_s, $K_s(x) = |x|^{-s}$. In this paper we show that similar results are true for capacities associated to very general kernels K. We assume that $K : [0, \infty) \to (0, \infty]$ is strictly decreasing, continuous for $r > 0$, $K(r) \to \infty$ as $r \to 0$, and

$$\int_0^1 K(t)\, t^{N-1}\, dt < \infty.$$

The capacity C_K is defined for $E \subset \mathbf{R}^N$ as

$$C_K(E) = \sup\{\mu(E) \mid \mu \in \mathcal{A}_K(E)\},$$

where $\mathcal{A}_K(E)$ consists of finite, positive Borel measures μ with compact support, $\operatorname{supp}(\mu)$, contained in E, and such that

$$\int K(|x-y|)\, d\mu(y) \le 1 \quad \text{for } x \in \mathbf{R}^N.$$

We are interested in the following question: if $C_K(E) > 0$, how quickly can $C_K\big(B(a,\delta) \cap E\big)$ tend to zero as $\delta \to 0$ for typical points $a \in E$? Here $B(a,\delta)$ is the open ball centered at a and with radius δ. A trivial upper bound is given by $1/K(\delta)$. For nice kernels K (for example, Riesz and logarithmic kernels) and open sets E, this is sharp: $C_K\big(B(a,\delta)\big) \asymp 1/K(\delta)$ for $a \in E$. But it is not sharp for general, say, compact sets E. As in [MP] for the Riesz kernels, we show here with mild regularity conditions on the kernel K and a non-decreasing, continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ that

$$\int_0^1 h(t)\, dK(t) = -\infty \tag{1}$$

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is a necessary and sufficient condition in order that there exists a compact set E such that $C_K(E) > 0$ and

$$C_K\big(B(a,\delta)\cap E\big) \asymp h(\delta) \quad \text{for } a \in E, \text{ as } \delta \to 0.$$

Note that the condition (1) appears also in the well-known results comparing capacities and Hausdorff measures, see [C] and [E]: $-\int_0^1 h(t)\,dK(t) < \infty$ is essentially the optimal condition guaranteeing that positive Hausdorff h-measure implies positive K-capacity.

Some of the arguments from [MP] generalize in a straightforward manner and we omit them. But for some others more care is needed, in particular, if $K(t)$ grows rather slowly when $t \to 0$. This is so in the case of the classical logarithmic capacity C_0; then $K(t) = -\log t$ for $0 < t < t_0$, $t_0 \in (0,1)$. In the following theorem the assumptions on K and h are rather natural. We have not tried to find the optimal conditions.

Our main result is

Theorem. *Let K and h be as above.*

(a) Let X be a compact set in $\mathbf{R}^N$ with $C_K(X) > 0$. If

$$-\int_0^1 h(t)\,dK(t) < \infty, \tag{2}$$

then for C_K-almost all $a \in X$ one has

$$\limsup_{\delta\to 0} \frac{C_K\big(B(a,\delta)\cap X\big)}{h(\delta)} = \infty. \tag{3}$$

(b) Suppose that there exist positive constants A, B and C such that $C < 2^N$ and for $t > 0$,

$$Ah(t) \le 1/K(t), \quad K(t) \le BK(2t) \quad \text{and} \quad h(2t) \le Ch(t). \tag{4}$$

Assume also that K is absolutely continuous and that

$$\int_0^1 h(t)\,K'(t)\,dt = -\infty. \tag{5}$$

Then there exists a Cantor set X_1 such that for some $0 < A_1 < A_2 < \infty$,

$$A_1 h(\delta) \le C_K\big(B(a,\delta)\cap X_1\big) \le A_2 h(\delta)$$

for all $a \in X_1$ and $\delta \in (0,1)$.

