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THE ROLE OF THE DOMAIN TOPOLOGY ON THE NUMBER OF POSITIVE SOLUTIONS TO ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we are concerned with the asymptotically linear elliptic problem

$$-\Delta u = \lambda f(u) \text{ in } H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a smooth bounded domain and $\lim\limits_{t\to+\infty} \frac{f(t)}{t} = l$, $0 < l < +\infty$. We prove under natural assumptions that the problem possesses at least $\mathrm{cat}\Omega+1$ distinct positive solutions as long as $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$ is large enough, where $\{\lambda_k\}$ is the sequence of eigenvalues of $-\Delta$ with zero Dirichlet data on $\partial\Omega$ and $\mathrm{cat}\Omega$ is the Ljusternik–Schnirelmann category of $\bar{\Omega}$ relative to $\bar{\Omega}$ itself.

1. Introduction and main result

In this paper, we study the following elliptic problem with asymptotically linear term at infinity

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 3$, $\lambda > 0$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies

(f_1) $\lim\limits_{t\to+\infty} \dfrac{f(t)}{t} = l, \quad 0 < l < +\infty, \quad \lim\limits_{t\to 0+} \dfrac{f(t)}{t^{\frac{N+2}{N-2}}} = 0;$

(f_2) $\dfrac{f(t)}{t}$ is strictly increasing in $t > 0$;

(f_3) $f(t) \geqslant 0$ for $t > 0$.

We are interested in the role of the domain topology on the number of solutions to (1.1).

For a wide class of semilinear elliptic problems, the relation between the topology of the domain and the existence and multiplicity of positive solutions has been studied systematically by many authors, see e.g. [1], [2], [3], [5], [6], [8], [10]. In [1], using the homology group of the domain, A. Bahri and J.M. Coron studied the effect of the

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topology of the domain on the existence of a positive solution to semilinear elliptic equations involving Sobolev critical exponent

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded regular and connected open set in $\mathbb{R}^N$ with $N \geqslant 3$. In [5], among other things, Brezis and Nirenberg showed the existence of the two positive solutions to the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an annulus and $\lambda > 0$ is near 0. In [2], Benci and Cerami obtained more precise conclusion for the semilinear problem

$$\begin{cases} -\Delta u + \lambda u = u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a smooth bounded domain, $2 < p < 2^* = \frac{2N}{N-2}$. They concluded that the topology of Ω is relevant when the nonlinearity acts strongly in some sense on the problem, i.e. for λ large enough or p close enough to 2^*, (1.4) has at least $\text{cat}\Omega$ positive solutions, where $\text{cat}\Omega$ denotes the Ljusternik–Schnirelmann category of $\bar{\Omega}$ in itself (see the following definition).

Definition 1.1. (see [9], [19]) *Let M be a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\text{cat}_M A = k$), if A is covered by k closed sets $A_j, 1 \leqslant j \leqslant k$, which are contractible in M, and if k is minimal with this property. If no such finite covering exists, we let $\text{cat}_M A = +\infty$.*

Later, Benci and Cerami improved the result in [2] to show the existence of at least one more solution in [3], [8]. In [6], Candela extends the result in [2] to more general problem

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $at^{p-1} \leqslant f(t) \leqslant bt^{p-1}$, $t \geqslant 0$, $a > 0$, $b > 0$, $1 < \frac{b}{a} < 1 + \varepsilon$ for ε suitably small. In [7], Cao–Li–Zhong studied the problem

$$\begin{cases} -\Delta u + \lambda u = u^{p-1} + f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a smooth bounded domain, $2 < p < 2^* = \frac{2N}{N-2}$, $f \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies

$$(f_1)' \ \lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = 0;$$

$$(f_2)' \ \exists \gamma > 0, \text{ such that } \frac{d}{dt}\left(\frac{f(t)}{t^{1+\gamma}}\right) \geqslant 0 \text{ for } t > 0;$$

$$(f_3)' \ f(t) > 0 \text{ for } t > 0.$$

They observed that the effect of lower order term $f(t)$ on the number of solutions to (1.6) becomes weaker as λ increases and obtained that (1.6) possesses at least $\mathrm{cat}\Omega$ distinct solutions for $\lambda > 0$ large enough.

However, none of the above mentioned papers studied (1.1) under the assumptions $(f_1) - (f_3)$. The case we are interested in this paper is the so-called "asymptotically linear" case. Our main result is the following

Theorem 1.1. *Suppose f satisfies the assumptions $(f_1) - (f_3)$ and let $\{\lambda_k\}$ be the sequence of eigenvalues of $-\Delta$ with zero Dirichlet data on $\partial\Omega$. Then there exists $\bar{\lambda} > 0$, such that for any $\lambda \geqslant \bar{\lambda}$ and $\lambda \notin \{\frac{\lambda_k}{l}\}$,* (1.1) *has at least (catΩ) distinct solutions. In addition, if Ω is not contractible, then there exists $\tilde{\lambda} > 0$ such that* (1.1) *has at least (catΩ+1) distinct solutions for any $\lambda \geqslant \tilde{\lambda}$ and $\lambda \notin \{\frac{\lambda_k}{l}\}$.*

The main idea to prove our main result is motivated by [2], [7], and [8]. Solutions to (1.1) corresponds to critical points of energy functional $I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla|^2\, dx - \lambda \int_\Omega F(u)\, dx$ related to (1.1). By Ljusternik–Schnirelmann theory, we know that if a smooth functional I on a Finsler manifold M satisfies the (P-S) condition and bounded from blow, then I has at least $\mathrm{cat}M$ distinct critical points. However, as the problem is asymptotically linear at infinity, the usual Ambrosetti–Rabinowitz condition

$$F(t) = \int_0^t f(s)\, ds \leqslant \theta t f(t) \text{ for } t \geqslant 0,$$

where $\theta = \frac{1}{2+\gamma}$, $\gamma > 0$, which is important for verifying the (P-S) conditions, does not hold. Thus we need different approach from [2], [3], [6], [7], [8] in verifying the (P-S) conditions of the energy functional. Here we adapt the approach in [13], [16], [17]. Also due to the asymptotic linearity at infinity, it is not convenient to use the method in [8] to constrain the functional on $\varphi_\Omega = \{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} = 1\} \setminus \{v \in H_0^1(\Omega) : v \leqslant 0 \text{ a.e.}\}$. It is not convenient to consider the constrained minimization problem

$$\inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega F(u) dx = 1\right\}$$

either, because $F(u)$ is not homogeneous. We use the method in [7] to study the critical points of $I_\lambda(u)$ with the natural constraint. To this end, we define

$$M_{\lambda,\Omega} = \left\{u \in H_0^1(\Omega) \setminus \{0\} : \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u f(u) dx = 0\right\}$$

which is a Finsler manifold with a natural Finsler structure, and verify that $I_\lambda(u)$ satisfies (P-S) condition on $M_{\lambda,\Omega}$. We also show that $I_\lambda(u)$ is bounded from below on

$M_{\lambda,\Omega}$ and for some $c > 0$ (in fact, $c = m(\lambda, B_r(0))$)

$$cat(I_\lambda^c) \geqslant cat(\Omega), \tag{1.7}$$

where $I_\lambda^c = \{u \in M_{\lambda,\Omega} : I_\lambda(u) \leqslant c\}$. Then, Ljusternik–Schnirelmann theory gives at least $cat\Omega$ positive critical points of I. Finally, we show that for some higher energy level γ, the topological type of I_λ^c is different from I_λ^γ when Ω is not contractible. Hence as in [8], we get one more critical point of I_λ.

To carry out the above procedure, in particular to prove (1.7), we should notice that there is a one to one correspondence between the solution to (1.1) and the solution to

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_\lambda \\ u > 0 & \text{in } \Omega_\lambda \\ u = 0 & \text{on } \partial\Omega_\lambda, \end{cases} \tag{1.8}$$

where $\Omega_\lambda = \left\{x \in \mathbb{R}^N : \frac{x}{\sqrt{\lambda}} \in \Omega\right\}$. Following [2], [3], [6], [7], we need study the relation between the least energy solutions (ground state) to (1.8) and those to

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.9}$$

(1.9) is a problem with the so-called "zero mass" conditions, which is different from the cases dealt with in [3], [6], [7], [8]. Similar to [2], we thus need a compactness result as in [20] to analysis the relations between the energy levels (see Lemma 4.1 below).

In Section 2, we give some preliminary results and notation. In Section 3, we prove a compactness result, and in Section 4, we prove our main theorem.

Throughout this paper, we use standard notations.

$H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm $\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}$.

$H^1(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}$ is a Hilbert space with the norm $\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}}$.

$\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{1}{2}}$.

$\langle\cdot,\cdot\rangle$ denotes the dual paring between a Banach space and its dual space.

We denote the support of u by supp$u$, $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$. And we set $\mathbb{R}^+ \equiv \{x \in \mathbb{R} : x > 0\}$, $B_r \equiv B_r(0)$.

2. Preliminary results and notation

Without loss of generality, we assume that $f(t) = 0$ for $t \leqslant 0$. Then by (f_1), (f_2), for any $t \in \mathbb{R}$,

$$|f(t)| \leqslant l|t| \quad \text{and} \quad |F(t)| \leqslant \frac{l}{2}|t|^2, \tag{2.1}$$

$$\frac{|f(t)|}{|t|^{\frac{N+2}{N-2}}} \leqslant M, \tag{2.2}$$

where M is a positive number. We also deduce from (f_2) that

$$F(t) \leqslant \frac{1}{2} t f(t) \quad \forall t \in \mathbb{R}. \tag{2.3}$$

It is well known that solutions to (1.1) are critical points of the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega F(u)\, dx \tag{2.4}$$

defined on $H_0^1(\Omega)$, where $F(u) = \int_0^u f(t)\, dt$. It is easy to see that $I_\lambda \in C^{1,1}(H_0^1(\Omega), \mathbb{R})$, i.e. the Frechét derivative I_λ' of I_λ is Lipschitz continuous. Let

$$M_{\lambda,\Omega} = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : g(u) \triangleq \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u f(u) dx = 0 \right\}. \tag{2.5}$$

As $f \in C^1(\mathbb{R}, \mathbb{R})$ and (f_2), $M_{\lambda,\Omega}$ is a complete Finsler manifold with a natural Finsler structure (see [9], [14]). Similar to Lemma 2.6 of [14], we have

Lemma 2.1. *For any $u \in M_{\lambda,\Omega}$, $v \in H_0^1(\Omega)$, we have*

$$\langle dI_\lambda \left|_{M_{\lambda,\Omega}}(u)\right., \pi v\rangle = \langle I_\lambda'(u), v\rangle,$$

where π is a project of $H_0^1(\Omega)$ to the tangent space $T_u M_{\lambda,\Omega}$.

We recall that I_λ satisfies $(\text{P-S})_c$ condition on $H_0^1(\Omega)$ if $\{u_n\} \subset H_0^1(\Omega)$ satisfies $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$, then there is some $u_0 \in H_0^1(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_0$ in $H_0^1(\Omega)$ as $n \to \infty$.

Lemma 2.2. *The functional I_λ satisfies $(P\text{-}S)_c$ condition on $H_0^1(\Omega)$ provided $0 < \lambda \notin \left\{\frac{\lambda_k}{l}\right\}$ where $\{\lambda_k\}$ is the sequence of eigenvalues of $-\Delta$ with zero Dirichlet data on $\partial\Omega$.*

This lemma was essentially proved in [13]. We omit the details.

From now on we suppose $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$. For any $\lambda > 0$, we denote

$$m(\lambda, \Omega) = \inf\left\{I_\lambda(u) : u \in M_{\lambda,\Omega}\right\}. \tag{2.6}$$

Similar to [13], we know that $M_{\lambda,\Omega} \neq \varnothing$ for λ large enough. Without loss of generality, we assume $M_{\lambda,\Omega} \neq \varnothing$ for any$\lambda > 0$.

It is easy to show that $m(\lambda, \Omega) \geqslant 0$. So by Lemma 2.1, Lemma 2.2 and Ekeland's variational principle on Finsler manifold (see Lemma 2.5 in [14]), we know that $m(\lambda, \Omega)$ is achieved by a positive function u, which is a ground state solution to (1.1) (i.e. solution with least energy). Moreover, when Ω is a ball $B_r(x_0) = \left\{y \in \mathbb{R}^N : |y - x_0| < r\right\}$, it follows from [12] that this function is spherically symmetric about the center x_0 and decreases when the radial coordinate increases. Furthermore, it is obviously that $m(\lambda, B_r(x_0))$ depends only on the radius r. Here, $m(\lambda, B_r(x_0)) = \inf\{I_\lambda(u) : u \in M_{\lambda, B_r(x_0)}\}$. So we could set

$$m(\lambda, B_r(x_0)) = m(\lambda, B_r(0)) \text{ for } x_0 \in \mathbb{R}^N. \tag{2.7}$$

We can also deduce that

$$m(\lambda, B_{r_2}(0)) < m(\lambda, B_{r_1}(0)) \text{ if } r_1 < r_2, \tag{2.8}$$

$$m(\lambda, \Omega) < m(\lambda, B_r(0)) \text{ if } \Omega \supsetneqq B_r(x). \tag{2.9}$$

Now we rescale the problem (1.1). In fact, there is a one-to-one correspondence between the solutions to (1.1) and the solutions to

$$\left\{\begin{array}{ll} -\Delta u = f(u) & \text{in } \Omega_\lambda \\ u > 0 & \text{in } \Omega_\lambda \\ u = 0 & \text{on } \partial\Omega_\lambda, \end{array}\right. \tag{2.10}$$

where $\Omega_\lambda = \left\{x \in \mathbb{R}^N : \frac{x}{\sqrt{\lambda}} \in \Omega\right\}$.

This conclusion is simple to prove if we define a one-to-one map

$$T : H_0^1(\Omega) \to H_0^1(\Omega_\lambda)$$

by $Tu(x) = u\left(\frac{x}{\sqrt{\lambda}}\right)$.

Equation (2.10) is associated with the functional

$$\widetilde{I_\lambda}(u) = \frac{1}{2}\int_{\Omega_\lambda} |\nabla u|^2 \, dx - \int_{\Omega_\lambda} F(u)\, dx$$

constrained to lie upon

$$\widetilde{M_{1,\Omega_\lambda}} = \left\{u \in H_0^1(\Omega_\lambda) \setminus \{0\} : \int_{\Omega_\lambda} |\nabla u|^2 \, dx - \int_{\Omega_\lambda} uf(u)\, dx = 0\right\}.$$

We set

$$m(1,\Omega_\lambda) = \inf\left\{\widetilde{I_\lambda}(u) : u \in \widetilde{M_{1,\Omega_\lambda}}\right\}.$$

Similar to [7], we can easily obtain

Lemma 2.3. *For any fixed* $\lambda > 0$ *and* $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$,

$$m(\lambda,\Omega) = \lambda^{1-\frac{N}{2}} m(1,\Omega_\lambda).$$

Denote

$$\begin{aligned} m(1,\mathbb{R}^N) \;\; = \;\; \inf\Bigg\{&\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(u)\, dx : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \\ &\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} uf(u)\, dx, u \neq 0\Bigg\}. \end{aligned}$$

Using the methods of [4] and [14], we can show the following

Lemma 2.4. $m(1,\mathbb{R}^N)$ *is achieved by a positive function which is a ground state solution to* (1.9).

If $u \in H_0^1(D)$ $(D \subset \mathbb{R}^N)$, we will use the same symbol u as its extension to $\mathbb{R}^N$, with $u = 0$ outside of D. For any $u \in H^1_{\text{comp}}(\mathbb{R}^N)$ (the subspace of $H_0^1(\mathbb{R}^N)$ with compact support), we define as in [2]

$$\beta(u) = \frac{\int_{R^N} x \cdot |\nabla u|^2 \, dx}{\int_{R^N} |\nabla u|^2 \, dx}.$$

For any $\rho > 0$, we denote

$$\begin{aligned} \Omega_\rho^+ &= \left\{x \in \mathbb{R}^N : \operatorname{dist}(x,\Omega) \leqslant \rho\right\}, \\ \Omega_\rho^- &= \left\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geqslant \rho\right\}. \end{aligned}$$

For $\lambda > 0$ and $\lambda \notin \{\frac{\lambda_k}{l}\}$, $\rho > 0$, the operator

$$\Phi_{\lambda,\rho} : \Omega_{2\rho}^{-} \to H_0^1(\Omega)$$

is given by

$$[\Phi_{\lambda,\rho}(y)](x) = \begin{cases} u_{\lambda,\rho}(|x-y|), & \forall x \in B_\rho(y), \\ 0, & \forall x \in \Omega \setminus B_\rho(y), \end{cases}$$

where $u_{\lambda,\rho}(|x|)$ is a positive radially symmetric about the origin function such that $m(\lambda, B_\rho(0))$ is achieved by $u_{\lambda,\rho}$.

We will denote $I_\lambda^c = \{u \in M_{\lambda,\Omega} : I_\lambda(u) \leqslant c\}$ and for $\rho > 0$ and $\alpha \geqslant 1$ we define

$$\begin{aligned} m^*(\lambda,\rho,\alpha) &= \inf\left\{I_\lambda(u) : u \in M_{\lambda,B_{\alpha\rho}\setminus B_\rho}, \beta(u) = 0\right\}, \\ m^*\left(1,\sqrt{\lambda}\rho,\alpha\right) &= \inf\left\{\widetilde{I}_\lambda(u) : u \in \widetilde{M}_{1,B_{\alpha\sqrt{\lambda}\rho}\setminus B_{\sqrt{\lambda}\rho}}, \beta(u) = 0\right\}, \\ m^*(1,\rho,\alpha) &= \inf\left\{\frac{1}{2}\int_{B_{\alpha\rho}\setminus B_\rho} |\nabla u|^2\,dx - \int_{B_{\alpha\rho}\setminus B_\rho} F(u)\,dx : u \neq 0,\right. \\ &\left. u \in H_0^1(B_{\alpha\rho}\setminus B_\rho), \int_{B_{\alpha\rho}\setminus B_\rho} |\nabla u|^2\,dx = \int_{B_{\alpha\rho}\setminus B_\rho} uf(u)\,dx, \beta(u) = 0\right\}. \end{aligned}$$

3. A compactness result

We need a compactness result when we study nonlinear elliptic equation in $\mathbb{R}^N$ via critical point theory. Here, we discuss a special compactness result to deal with the "zero mass" situation.

Consider elliptic Dirichlet problem (1.9), i.e.

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases}$$

where $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies $(f_1), (f_2), (f_3)$ and $f(t) = 0$ for $t \leqslant 0$.

The uniformly convex Banach space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ can be characterized by

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u : |\nabla u| \in L^2\left(\mathbb{R}^N\right), u \in L^{\frac{2N}{N-2}}\left(\mathbb{R}^N\right)\right\}$$

due to Sobolev imbedding.

The energy functional associated with problem (1.9) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

We have the following lemma (see e.g. [15] and [20]).

Lemma 3.1. *Let $\{\rho_n\} \subset L^1(\mathbb{R}^N)$ be a bounded sequence and $\rho_n \geqslant 0$, then there exists a subsequence, still denoted by $\{\rho_n\}$, such that one of the following two possibilities occurs:*

(i) (vanishing): $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{y+B_R} \rho_n\,dx = 0$ for all $0 < R < +\infty$.

(ii) (nonvanishing): There exists $\alpha > 0$, $R < +\infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{y_n+B_R} \rho_n\,dx \geqslant \alpha > 0.$$

Lemma 3.2. *Suppose that $f(u)$ satisfies*

$$\lim_{s\to 0}\frac{|f(s)|}{|s|^{\frac{N+2}{N-2}}}=0 \quad \textit{and} \quad \lim_{s\to \infty}\frac{|f(s)|}{|s|^{\frac{N+2}{N-2}}}=0.$$

Assume that $\{u_n\}$ is a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $u_0\in\mathcal{D}^{1,2}(\mathbb{R}^N)$, such that

$$u_n \rightharpoonup u_0 \ \textit{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N),$$
$$u_n \to u_0 \ \textit{a.e. in } \mathbb{R}^N.$$

Then

$$\lim_{n\to\infty}\left[\int_{\mathbb{R}^N}F(u_n)\,dx-\int_{\mathbb{R}^N}F(u_0)\,dx-\int_{\mathbb{R}^N}F(u_n-u_0)\,dx\right]=0. \tag{3.1}$$

Proof For any $R<+\infty$, by the mean value theorem,

$$\begin{aligned}\int_{\mathbb{R}^N}F(u_n)\,dx&=\int_{B_R}F(u_n)\,dx+\int_{\mathbb{R}^N\setminus B_R}F(u_0+(u_n-u_0))\,dx\\&=\int_{B_R}F(u_n)\,dx+\int_{\mathbb{R}^N\setminus B_R}(F(u_n-u_0)+f(\theta u_0+(u_n-u_0))\,u_0)\,dx,\end{aligned}$$

where θ depends on R, satisfying $0<\theta<1$. Then

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}F(u_n)\,dx-\int_{\mathbb{R}^N}F(u_0)\,dx-\int_{\mathbb{R}^N}F(u_n-u_0)\,dx\right|\\&\leqslant\left|\int_{B_R}(F(u_n)-F(u_0))\,dx\right|+\left|\int_{\mathbb{R}^N\setminus B_R}F(u_0)\,dx\right|\\&+\left|\int_{B_R}F(u_n-u_0)\,dx\right|+\left|\int_{\mathbb{R}^N\setminus B_R}f(\theta u_0+(u_n-u_0))u_0\,dx\right|.\end{aligned} \tag{3.2}$$