Proof. The proof of (a) is quite similar to that in [MP] and we leave out some details. It is enough to find at least one point a for which (3) holds. Let $\mu \in \mathcal{A}_K(X)$ with $\mu(X) > 0$. If for some $a \in X$

$$\limsup_{\delta\to 0} \frac{\mu\big(B(a,\delta)\big)}{h(\delta)} = \infty,$$

a is the required point since $\mu|_{\overline{B}(a,\delta)} \in \mathcal{A}_K\big(\overline{B}(a,\delta)\cap X\big)$ for all $\delta > 0$, whence $C_K\big(\overline{B}(a,\delta)\cap X\big) \geq \mu\big(\overline{B}(a,\delta)\big)$. If $\limsup_{\delta\to 0}\mu\big(B(a,\delta)\big)/h(\delta) < \infty$ for all $a \in X$, then the Hausdorff content $M_h(X) > 0$. By standard density results for the Hausdorff content, see [F, 2.10.17(3)], there is $a \in X$ such that

$$\limsup_{\delta\to 0}\frac{M_h\big(B(a,\delta)\cap X\big)}{h(\delta)} \geq 1. \tag{6}$$

We could now finish the proof of (a) as in [MP]. But we can also use a result of Eiderman in [E] as follows. First, (2) implies that $\lim_{\delta\to 0} h(\delta)\,K(\delta) = 0$. (We would like to thank Vladimir Eiderman for this observation. We leave the simple proof to the reader.) Hence also $\int_0^1 K(t)\,dh(t) < \infty$. Define $r_\delta > 0$ for small $\delta > 0$ by

$$h(r_\delta) = C_K\big(B(a,\delta)\cap X\big)\int_0^{r_\delta} K(t)\,dh(t).$$

Then $r_\delta \to 0$ as $\delta \to 0$. By Proposition 3.1 of [E]

$$M_h\big(B(a,\delta)\cap X\big) \leq A(N)\int_0^{r_\delta} K(t)\,dh(t)\,C_K\big(B(a,\delta)\cap X\big),$$

which by (6) yields (3).

We now prove (b). Choose $\alpha > 0$ such that

$$CB^\alpha < 2^N, \tag{7}$$

and define k and g for $r > 0$ by

$$k(r) = \alpha A h(r) \quad\text{and}\quad g(r) = k(r)\exp\Big(\int_r^1 k(t)\,K'(t)\,dt\Big). \tag{8}$$

Then $g(r)/k(r) \to 0$ as $r \to 0$,

$$\Big(\frac{g(t)}{k(t)}\Big)' = -K'(t)\,g(t) \quad\text{for almost all } t,$$

and so

$$\frac{g(r)}{k(r)} = -\int_0^r g(t)\,K'(t)\,dt. \tag{9}$$

We also note that g is strictly increasing and set for $j = 1, 2, \dots,$

$$l_j = g^{-1}(2^{-Nj}),\ \text{that is,}\quad g(l_j) = 2^{-Nj}. \tag{10}$$

(Of course we may assume that $g(t) > 2^{-N}$ for some $t > 0$, so that such l_j's exist.) We now check that

$$2l_{j+1} < l_j \quad\text{for } j = 1, 2, \dots. \tag{11}$$

This is equivalent to

$$g\big(2g^{-1}(2^{-N(j+1)})\big) < 2^{-Nj}.$$

For this it is sufficient that g satisfies the doubling condition

$$g(2r) \le Dg(r) \quad \text{for } r > 0$$

with some constant $D < 2^N$. By (8) this means that

$$\text{(12)} \qquad \frac{h(2r)}{h(r)} = \frac{k(2r)}{k(r)} \le D \exp\Big(\int_r^{2r} k(t)\, K'(t)\, dt\Big).$$

But using (4),

$$\begin{aligned}\int_r^{2r} k(t)\, K'(t)\, dt = \alpha A \int_r^{2r} h(t)\, K'(t)\, dt &\ge \alpha \int_r^{2r} K(t)^{-1}\, K'(t)\, dt \\ &= \alpha \int_r^{2r} \frac{d}{dt}\big(\log K(t)\big)\, dt = \alpha \log \frac{K(2r)}{K(r)} \ge \alpha \log(1/B) \\ &= \log B^{-\alpha}.\end{aligned}$$