By the assumption on $f(u)$ and Strauss Lemma (see [4], [18]), we have for fixed $R<+\infty$

$$\left|\int_{B_R}(F(u_n)-F(u_0))\,dx\right|\to 0,\quad \left|\int_{B_R}F(u_n-u_0)\,dx\right|\to 0 \quad \text{as } n\to+\infty. \tag{3.3}$$

By (2.2), Hölder inequality and boundness of $\{u_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N\setminus B_R}f(\theta u_0+(u_n-u_0))\,u_0\,dx\right|\\&\leqslant C_1\int_{\mathbb{R}^N\setminus B_R}|\theta u_0+(u_n-u_0)|^{\frac{N+2}{N-2}}\,|u_0|\,dx\\&\leqslant C_1\left(\int_{\mathbb{R}^N\setminus B_R}|\theta u_0+(u_n-u_0)|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N+2}{2N}}\left(\int_{\mathbb{R}^N\setminus B_R}|u_0|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}\\&\leqslant C_2\left(\int_{\mathbb{R}^N\setminus B_R}|u_0|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}.\end{aligned}$$

So by (3.2), (3.3) and the above inequality, let $n\to\infty$, then $R\to+\infty$, we deduce

$$\lim_{n\to\infty}\left[\int_{\mathbb{R}^N}F(u_n)\,dx-\int_{\mathbb{R}^N}F(u_0)\,dx-\int_{\mathbb{R}^N}F(u_n-u_0)\,dx\right]=0.$$

Hence, the Lemma is proved.

Now, we give the following compactness result

Theorem 3.3. *Suppose that* $(f_1), (f_2)$ *and* (f_3) *hold. Assume that for fixed* $c \in \mathbb{R}^+$, $\{u_n\}$ *is a* $(P-S)_c$ *sequence in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$,

i.e. $I(u_n) = c + o(1)$ *and* $I'(u_n) \to 0$ *in* $\left(\mathcal{D}^{1,2}(\mathbb{R}^N)\right)^*$.

Moreover, assume that $\langle I'(u_n), u_n\rangle = 0$. *Then there exist a nonnegative integer* k, *solution* u^0 *to* (1.9), *nontrivial solutions* $u^1, u^2, \cdots, u^k$ *to problem* (1.9), *and sequences* $\{x_m^1\}, \cdots, \{x_m^k\} \subset \mathbb{R}^N$ *with* $|x_m^i| \to \infty$ *as* $m \to \infty$, $\quad i = 1, \cdots, k$, *such that for some subsequence* $\{u_m\}$ *of* $\{u_n\}$, *as* $m \to \infty$

$$\left\{\begin{array}{ll} u_m^0 \equiv & u_m \rightharpoonup u^0 \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N) , \\ u_m^j \equiv & (u_m^{j-1} - u^{j-1})(x + x_m^j) \rightharpoonup u^j \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N), j = 1, \cdots, k. \end{array}\right. \tag{3.4}$$

Furthermore,

$$\|u_m\|^2 \to \sum_{j=0}^{k} \left\|u^j\right\|^2, \tag{3.5}$$

$$c = I\left(u^0\right) + \sum_{j=1}^{k} I\left(u^j\right). \tag{3.6}$$

Proof First, we use similar method as in [14] to show that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

If $\|u_n\| \to +\infty$, then we let

$$t_n = \frac{2\sqrt{c}}{\|u_n\|}, \quad \omega_n(x) = t_n u_n(x) = \frac{2\sqrt{c}u_n}{\|u_n\|}.$$

So $\|\omega_n\| = 2\sqrt{c}$.

Now we apply Lemma 3.1 to $\rho_n = |\nabla \omega_n|^2 + |\omega_n|^{\frac{2N}{N-2}}$.

If "vanishing" occurs, then

$$\sup_{\mathbb{R}^N} \int_{y+B_R} \rho_n(x)\, dx \to 0 \quad \text{as } n \to \infty \quad \forall 0 < R < +\infty.$$

By Lemma II 2 in [15] we obtain that

$$\int_{\mathbb{R}^N} \omega_n f(\omega_n)\, dx \to 0, \quad \int_{\mathbb{R}^N} F(\omega_n)\, dx \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Then

$$I(\omega_n) = \frac{1}{2}\|\omega_n\|^2 - \int_{\mathbb{R}^N} F(\omega_n)\, dx = \frac{(2\sqrt{c})^2}{2} + o(1) = 2c + o(1). \tag{3.8}$$

On the other hand, for any $t > 0$,

$$I(tu_n) = \frac{t^2}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(tu_n)\, dx, \tag{3.9}$$

$$\langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_{\mathbb{R}^N} u_n f(u_n)\, dx = 0. \tag{3.10}$$

By (3.9), (3.10) we deduce that

$$I\left(tu_n\right)=\frac{t^2}{2}\int_{\mathbb{R}^N}u_nf(u_n)\,dx-\int_{\mathbb{R}^N}F(tu_n)\,dx.$$

Set

$$h(t)=\frac{1}{2}t^2sf(s)-F(ts),$$

by $(f_1),(f_2)$, it is easy to see that

$$h'(t)=tsf(s)-f(ts)\begin{cases}\geqslant 0 & \text{if } 0<t\leqslant 1,\ s>0\\ \leqslant 0 & \text{if } 0<t\geqslant 1,\ s>0\\ =0 & \text{if } t>0,\ s\leqslant 0\end{cases}$$

which means that

$$h\left(t\right)\leqslant h(1)\quad \forall t>0.$$

So

$$I(tu_n)\leqslant\frac{1}{2}\left\|u_n\right\|^2-\int_{\mathbb{R}^N}F(u_n)\,dx=I(u_n).$$

Therefore,

$$I\left(\omega_n\right)=I\left(t_nu_n\right)\leqslant I\left(u_n\right)=c+o\left(1\right) \tag{3.11}$$

which is impossible by (3.8).

Hence, "nonvanishing" occurs, i.e. there exists $\eta>0$, $R>0$, $\{y_n\}\subset\mathbb{R}^N$ such that

$$\lim_{n\to\infty}\int_{y_n+B_R}\left(\left|\nabla\omega_n\right|^2+\left|\omega_n\right|^{\frac{2N}{N-2}}\right)dx\geqslant\eta>0. \tag{3.12}$$

Let $\widetilde{\omega}_n\left(x\right)=\omega_n\left(x+y_n\right)$. Notice that $\|\widetilde{\omega}_n\|=\|\omega_n\|$. Therefore, there is some $\widetilde{\omega}\in\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, such that

$$\begin{cases}\widetilde{\omega}_n\rightharpoonup\widetilde{\omega} & \text{weakly in } \mathcal{D}^{1,2}\left(\mathbb{R}^N\right),\\ \widetilde{\omega}_n\to\widetilde{\omega} & \text{strongly in } L^p_{loc}\left(\mathbb{R}^N\right),\ 2\leqslant p<2^*,\\ \widetilde{\omega}_n\to\widetilde{\omega} & \text{a.e. in } \mathbb{R}^N.\end{cases} \tag{3.13}$$

Since $I'(u_n)\to 0$ in $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$, we have for any $\varphi\in\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$

$$\int_{\mathbb{R}^N}\nabla u_n\nabla\varphi\,dx=\int_{\mathbb{R}^N}f\left(u_n\right)\varphi\,dx+o\left(1\right). \tag{3.14}$$

Set $\widetilde{u}_n(x)=u_n(x+y_n)$.

From the translation invariance of I' and (3.14), we have for any $\varphi\in\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$

$$\int_{\mathbb{R}^N}\nabla\widetilde{u}_n\nabla\varphi\,dx=\int_{\mathbb{R}^N}f\left(\widetilde{u}_n\right)\varphi\,dx+o\left(1\right)\left\|\varphi\right\|.$$

So for any $\varphi\in\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$

$$\int_{\mathbb{R}^N}\nabla\widetilde{\omega}_n\nabla\varphi\,dx=\int_{\mathbb{R}^N}\frac{2\sqrt{c}f\left(\widetilde{u}_n\right)}{\|\widetilde{u}_n\|}\varphi\,dx+o\left(1\right)\left\|\varphi\right\|. \tag{3.15}$$

Let $\varphi = \xi^2(x)\,\widetilde{\omega}_n$, where $\xi \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \xi \leqslant 1$, $\operatorname{supp}\xi \subset B_{2R}$, and $\xi = 1$ on B_R. Then by (3.15), we have

$$\int_{\mathbb{R}^N} \nabla\widetilde{\omega}_n \nabla\left(\xi^2(x)\,\widetilde{\omega}_n\right) dx = \int_{\mathbb{R}^N} \frac{2\sqrt{c} f(\widetilde{u}_n)}{\|\widetilde{u}_n\|}\left(\xi^2(x)\,\widetilde{\omega}_n\right) dx + o(1)\left\|\xi^2(x)\,\widetilde{\omega}_n\right\|.$$

So

$$\begin{aligned}\int_{B_R} |\nabla\widetilde{\omega}_n|^2\,dx &\leqslant \int_{B_{2R}} |\nabla(\xi(x)\,\widetilde{\omega}_n)|^2\,dx \\ &= \int_{B_{2R}} \left(\frac{2\sqrt{c} f(\widetilde{u}_n)}{\|\widetilde{u}_n\|}\xi^2(x)\,\widetilde{\omega}_n + \widetilde{\omega}_n^2\,|\nabla\xi(x)|^2\right) dx + o(1) \\ &\leqslant \left[\sup_{x\in\mathbb{R}^N} |\nabla\xi(x)|^2 + l\right] \int_{B_{2R}} \widetilde{\omega}_n^2\,dx + o(1).\end{aligned}$$

We claim that $\widetilde{\omega} \neq 0$. Otherwise $\widetilde{\omega} = 0$. By (3.13) and the above inequality, we have

$$\lim_{n\to\infty} \int_{B_R} |\nabla\widetilde{\omega}_n|^2\,dx = 0.$$

This implies

$$\lim_{n\to\infty} \int_{B_R} \left(|\nabla\widetilde{\omega}_n|^2 + |\widetilde{\omega}_n|^{\frac{2N}{N-2}}\right) dx = 0 \text{ due to Sobolev imbedding,}$$

i.e.

$$\lim_{n\to\infty} \int_{y_n+B_R} \left(|\nabla\omega_n|^2 + |\omega_n|^{\frac{2N}{N-2}}\right) dx = 0$$

which contradicts (3.12). So $\widetilde{\omega} \neq 0$. Set

$$p_n(x) = \begin{cases} \frac{f(\widetilde{u}_n)}{\widetilde{u}_n}, & \widetilde{u}_n(x) > 0 \\ 0, & \widetilde{u}_n(x) \leqslant 0. \end{cases} \tag{3.16}$$

Then by (3.13), (3.15), we know that

$$\int_{\mathbb{R}^N} \nabla\widetilde{\omega}\nabla\varphi\,dx = \int_{\mathbb{R}^N} p_n(x)\,\widetilde{\omega}_n\varphi\,dx + o(1) \text{ for all } \varphi \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right). \tag{3.17}$$

By (2.1), $0 \leqslant p_n(x) \leqslant l$ and the fact that $\{p_n(x)\}$ is bounded in $L^2_{loc}\left(\mathbb{R}^N\right)$, there is some $h \in L^2_{loc}\left(\mathbb{R}^N\right)$ such that

$$p_n(x) \rightharpoonup h(x) \text{ weakly in } L^2_{loc}\left(\mathbb{R}^N\right). \tag{3.18}$$

By using (3.13), we know that for all $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} p_n(x)\,\widetilde{\omega}_n(x)\,\varphi(x)\,dx \to \int_{\mathbb{R}^N} h(x)\,\widetilde{\omega}\varphi\,dx.$$

Thus, (3.17) implies

$$\int_{R^N} \nabla\widetilde{\omega}\,\nabla\varphi\,dx = \int_{R^N} h(x)\,\widetilde{\omega}\varphi\,dx \quad \text{for all} \quad \varphi \in C_0^\infty(R^N). \tag{3.19}$$

Set $A = \left\{x \in \mathbb{R}^N : \widetilde{\omega}\,(x) > 0\right\}$. It is easy to know that $\widetilde{u}_n\,(x) \to +\infty$ a.e. in A. Then $h\,(x) = l$ a.e. in A. Similarly, we have $h(x) = 0$ a.e. in $\{x \in \mathbb{R}^N : \widetilde{\omega}(x) < 0\}$. So $\forall \varphi \in C_0^\infty\left(R^N\right)$

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla\widetilde{\omega}\,\nabla\varphi\,dx &= \int_{\{x\in\mathbb{R}^N:\widetilde{\omega}(x)<0\}} h(x)\,\widetilde{\omega}\varphi\,dx + \int_A h(x)\,\widetilde{\omega}\varphi\,dx \\ &= \int_A l\widetilde{\omega}\varphi\,dx = \int_{\mathbb{R}^N} l\widetilde{\omega}^+\varphi\,dx. \end{aligned} \tag{3.20}$$

We claim that $\widetilde{\omega} \in L^2(\mathbb{R}^N)$. Indeed, it is easy to see that (3.20) remain holds for any $\varphi \in H^1_{comp}(\mathbb{R}^N)$. For any $R > 0$, let $\varphi(x) = \varphi_R(x) = \xi(\frac{x}{R})\,\widetilde{\omega}(x)$, where $\xi \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \xi \leqslant 1$, supp$\xi \subset B_2(0)$, and $\xi = 1$ on $B_1(0)$, $|\nabla\xi| \leqslant M$ for some $M > 0$, then

$$\begin{aligned} & l\int_{B_R} (\widetilde{\omega}^+)^2\,dx \leqslant l\int_{\mathbb{R}^N} \widetilde{\omega}^+\varphi_R\,dx = \int_{\mathbb{R}^N} \nabla\widetilde{\omega}\nabla\varphi_R\,dx \\ & \leqslant \left(\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla\varphi_R|^2\,dx\right)^{\frac{1}{2}} \\ & \leqslant C + \int_{\mathbb{R}^N} |\nabla\varphi_R|^2\,dx \leqslant C + \int_{\mathbb{R}^N} [|\nabla_x\,\xi(\frac{x}{R})|^2\,\widetilde{\omega}^2 + \nabla(\xi^2(\frac{x}{R})\widetilde{\omega})\,\nabla\widetilde{\omega}]\,dx \\ & \leqslant C + M\int_{B_{2R}} \frac{\widetilde{\omega}^2}{R^2}\,dx + \int_{\mathbb{R}^N} |\nabla(\xi^2(\frac{x}{R})\,\widetilde{\omega})|\,|\nabla\widetilde{\omega}|\,dx \\ & \leqslant C + M\int_{B_{2R}} \frac{\widetilde{\omega}^2}{R^2}\,dx + \int_{\mathbb{R}^N} [|\nabla_x\,\xi^2(\frac{x}{R})|\,|\widetilde{\omega}|\,|\nabla\widetilde{\omega}| + \xi^2(\frac{x}{R})\,|\nabla\widetilde{\omega}|^2]\,dx \\ & \leqslant C + M\int_{B_{2R}} \frac{\widetilde{\omega}^2}{R^2}\,dx + C\int_{B_{2R}} \frac{|\widetilde{\omega}|}{R}|\nabla\widetilde{\omega}|\,dx + C\int_{B_{2R}} |\nabla\widetilde{\omega}|^2\,dx \\ & \leqslant C + C\int_{B_{2R}} \frac{\widetilde{\omega}^2}{R^2}\,dx + C\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}|^2\,dx \\ & \leqslant C + C\int_{\mathbb{R}^N} |\widetilde{\omega}|^{2^*}\,dx + C\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}|^2\,dx \leqslant C. \end{aligned}$$

So $\widetilde{\omega}^+ \in L^2(\mathbb{R}^N)$. Putting $\phi(x) = \xi(\frac{x}{R})\,\widetilde{\omega}^-(x)$ in (3.20), we have

$$\int_{\mathbb{R}^N} \nabla\widetilde{\omega}\cdot\nabla(\widetilde{\omega}^-\xi(\frac{x}{R}))\,dx = \int_{\mathbb{R}^N} l\widetilde{\omega}^+\cdot\widetilde{\omega}^-\xi(\frac{x}{R})\,dx = 0.$$

So

$$\begin{aligned} & \int_{B_R(0)} |\nabla\widetilde{\omega}^-|^2\,dx \leqslant \int_{\mathbb{R}^N} |\nabla\widetilde{\omega}^-|^2\,\xi(\frac{x}{R})\,dx = -\int_{\mathbb{R}^N} \nabla\widetilde{\omega}\cdot(\widetilde{\omega}^-\nabla_x\xi(\frac{x}{R}))\,dx \\ & \leqslant \int_{\mathbb{R}^N} |\nabla\widetilde{\omega}|\,|\widetilde{\omega}^-|\,|\nabla_x\,\xi(\frac{x}{R})|\,dx \leqslant C\int_{B_{2R}(0)\setminus B_R(0)} \frac{|\nabla\widetilde{\omega}|\,|\widetilde{\omega}|}{R}\,dx \\ & \leqslant C\left(\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_{2R}(0)\setminus B_R(0)} \frac{|\widetilde{\omega}|^2}{R^2}\,dx\right)^{\frac{1}{2}} \\ & \leqslant C\left(\int_{B_{2R}(0)\setminus B_R(0)} |\widetilde{\omega}|^{2^*}\,dx\right)^{\frac{N-2}{N}}. \end{aligned}$$

Let $R \to +\infty$, we have $\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}^-|^2\,dx = 0$, i.e. $\widetilde{\omega}^- = 0$. Therefore $\widetilde{\omega} = \widetilde{\omega}^+ \in L^2(\mathbb{R}^N)$. This concludes $\widetilde{\omega} \in H^1(\mathbb{R}^N)$ and by (3.20), for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla\widetilde{\omega}\,\nabla\varphi\,dx = l\int_{\mathbb{R}^N} \widetilde{\omega}\varphi\,dx.$$

This means $\widetilde{\omega}$ is an eigenfunction of $-\Delta$ in $H^1\left(\mathbb{R}^N\right)$, which is impossible by Pohozaev identity (see [4]).

Therefore $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$.

Then we may assume that for some $u^0 \in D^{1,2}\left(\mathbb{R}^N\right)$, as $n \to \infty$

$$\left\{\begin{array}{ll} u_n \rightharpoonup u^0 & \text{weakly in } \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \\ u_n \to u^0 & \text{strongly in } L^p_{loc}(\mathbb{R}^N),\ 2 \leqslant p < 2^* \\ u_n \to u^0 & \text{a.e. in}\mathbb{R}^N. \end{array}\right. \tag{3.21}$$

Since $I'(u_n) \to 0$ in $(\mathcal{D}^{1,2}(\mathbb{R}^N))^*$, we get directly that u^0 is a solution to problem (1.9).

Denote $v_n^1 = u_n - u^0$.

We have

$$\left\|u_n - u^0\right\|^2 = \|u_n\|^2 + \left\|u^0\right\|^2 - 2\langle u_n, u^0\rangle = \|u_n\|^2 - \left\|u^0\right\|^2 + o\left(1\right). \tag{3.22}$$

By Lemma 3.2 and (3.22), we obtain that

$$I(v_n^1) = I(u_n) - I(u^0) + o(1). \tag{3.23}$$

Next, we claim that

$$I'(v_n^1) = I'(u_n) - I'(u^0) + o(1) = o(1). \tag{3.24}$$

In fact, for any $\varphi \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$,

$$\langle I'(u_n) - I'(u^0), \varphi\rangle = \int_{\mathbb{R}^N} \nabla v_n^1 \nabla\varphi\,dx - \int_{\mathbb{R}^N} (f(u_n) - f(u^0))\varphi\,dx,$$

$$\langle I'(v_n^1), \varphi\rangle = \int_{\mathbb{R}^N} \nabla v_n^1 \nabla\varphi\,dx - \int_{\mathbb{R}^N} f(v_n^1)\varphi\,dx.$$

Then we only need to show that

$$\left|\int_{\mathbb{R}^N} (f(u_n) - f(u^0) - f(v_n^1))\varphi\,dx\right| = o(1)\,\|\varphi\|\,. \tag{3.25}$$

Indeed, for any $R < +\infty$,

$$\begin{aligned}\int_{\mathbb{R}^N} f(u_n)\varphi\,dx &= \int_{B_R} f(u_n)\varphi\,dx + \int_{\mathbb{R}^N\setminus B_R} f(u^0 + (u_n - u^0))\varphi\,dx \\ &= \int_{B_R} f(u_n)\varphi\,dx + \int_{\mathbb{R}^N\setminus B_R} (f(u_n - u^0) + f'(\theta u^0 + (u_n - u^0))u^0)\varphi\,dx,\end{aligned}$$

where θ depends on R, satisfying $0 < \theta < 1$.