Thus (12) follows by (4) from

$$D \exp\Big(\int_r^{2r} k(t)\, K'(t)\, dt\Big) \ge D/B^{\alpha} = C \ge h(2r)/h(r)$$

if we choose $D = CB^{\alpha} < 2^N$, using (7). Now (11) allows us to construct the standard N-dimensional Cantor set

$$X_1 = \bigcap_{n=0}^{\infty} \bigcup_{m=1}^{2^{Nn}} Q_n^m,$$

where each Q_n^m is a closed cube of side-length l_n. For each n,

$$X_1 = \bigcup_{m=1}^{2^{Nn}} X_n^m,$$

where $X_n^m = X_1 \cap Q_n^m$, $m = 1, \dots, 2^{Nn}$, are congruent Cantor sets with parameters $\{l_j\}_{j=n}^{\infty}$. By [E, Corollary 1.1] and (10), one has

$$\begin{aligned}\text{(13)} \qquad C_K(X_n^m) \asymp \Big(\sum_{j=0}^{\infty} 2^{-Nj} K(l_{j+n})\Big)^{-1} &= \Big(\sum_{j=n}^{\infty} 2^{Nn}\, 2^{-Nj}\, K(l_j)\Big)^{-1} \\ &= g(l_n)\Big(\sum_{j=n}^{\infty} 2^{-Nj}\, K(l_j)\Big)^{-1}.\end{aligned}$$

Since $\int_{l_{j+1}}^{l_j} dg(t) = 2^{-Nj}(1-2^{-N})$ and since $g(t)\,K(t) \to 0$ as $t \to 0$ (which follows from (4), (5) and (8)), we get

$$\begin{aligned}\sum_{j=n}^{\infty} 2^{-Nj}\,K(l_j) &= (1-2^{-N})^{-1}\sum_{j=n}^{\infty} K(l_j)\int_{l_{j+1}}^{l_j} dg(t)\\ &\lesssim \int_0^{l_n} K(t)\,dg(t)\\ &= K(l_n)\,g(l_n) - \int_0^{l_n} g(t)\,K'(t)\,dt.\end{aligned}$$

Hence by (13), (9), (8) and (4),

$$\begin{aligned}C_K(X_n^m) &\gtrsim g(l_n)\Bigl(K(l_n)\,g(l_n) - \int_0^{l_n} g(t)\,K'(t)\,dt\Bigr)^{-1}\\ &= k(l_n)/\bigl(K(l_n)\,k(l_n)+1\bigr) \asymp h(l_n).\end{aligned}$$

Similar estimates yield

$$C_K(X_n^m) \lesssim h(l_{n-1}).$$

But by (8) and (10),

$$\frac{h(l_n)}{h(l_{n-1})} = \frac{g(l_n)\exp\Bigl(\int_{l_{n-1}}^1 k(t)\,K'(t)\,dt\Bigr)}{g(l_{n-1})\exp\Bigl(\int_{l_n}^1 k(t)\,K'(t)\,dt\Bigr)} \geq 2^{-N}, \tag{14}$$

whence

$$C_K(X_n^m) \asymp h(l_n). \tag{15}$$

Finally (14) and (15) together imply easily that

$$C_K\bigl(B(a,\delta)\cap X_1\bigr) \asymp h(\delta) \quad \text{for } a \in X_1,\ 0<\delta<1.$$

This completes the proof of the theorem.

Remarks. The doubling condition on h was only needed to prove (11). On the other hand, (11) together with the doubling condition for $1/K$ imply the doubling condition for h as in [MP]. In [MP] we also assumed that for some $\varepsilon > 0$ $\;l_{j+1} \geq \varepsilon l_j$ for all j, but this is not needed as the above proof shows.

In [M] Martio studied densities of variational p-capacities and compared them with the Hausdorff h-measure densities.

Note added after the completion of this paper: Recently Tolsa has proved in [T] that γ and γ_+ are comparable. Hence our results on γ_+ are also valid for γ.

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