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^N} (f(u_n) - f(u^0) - f(v_n^1))\varphi\,dx\right| &\leqslant \left|\int_{B_R} (f(u_n) - f(u^0))\varphi\,dx\right| \\ &+ \left|\int_{\mathbb{R}^N\setminus B_R} f(u^0)\varphi\,dx\right| + \left|\int_{B_R} f(v_n^1)\varphi\,dx\right| + \left|\int_{\mathbb{R}^N\setminus B_R} f'(\theta u^0 + v_n^1)\,u^0\varphi\,dx\right|.\end{aligned}$$

By (f_1), we know that

$$\frac{f'(t)}{t^{\frac{4}{N-2}}} \to 0 \quad \text{as} \quad t \to 0+, \quad \frac{f'(t)}{t^{\frac{4}{N-2}}} \to 0 \quad \text{as} \quad t \to +\infty.$$

So we have $|f'(t)| \leqslant C_1 |t|^{\frac{4}{N-2}}$ for some $C_1 > 0$.

Hence

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (f(u_n) - f(u^0) - f(v_n^1))\varphi \, dx \right| \\
&\leqslant \left[\left(\int_{B_R} \left| f(u_n) - f\left(u^0\right) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} + \left(\int_{B_R} \left| f\left(v_n^1\right) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} \right] \cdot C \|\varphi\| \\
&+ \int_{\mathbb{R}^N \setminus B_R} \left| f\left(u^0\right) \varphi \right| dx + C \int_{\mathbb{R}^N \setminus B_R} \left| \theta u^0 + v_n^1 \right|^{\frac{4}{N-2}} \left| u^0 \varphi \right| dx \\
&\leqslant \left[\left(\int_{B_R} \left| f(u_n) - f\left(u^0\right) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} + \left(\int_{B_R} \left| f\left(v_n^1\right) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} \right] \cdot C \|\varphi\| \\
&+ \left[\left(\int_{\mathbb{R}^N \setminus B_R} \left| f\left(u^0\right) \right|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} + C \left(\int_{R^N \setminus B_R} \left| u^0 \right|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \right] \cdot C \|\varphi\|.
\end{aligned}$$

By Strauss Lemma (see [4], [18]), letting $n \to \infty$ and then $R \to \infty$, we obtain (3.25).

If $\|v_n^1\| \to 0$, as $n \to \infty$, by (3.22)–(3.25), we have completed the proof.

Without loss of generality, we may assume that $\|v_n^1\|^2 \to l_1 > 0$ as $n \to \infty$.

Let $\rho_n = |\nabla v_n^1|^2 + |v_n^1|^{\frac{2N}{N-2}}$. We apply Lemma 3.1 to deduce that there exists a subsequence, still denoted by $\{v_n^1\}$, such that only one of the two cases holds: case (i) “vanishing”, case(ii) “nonvanishing”.

If case (i) occurs, then

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} \left(\left| \nabla v_n^1 \right|^2 + \left| v_n^1 \right|^{\frac{2N}{N-2}} \right) dx \to 0,$$

as $n \to \infty$ $\forall R < +\infty$.

Applying Lemma II 2 in [15], we have

$$\int_{\mathbb{R}^N} f\left(v_n^1\right) v_n^1 \, dx \to 0.$$

By (3.24), we have

$$\int_{\mathbb{R}^N} \left| \nabla v_n^1 \right|^2 dx - \int_{\mathbb{R}^N} f(v_n^1)\, v_n^1 \, dx = o(1).$$

So $\int_{\mathbb{R}^N} |\nabla v_n^1|^2 \, dx \to 0$ as $n \to \infty$, which is a contradiction to $l_1 > 0$.

So only “nonvanishing” occurs, i.e. $\exists \alpha > 0, R < +\infty, \{y_n\} \subset \mathbb{R}^N$, such that

$$\lim_{n \to \infty} \int_{y_n + B_R} \left(\left| \nabla v_n^1 \right|^2 + \left| v_n^1 \right|^{\frac{2N}{N-2}} \right) dx \geqslant \alpha > 0. \tag{3.26}$$

Since $v_n^1 \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, we may assume that $|y_n| \to +\infty$.

Indeed, suppose that $\{y_n\}$ is bounded in $\mathbb{R}^N$. We choose R_0 large enough such that $\left|\sup_n \{|y_n|\} + R\right| < R_0$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi(x) \leqslant 1$, supp$\varphi \subset B_{2R_0}$ and $\varphi(x) = 1$ on B_{R_0}.

Multiplying (3.24) by $\varphi^2(x)\, v_n^1$, we get

$$\int_{\mathbb{R}^N} \nabla v_n^1 \nabla(\varphi^2 v_n^1)\, dx = \int_{\mathbb{R}^N} \varphi^2 f(v_n^1)\, v_n^1\, dx + o(1) \text{ as } n \to \infty.$$

So

$$\begin{aligned}
\int_{y_n+B_R} \left|\nabla v_n^1\right|^2 dx &\leqslant \int_{B_{R_0}} \left|\nabla v_n^1\right|^2 dx \leqslant \int_{B_{2R_0}} \left|\nabla(\varphi v_n^1)\right|^2 dx \\
&= \int_{B_{2R_0}} f(v_n^1) v_n^1 \varphi^2\, dx + \int_{B_{2R_0}} (v_n^1)^2 \left|\nabla\varphi\right|^2 dx + o(1) \\
&\leqslant \left[\sup_{x\in\mathbb{R}^N} |\nabla\varphi|^2 + l\right] \int_{B_{2R_0}} \left(v_n^1\right)^2 dx + o(1).
\end{aligned}$$

Therefore $\lim\limits_{n\to\infty} \int_{y_n+B_R} \left|\nabla v_n^1\right|^2 dx = 0$. By the Sobolev imbedding theorem,

$$\lim_{n\to\infty} \int_{y_n+B_R} \left(\left|\nabla v_n^1\right|^2 + \left|v_n^1\right|^{\frac{2N}{N-2}}\right) dx = 0 \tag{3.27}$$

which is a contradiction to (3.26). So $|y_n| \to +\infty$.

Denote

$$\begin{aligned}
x_n^1 &= y_n, \\
u_n^1(x) &= v_n^1\left(x + x_n^1\right) = \left(u_n - u^0\right)\left(x + x_n^1\right).
\end{aligned} \tag{3.28}$$

We can extract a subsequence of $\{u_n^1\}$, still denoted by $\{u_n^1\}$, such that for some $u^1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u_n^1 \rightharpoonup u^1$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$.

From the translation invariance of I' and (3.24), we have

$$I'\left(u_n^1\right) \to 0 \text{ in } \left(\mathcal{D}^{1,2}\left(\mathbb{R}^N\right)\right)^* \text{ as } n \to \infty. \tag{3.29}$$

Because

$$\lim_{n\to\infty} \int_{B_R} \left(\left|\nabla u_n^1\right|^2 + \left|u_n^1\right|^{\frac{2N}{N-2}}\right) dx \geqslant \alpha > 0,$$

by the similar argument as the deducing of (3.27), we get $u^1 \neq 0$.

By (3.29), we also get u^1 is a nontrivial solution to problem (1.9).

Denote $v_n^2 = u_n^1 - u^1$. Then

$$\left\|v_n^2\right\|^2 = \left\|u_n^1\right\|^2 - \left\|u^1\right\|^2 + o(1) \text{ as } n \to \infty. \tag{3.30}$$

Without loss of generality, we may assume that $u_n^1 \to u^1$ a.e. in R^N as $n \to \infty$.

By Lemma 3.2, we have that

$$\int_{\mathbb{R}^N} F(u_n^1)\, dx = \int_{\mathbb{R}^N} F(u^1)\, dx + \int_{\mathbb{R}^N} F(v_n^2)\, dx + o(1),$$

and then

$$\begin{aligned}
I\left(v_n^2\right) &= I\left(u_n^1\right) - I\left(u^1\right) + o(1) = I\left(v_n^1\right) - I\left(u^1\right) + o(1) \\
&= I(u_n) - I\left(u^0\right) - I\left(u^1\right) + o(1).
\end{aligned} \tag{3.31}$$

If $\|v_n^2\| \to 0$, we have completed the proof. We may assume that $\|v_n^2\|^2 \to l_2 > 0$, as $n \to \infty$.

By the same argument as above, we can get $\{x_n^2\} \subset \mathbb{R}^N$ with $|x_n^2| \to +\infty$, a subsequence of $\{v_n^2\}$ (still denoted by $\{v_n^2\}$), some $u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $u^2 \neq 0$, such that

$$\begin{aligned} u_n^2(x) &= v_n^2(x + x_n^2) \rightharpoonup u^2 \text{ weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N), \\ I'(v_n^2) &= I'(u_n^1) - I'(u^1) + o(1), \end{aligned} \tag{3.32}$$

then u^2 is a nontrivial solution to problem (1.9).

We prove the theorem by iteration. We obtain sequence $v_n^j = u_n^{j-1} - u^{j-1}, j \geqslant 2$ and $\{x_n^j\} \subset \mathbb{R}^N$ with $|x_n^j| \to +\infty$ such that $u_n^j(x) = v_n^j(x + x_n^j) \rightharpoonup u^j$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$ for some $u^j \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $u^j \neq 0$, u^j is a nontrivial solution to problem (1.9). Moreover, as $n \to \infty$

$$\begin{aligned} \|u_n^j\|^2 &= \|v_n^j\|^2 = \|u_n^{j-1}\|^2 - \|u^{j-1}\|^2 + o(1) = \cdots \\ &= \|u_n\|^2 - \sum_{i=0}^{j-1} \|u^i\|^2 + o(1), \end{aligned} \tag{3.33}$$

$$\begin{aligned} I(u_n^j) &= I(v_n^j) = I(u_n^{j-1}) - I(u^{j-1}) + o(1) = \cdots \\ &= I(u_n) - I(u^0) - \sum_{i=0}^{j-1} I(u^i) + o(1) \\ &= c - I(u^0) - \sum_{i=0}^{j-1} I(u^i) + o(1). \end{aligned} \tag{3.34}$$

Applying Lemma 2.4, we know that equation (1.9) possesses a ground state ψ (i.e. ψ is a solution to problem (1.9) and for any nontrivial solution u of (1.9), we have $I(u) \geqslant I(\psi) > 0$).

From (3.34), we know that the above iterative process must end at finite steps. Therefore, we have completed the proof of theorem.

Remark 3.1. *If the conditions in Theorem 3.3 hold and we also assume that $c = m(1, \mathbb{R}^N)$, then by* (3.6)*, we know* (3.4)–(3.6) *hold for $k = 0$ or $k = 1$. If $k = 0$, combining with* (3.5) *we can deduce that $u_m \to \psi$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, where ψ is a ground state solution to* (1.9) *realizing $m(1, \mathbb{R}^N)$. If $k = 1$, we know that $I(u^0) = 0$ and $u^1 = \psi$. So by (f_2), we have $u^0 = 0$. Also by* (3.6)*, we know that $u_m^1 \to \psi$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. By* (3.4)*, $u_m = \omega_m(x) + \psi(x - x_m)$ where $\omega_m \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Nevertheless, u_m has the form $\omega_m(x) + \psi(x - x_m)$ with $\omega_m \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

4. Proof of Theorem 1.1

Firstly, we will show that when λ goes to infinity the least energy of the ground state solution to (2.10) (i.e. $m(1, \Omega_\lambda)$) converges to that of the ground state solution to (1.9).

Lemma 4.1. *For any fixed* $\alpha > 1$, $\inf_{\rho>0} m^*(1,\rho,\alpha) > m(1,\mathbb{R}^N)$.

Proof It is easy to know that $m^*(1,\rho,\alpha) \geqslant m(1,\mathbb{R}^N)$. To prove the strict inequality, we argue by contradiction and we suppose that equality holds, i.e.

$$m^*(1,\rho_n,\alpha) \to m(1,\mathbb{R}^N) \text{ as } n\to\infty. \tag{4.1}$$

It is obvious that $\{\rho_n\}$ is unbounded. Otherwise, we suppose it were bounded by L we should have $m^*(1,\rho_n,\alpha) \geqslant m(1,B_{\alpha L}(0)) > m(1,\mathbb{R}^N)$, this contradicts (4.1). So $\{\rho_n\}$ is unbounded. Then there exists a sequence of functions $\{u_n\}$ such that $u_n \in H_0^1\left(B_{\alpha\rho_n}(0)\setminus B_{\rho_n}(0)\right)$,

$$\begin{aligned}
\int_{B_{\alpha\rho_n}(0)\setminus B_{\rho_n}(0)} |\nabla u_n|^2\,dx &= \int_{B_{\alpha\rho_n}(0)\setminus B_{\rho_n}(0)} u_n f(u_n)\,dx,\\
\beta(u_n) &= 0,\\
\frac{1}{2}\int_{B_{\alpha\rho_n}(0)\setminus B_{\rho_n}(0)} |\nabla u_n|^2\,dx &- \int_{B_{\alpha\rho_n}(0)\setminus B_{\rho_n}(0)} F(u_n)\,dx \to m(1,\mathbb{R}^N) \text{ as } n\to\infty.
\end{aligned}$$

So $\{u_n\}$ is a minimizing sequence of $m(1,\mathbb{R}^N)$. By Ekeland's variational principle on Finsler manifold (see Lemma 2.5 in [14]), there are v_n with $\|u_n - v_n\| \to 0$,

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \int_{\mathbb{R}^N} F(v_n)\,dx &\to m(1,\mathbb{R}^N) \text{ as } n\to\infty,\\
\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx &= \int_{\mathbb{R}^N} v_n f(v_n)\,dx,\\
\int_{\mathbb{R}^N} \nabla v_n \nabla\varphi\,dx - \int_{\mathbb{R}^N} f(v_n)\varphi\,dx &= o(1)\,\|\varphi\| \quad \forall \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N).
\end{aligned}$$

By Remark 3.1, there are $x_n \in \mathbb{R}^N$ and $\omega_n \in D^{1,2}(\mathbb{R}^N)$ such that

$$v_n(x) = \psi(x - x_n) + \omega_n(x) \text{ with } \|\omega_n\| \to 0 \text{ as } n\to\infty,$$

where ψ is the ground state to (1.9) realizing $m\left(1,\mathbb{R}^N\right)$. Therefore we can obtain $u_n(x) = \psi(x - x_n) + \widetilde{\omega}_n(x)$ with $\|\omega_n\| \to 0$ as $n\to\infty$.

Since $u_n \in H_0^1(\mathbb{R}^N \setminus B_{\rho_n})$, we have $|x_n| \to +\infty$.

To get a contradiction to $\beta(u_n) = 0$, we can use the argument similar to the proof of Lemma 2.2 in [2]. We omit the details here. Hence the lemma is proved.

In what follows, without any loss of generality, we assume that $0 \in \Omega$. Moreover, we denote a number by $r \in \mathbb{R}^+$ such that Ω_r^+, $\Omega_{r/2}^+$, Ω_{2r}^-, Ω are homotopically equivalent and $B_{2r}(0) \subset \Omega$ which is possible because Ω is smooth domain.

Lemma 4.2. $m(1,\Omega_\lambda) = m(1,\mathbb{R}^N) + o(1)$ *where* $o(1) \to 0$ *as* $\lambda \to +\infty$.

Proof It is obvious that $m(1,\Omega_\lambda) \geqslant m(1,\mathbb{R}^N)$. So it suffices to show

$$m(1,\Omega_\lambda) \leqslant m(1,\mathbb{R}^N) + o(1). \tag{4.2}$$

Set $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B_{\frac{1}{2}}(0)$, $\varphi = 0$ on $B_1(0)^c$, $|\nabla\varphi| \leqslant 4$.

Letting $\varphi_\lambda(\cdot) = \varphi\left(\frac{\cdot}{\sqrt{\lambda}r}\right)$, we conclude that there exists a $t_\lambda \in \mathbb{R}^+$ such that $v_\lambda = t_\lambda \varphi_\lambda \psi \in H_0^1\left(B_{\sqrt{\lambda}}r\right) \subset H_0^1(\Omega_\lambda)$, $v_\lambda \in \widetilde{M}_{1,\Omega_\lambda}$ for λ large enough, i.e.

$$\int_{\Omega_\lambda} |\nabla v_\lambda|^2\,dx = \int_{\Omega_\lambda} v_\lambda f(v_\lambda)\,dx \quad \text{and} \tag{4.3}$$

$$t_\lambda \to 1 \quad \text{as } \lambda \to +\infty, \tag{4.4}$$

where ψ is the ground state of (1.9) achieving $m(1,\mathbb{R}^N)$.

Indeed, for $t \in \mathbb{R}^+$, we define $h_\lambda : \mathbb{R}^+ \to \mathbb{R}$ by

$$h_\lambda(t) = \int_{\Omega_\lambda} |\nabla(\varphi_\lambda\psi)|^2\,dx - \frac{1}{t}\int_{\Omega_\lambda} f(t\varphi_\lambda\psi)\,\varphi_\lambda\psi\,dx.$$

For any fixed $\lambda \in \mathbb{R}^+$, by (f_1) and Lebesgue's Dominated Convergence theorem

$$\lim_{t\to 0+} \frac{1}{t}\int_{\Omega_\lambda} f(t\varphi_\lambda\psi)\,\varphi_\lambda\psi\,dx = 0.$$

So

$$\lim_{t\to 0+} h_\lambda(t) = \int_{\Omega_\lambda} |\nabla(\varphi_\lambda\psi)|^2\,dx > 0. \tag{4.5}$$

Also by (f_1) and Lebesgue's Dominated Convergence theorem

$$\lim_{t\to +\infty} h_\lambda(t) = \int_{\Omega_\lambda} |\nabla(\varphi_\lambda\psi)|^2\,dx - l\int_{\Omega_\lambda} \varphi_\lambda^2\,\psi^2\,dx. \tag{4.6}$$

Since ψ is a solution to (1.9), we have

$$\int_{\mathbb{R}^N} \nabla\psi\,\nabla(\psi\varphi_\lambda^2)\,dx = \int_{\mathbb{R}^N} f(\psi)\,\psi\varphi_\lambda^2\,dx, \tag{4.7}$$

$$\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx = \int_{R^N} f(\psi)\,\psi\,dx. \tag{4.8}$$

Then by (4.7) we have

$$\int_{\Omega_\lambda} |\nabla(\varphi_\lambda\psi)|^2\,dx = \int_{\Omega_\lambda} f(\psi)\,\psi\,\varphi_\lambda^2\,dx + \int_{\Omega_\lambda} |\nabla\varphi_\lambda|^2\,\psi^2\,dx. \tag{4.9}$$

Denote $a_\lambda = \int_{\Omega_\lambda} |\nabla(\varphi_\lambda\psi)|^2\,dx - l\int_{\Omega_\lambda} \varphi_\lambda^2\,\psi^2\,dx$.

So

$$a_\lambda = \int_{\Omega_\lambda} f(\psi)\,\psi\varphi_\lambda^2\,dx + \int_{\Omega_\lambda} |\nabla\varphi_\lambda|^2\,\psi^2\,dx - l\int_{\Omega_\lambda} \varphi_\lambda^2\,\psi^2\,dx. \tag{4.10}$$

It is clear that

$$\lim_{\lambda\to+\infty} \int_{\Omega_\lambda} f(\psi)\,\psi\,\varphi_\lambda^2\,dx = \int_{R^N} f(\psi)\,\psi\,dx. \tag{4.11}$$

We distinguish two cases:

(1) $\psi \in L^2(\mathbb{R}^N)$. In this case, it is easy to show that

$$\lim_{\lambda\to+\infty} \int_{\Omega_\lambda} |\nabla\varphi_\lambda|^2\,\psi^2\,dx = 0 \quad \text{and} \quad \lim_{\lambda\to+\infty} \int_{\Omega_\lambda} \varphi_\lambda^2\psi^2\,dx = \int_{\mathbb{R}^N} \psi^2\,dx.$$

So combining the above equalities with (4.10), (4.11), we have

$$\lim_{\lambda\to+\infty} a_\lambda = \int_{\mathbb{R}^N} f(\psi)\,\psi\,dx - l\int_{\mathbb{R}^N}\psi^2\,dx.$$

By (2.1),

$$\int_{\mathbb{R}^N} f(\psi)\,\psi\,dx - l\int_{\mathbb{R}^N}\psi^2\,dx \leqslant 0$$

and if

$$\int_{\mathbb{R}^N} f(\psi)\,\psi\,dx - l\int_{\mathbb{R}^N}\psi^2\,dx = 0,$$

then $f(\psi) = l\psi$ which contradicts (f_1) (f_2). So $\lim\limits_{\lambda\to+\infty} a_\lambda < 0$.

(2) $\psi \notin L^2(\mathbb{R}^N)$. In this case,

$$\lim_{\lambda\to+\infty}\int_{\Omega_\lambda}\varphi_\lambda^2\,\psi^2\,dx = +\infty, \tag{4.12}$$

$$\begin{aligned}
\int_{\Omega_\lambda}|\nabla\varphi_\lambda|^2\,\psi^2 dx &= \int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}|\nabla\varphi_\lambda|^2\,\psi^2 dx \\
&\leqslant \left(\int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}|\nabla\varphi_\lambda|^N\,dx\right)^{\frac{2}{N}}\left(\int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}|\psi|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}} \\
&= \left(\frac{1}{\sqrt{\lambda}r}\right)^2\left(\int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}\left|\nabla\varphi\left(\frac{x}{\sqrt{\lambda}r}\right)\right|^N dx\right)^{\frac{2}{N}}\left(\int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}|\psi|^{2^*}\,dx\right)^{\frac{N-2}{N}} \\
&= \left(\frac{1}{\sqrt{\lambda}r}\right)^2\left(\int_{B_1\setminus B_{\frac12}}|\nabla\varphi(x)|^N\left(\sqrt{\lambda}r\right)^N dx\right)^{\frac{2}{N}}\left(\int_{B_{\sqrt{\lambda}r}\setminus B_{\frac12\sqrt{\lambda}r}}|\psi|^{2^*}\,dx\right)^{\frac{N-2}{N}} \\
&\leqslant 4^2\left|B_1\setminus B_{\frac12}\right|^{\frac{2}{N}}\left(\int_{B_{\sqrt{\lambda}r}(0)\setminus B_{\frac12\sqrt{\lambda}r}(0)}|\psi|^{2^*}\,dx\right)^{\frac{N-2}{N}} \\
&\to 0 \quad \text{as } \lambda\to+\infty.
\end{aligned} \tag{4.13}$$

Then by (4.10)–(4.13) we have

$$\lim_{\lambda\to+\infty} a_\lambda = -\infty.$$

Therefore, by (4.6), (4.10) and the above argument we get

$$\lim_{t\to+\infty} h_\lambda(t) < 0 \quad \text{for } \lambda \quad \text{large enough.} \tag{4.14}$$

Combining (4.5) with (4.14), and noticing that $h_\lambda(t) \in C^1(\mathbb{R},\mathbb{R})$, we know there exists $t_\lambda \in (0,+\infty)$ such that

$$h_\lambda(t_\lambda) = 0 \text{ for } \lambda > \overline{\lambda}. \tag{4.15}$$

So (4.3) holds.

Notice that no matter which ψ belongs to, (4.13) always holds. By (4.15) and (4.9), we have

$$\int_{\Omega_\lambda} f(\psi)\,\psi\varphi_\lambda^2\,dx + \int_{\Omega_\lambda} |\nabla\varphi_\lambda|^2\,\psi^2\,dx = \frac{1}{t_\lambda}\int_{\Omega_\lambda} f(t_\lambda\psi\varphi_\lambda)\,\psi\varphi_\lambda\,dx.$$

So

$$\int_{R^N} f(\psi)\,\psi\,dx = \lim_{\lambda\to+\infty}\frac{1}{t_\lambda}\int_{\Omega_\lambda} f(t_\lambda\psi\varphi_\lambda)\,\psi\varphi_\lambda\,dx. \tag{4.16}$$

We can easily know that $t_\lambda \nrightarrow +\infty$.

Then by (f_1) (f_2) and the above equality we know $\lim\limits_{\lambda\to+\infty} t_\lambda$ exists and it is finite. We may denote it by α. So $\int_{\mathbb{R}^N} f(\psi)\,\psi\,dx = \frac{1}{\alpha}\int_{\mathbb{R}^N} f(\alpha\psi)\,\psi\,dx$ i.e.

$$\int_{\mathbb{R}^N}\left(\frac{f(\psi)}{\psi} - \frac{f(\alpha\psi)}{\alpha\psi}\right)\psi^2\,dx = 0.$$

Also by (f_2) we know $\alpha = 1$, i.e. (4.4) holds.

Therefore

$$\begin{aligned} m(1,\Omega_\lambda) &\leqslant \int_{\Omega_\lambda}\Big(\frac{1}{2}t_\lambda\psi\varphi_\lambda f(t_\lambda\psi\varphi_\lambda) - F(t_\lambda\psi\varphi_\lambda\Big))dx \\ &\leqslant \int_{\Omega_\lambda}\left(\frac{1}{2}t_\lambda\psi f(t_\lambda\psi) - F(t_\lambda\psi)\right)dx \\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^N}\psi f(\psi)\,dx - \int_{\mathbb{R}^N} F(\psi)\,dx + o(1) \\ &= m(1,\mathbb{R}^N) + o(1), \end{aligned}$$

i.e. (4.2) holds then Lemma 4.2 is proved.

Similar to Lemma 2.3, we have

Lemma 4.3. *For any fixed $\lambda > 0$ and $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$*

$$m^*(\lambda,\rho,\alpha) = \lambda^{1-\frac{N}{2}} m^*\left(1,\sqrt{\lambda}\rho,\alpha\right).$$

Corollary 4.4. *There exists a $\overline{\lambda} > 0$ such that $m^*(\lambda,\rho,\alpha) > m(\lambda,B_\rho(0))$ for any $\lambda \geqslant \overline{\lambda}$ and $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$.*

Proof By Lemma 4.3, we know

$$m^*(\lambda,\rho,\alpha) = \lambda^{1-\frac{N}{2}} m^*\left(1,\sqrt{\lambda}\rho,\alpha\right) \geqslant \lambda^{1-\frac{N}{2}}\inf_{\rho>0} m^*(1,\rho,\alpha).$$

By Lemma 2.3 and Lemma 4.2, we have

$$m(\lambda,B_\rho(0)) = \lambda^{1-\frac{N}{2}}\left(m\left(1,\mathbb{R}^N\right) + o(1)\right).$$

Then by Lemma 4.1, there exists a $\overline{\lambda} > 0$ such that $m^*(\lambda,\rho,\alpha) > m(\lambda,B_\rho(0))$ for any $\lambda \geqslant \overline{\lambda}$ and $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$.

Lemma 4.5. *There exists $\overline{\lambda} > 0$ such that*
$u \in M_{\lambda,\Omega}, I_\lambda(u) \leqslant m(\lambda,B_r(0))$ imply $\beta(u) \in \Omega_r^+$ for $\lambda \geqslant \overline{\lambda}$ and $\lambda \notin \left\{\frac{\lambda_k}{l}\right\}$.

Proof Suppose that $\lambda \geqslant \overline{\lambda}$ and let $\widehat{u}$ be a function such that

$$\widehat{u} \in M_{\lambda,\Omega}, \quad I_\lambda(\widehat{u}) \leqslant m(\lambda, B_r(0)),$$

where $\bar{\lambda}$ is as that in Corollary 4.4. We argue by contradiction and we assume that $\widehat{x} = \beta(\widehat{u}) \notin \Omega_r^+$.

For any $y \in \Omega$,

$$|y - \widehat{x}| = \left| \int_{\mathbb{R}^N} (y - x) \cdot |\nabla \widehat{u}|^2 \, dx \right| / \int_{\mathbb{R}^N} |\nabla \widehat{u}|^2 \, dx \leqslant \text{diam}\Omega.$$

Then $\Omega \subset B_{\text{diam}\Omega}(\widehat{x}) \setminus B_r(\widehat{x}) = B_{\gamma r}(\widehat{x}) \setminus B_r(\widehat{x})$, where $\text{diam}\Omega = \gamma r$, $\gamma > 1$.

Therefore,

$$\begin{aligned} &\inf\left\{I_\lambda(u) : u \in M_{\lambda, B_{\gamma r}(0)\setminus B_r(0)}, \beta(u) = 0\right\} \\ &= \inf\left\{I_\lambda(u) : u \in M_{\lambda, B_{\gamma r}(\widehat{x})\setminus B_r(\widehat{x})}, \beta(u) = \widehat{x}\right\} \\ &\leqslant I_\lambda(\widehat{u}) \leqslant m(\lambda, B_r(0)) \end{aligned}$$

i.e. $m^*(\lambda, r, \gamma) \leqslant m(\lambda, B_r(0))$. This contradicts Corollary 4.4.

To prove our main result, we need the following important proposition which was proposed by Cerami in [8] and stemmed from [2].

Proposition 4.6. *Given H, Ω^+ and Ω^- closed sets with $\Omega^- \subset \Omega^+$ and two continuous maps $\beta : H \to \Omega^+$, $\Psi : \Omega^- \to H$ such that $\beta \circ \Psi$ is homotopically equivalent to the embedding $j : \Omega^- \to \Omega^+$, then $cat_H H \geqslant cat_{\Omega^+}\Omega^-$.*

Proof of Theorem 1.1 By Lemma 2.1 and Lemma 2.2, we know that (P-S) condition also holds on $I_\lambda^{m(\lambda,B_r(0))}$. By Ljusternik–Schnirelmann theory, we know that I_λ on $I_\lambda^{m(\lambda,B_r(0))}$ has at least $\text{cat}\left(I_\lambda^{m(\lambda,B_r(0))}\right)$ distinct critical points whose energy are less than or equal to $m(\lambda, B_r(0))$.

Applying Proposition 4.6 to $H = I_\lambda^{m(\lambda,B_r(0))}$ we have that

$$\text{cat}\left(I_\lambda^{m(\lambda,B_r(0))}\right) \geqslant \text{cat}(\Omega) \ \text{ for } \lambda \geqslant \overline{\lambda} \text{ and } \lambda \notin \left\{\frac{\lambda_k}{l}\right\}, \tag{4.17}$$

where $\overline{\lambda}$ is as in Lemma 4.5. In fact, if we take $H = I_\lambda^{m(\lambda,B_r(0))}$, then by Lemma 4.5 we know $\beta : H \to \Omega_r^+$. On the other hand, by the definition of $\Phi_{\lambda,\rho}$, if we take $\Psi = \Phi_{\lambda,r}$, then $\Psi : \Omega_{2r}^- \to H$. And we also know that $\beta \circ \Psi = j$. So Proposition 4.6 implies (4.17) since Ω_r^+ and Ω_{2r}^- are topologically equivalent to Ω (i.e. $\text{cat}_{\Omega_r^+}\Omega_{2r}^- = \text{cat}_{\overline{\Omega}}\overline{\Omega} = \text{cat}\Omega$).

So, I_λ has at least $\text{cat}\Omega$ distinct critical points whose energy are less than or equal to $m(\lambda, B_r(0))$. Then (1.1) has at least $\text{cat}\Omega$ positive solutions by the maximum principle.

To get one more critical point when Ω is not contractible, we could consider the set $\Gamma = \Psi(\Omega_{2r}^-)$. It is clear that $\Psi(\Omega_{2r}^-)$ is compact and closed.

We claim that Γ is not contractible in $I_\lambda^{m(\lambda,B_{\frac{r}{2}}(0))}$.

To this end, we suppose, by contradiction, that Γ is contractible in $I_\lambda^{m(\lambda,B_{\frac{r}{2}}(0))}$, i.e. there is a homotopic mapping $h : [0,1] \longrightarrow I_\lambda^{m(\lambda,B_{\frac{r}{2}}(0))}$ such that $h(0,x) = x$,

$h(1,x)=\omega\in I_\lambda^{m(\lambda,B_{\frac{r}{2}}(0))}$ $\forall x\in\Gamma$. By Lemma 4.5, there exists a $\widetilde{\lambda}>0$, such that for any $\lambda\geqslant\widetilde{\lambda}$ and $\lambda\notin\{\frac{\lambda_k}{l}\}$, $\beta(I_\lambda^{m(\lambda,B_{\frac{r}{2}}(0))})\subset\Omega^+_{r/2}$.

Define homotopic mapping $H:[0,1]\times\Omega^-_{2r}\to\Omega^+_{r/2}$ by

$$H(t,x)=\beta(h(t,\psi(x))).$$

Then

$$\begin{aligned}H(0,x)&=\beta(h(0,\psi(x)))=\beta(\psi(x))=j(x)=x\\ H(1,x)&=\beta(h(1,\psi(x)))=\beta(\omega)\in\Omega^+_{\frac{r}{2}}.\end{aligned}$$

This implies Ω^-_{2r} is contractible in $\Omega^+_{\frac{r}{2}}$, which contradicts to our assumption that Ω is not contractible.

Now we choose v^* to be the function achieving $m(\lambda,\frac{r}{4})$,

$$v^*(x)=\begin{cases}u_{\lambda,\frac{r}{4}}(|x-y_0|), & x\in B_{\frac{r}{4}}(y_0)\\ 0, & x\in\Omega\setminus B_{\frac{r}{4}}(y_0).\end{cases}$$

We also can choose suitable $y_0\in\Omega$ such that $B_{\frac{r}{4}}(y_0)\cap\Omega^-_r=\varnothing$, i.e. $\forall v\in\Gamma$,

$$\operatorname{supp}v^*\cap\operatorname{supp}v=\varnothing. \tag{4.18}$$

So $v^*\in M_{\lambda,\Omega}$, $v^*\notin\Gamma, v^*\geqslant0$. Define $\Theta=\{\theta v^*+(1-\theta)v: v\in\Gamma,\theta\in[0,1]\}$.

Claim 1 Θ is compact. In fact, $\forall\{\omega_n\}\subset\Theta$, there exist $\theta_n\in[0,1]$ and $v_n\in\Gamma$ such that $\omega_n=\theta_nv^*+(1-\theta_n)\,v_n$. Due to the compactness of Γ and $[0,1]$, there are subsequence of $\{\theta_n\}$ and $\{v_n\}$ which remain denoted by $\{\theta_n\}$ and $\{v_n\}$ such that $\theta_n\to\theta$, $v_n\to v$ in $H^1_0(\Omega)$ for some $\theta\in[0,1]$ and $v\in\Gamma$. Therefore, $\omega_n\to\theta v^*+(1-\theta)v\in\Theta$ in $H^1_0(\Omega)$. This implies Θ is compact.

Claim 2 $0\notin\Theta$. If $0\in\Theta$, then $\exists\theta\in[0,1]$, $v\in\Gamma$, such that $\theta v^*+(1-\theta)\,v=0$, i.e. $\theta v^*=-(1-\theta)\,v$. Then $v\leqslant0$ in Ω. This is impossible.

Claim 3 $\forall u\in\Theta$, there exists only one $t\in\mathbb{R}^+$ such that $tu\in M_{\lambda,\Omega}$, i.e. there is a mapping $t:\Theta\to\mathbb{R}^+$. Furthermore, t is a continuous mapping. In particular, $t(v^*)=1$, $t(v)=1$ for any $v\in\Gamma$.

Indeed, $\forall u\in\Theta$, let

$$h(t)=\int_\Omega|\nabla u|^2\,dx-\lambda\int_\Omega\frac{uf(tu)}{t}\,dx.$$

By (f_1), (f_2),

$$\left|\frac{uf(tu)}{t}\right|\leqslant lu^2,\quad \lim_{t\to0+}\frac{uf(tu)}{t}=0,\quad \lim_{t\to+\infty}\frac{uf(tu)}{t}=lu^2.$$

Lebesgue's Dominated Theorem implies that

$$\begin{aligned}\lim_{t\to0+}h(t)&=\int_\Omega|\nabla u|^2\,dx>0\\ \lim_{t\to+\infty}h(t)&=\int_\Omega|\nabla u|^2\,dx-\lambda l\int_\Omega u^2\,dx.\end{aligned}$$

Since $u\in\Theta$, there is $\theta\in[0,1]$, $v\in\Gamma$ such that $u=\theta v^*+(1-\theta)\,v$. By (4.18), we have

$$\int_\Omega\nabla v\nabla v^*\,dx=0,\quad\int_\Omega vv^*\,dx=0.$$

Therefore

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\,dx - \lambda l \int_\Omega u^2\,dx \\
&= \int_\Omega \left(\theta^2|\nabla v^*|^2 + (1-\theta)^2|\nabla v|^2\right)\,dx - \lambda l \left(\theta^2 \int_\Omega (v^*)^2\,dx + (1-\theta)^2|\nabla v|^2\,dx\right) \\
&= \theta^2 \left[\int_\Omega \lambda v^* f(v^*)\,dx - \lambda l \int_\Omega (v^*)^2 dx\right] + (1-\theta)^2 \left[\int_\Omega \lambda v f(v)\,dx - \lambda l \int_\Omega v^2\,dx\right] \\
&< 0
\end{aligned}$$

by (f_2), i.e. $\lim_{t\to+\infty} h(t) < 0$.

By the mean value theorem, there exists a $t \in \mathbb{R}^+$, such that $h(t) = 0$, i.e. $tu \in M_{\lambda,\Omega}$.

If there exist t_1, $t_2 \in \mathbb{R}^+$, $t_1 < t_2$ such that $t_1 u \in M_{\lambda,\Omega}$, $t_2 u \in M_{\lambda,\Omega}$, i.e. $h(t_1) = h(t_2) = 0$. This implies

$$\int_\Omega \frac{uf(t_1 u)}{t_1}\,dx = \int_\Omega \frac{uf(t_2 u)}{t_2}\,dx.$$

On the other hand, by (f_2), we know

$$\int_\Omega \frac{uf(t_1 u)}{t_1}\,dx < \int_\Omega \frac{uf(t_2 u)}{t_2}\,dx.$$

A contradiction! So t is unique.

Now suppose u, $u_n \in \Theta$ $(n = 1, 2, \cdots)$ such that $u_n \to u$ in $H_0^1(\Omega)$. Then

$$\int_\Omega |\nabla u_n|^2\,dx - \lambda \int_\Omega \frac{u_n f(t(u_n)u_n)}{t(u_n)}\,dx = 0.$$

Let $n \to \infty$, we get

$$\int_\Omega |\nabla u|^2\,dx - \lambda \lim_{n\to\infty} \int_\Omega \frac{u_n f(t(u_n)u_n)}{t(u_n)}\,dx = 0. \tag{4.19}$$

It is easy to know that $\limsup_{n\to\infty} t(u_n) \neq +\infty$.

Set $\liminf_{n\to\infty} t(u_n) = t_1$, $\limsup_{n\to\infty} t(u_n) = t_2$. We deduce from (4.19) that

$$\int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega \frac{uf(t_1 u)}{t_1}\,dx = 0$$

$$\int_\Omega |\nabla u|^2\,dx - \lambda \int_\Omega \frac{uf(t_2 u)}{t_2}\,dx = 0.$$

Then by the uniqueness, $t_1 = t_2 = t(u)$, i.e. $\lim_{n\to\infty} t(u_n) = t(u)$.

Denote $\Lambda = \{t(u)u : u \in \Theta\}$. It is easy to know that Λ is compact due to the compactness of Θ and $\Gamma \subset \Lambda \subset M_{\lambda,\Omega}$. Define $\gamma(\lambda) = \max\{I_\lambda(\omega) : \omega \in \Lambda\}$. Then $\gamma(\lambda) < +\infty, \gamma(\lambda) \geqslant I_\lambda(v^*) = m(\lambda, \frac{r}{4}) > m(\lambda, \frac{r}{2})$.

We claim that Γ is contractible in $I_\lambda^{\gamma(\lambda)}$.

To this end, define homotopic mapping $\bar{h} : [0,1] \times \Gamma \to I_\lambda^{\gamma(\lambda)}$ by $\bar{h}(s,v) = t(sv^* + (1-s)\,v)(sv^* + (1-s)\,v)$, where t is the mapping in Claim 3. $\bar{h}$ is continuous due to Claim 3. Furthermore,

$$\bar{h}(0,v) = t(v)v = v, \quad \bar{h}(1,v) = t(v^*)v^* = v^* \quad \forall v \in \Gamma.$$

So Γ is contractible in $I_\lambda^{\gamma(\lambda)}$.

Denote $K_c = \{u \in M_{\lambda,\Omega} : I_\lambda(u) = c,\ dI_\lambda\big|_{M_{\lambda,\Omega}}(u) = 0\}$.
(i) If $K_{m(\lambda,B_{r/2}(0))} \neq \varnothing$ or $K_{\gamma(\lambda)} \neq \varnothing$, then I_λ has at least catΩ +1 distinct critical points.
(ii) If $K_{m(\lambda,B_{r/2}(0))} = \varnothing$ and $K_{\gamma(\lambda)} = \varnothing$, then there must be at least one critical value $c \in [m(\lambda, B_{\frac{r}{2}}(0)), \gamma(\lambda)]$.

In fact, suppose that there is no critical value between $m(\lambda, B_{\frac{r}{2}}(0))$ and $\gamma(\lambda)$. Using deformation lemma on Finsler manifold (see [19]) and the theorem of finite covering of Heine–Borel, we have there exists $\delta > 0$ such that $\forall \bar{\varepsilon}, \varepsilon > 0$ with $0 < \varepsilon < \bar{\varepsilon} < \delta$, there exists a continuous mapping $\eta = \eta_{\varepsilon,\bar{\varepsilon}} : [0,1] \times I_\lambda^{\gamma(\lambda)} \to I_\lambda^{\gamma(\lambda)}$ such that

$$\begin{aligned} &(1) \qquad \eta(0,x) = x \quad \forall x \in I_\lambda^{\gamma(\lambda)} \\ &(2) \qquad \eta(t,x) = x \quad \forall x \in I_\lambda^{m(\lambda,B_{r/2}(0))-\bar{\varepsilon}} \\ &(3) \qquad \eta(1, I_\lambda^{\gamma(\lambda)}) \subset I_\lambda^{m(\lambda,B_{r/2}(0))-\varepsilon}. \end{aligned}$$

We can choose δ satisfying $0 < \delta \leqslant m(\lambda, B_{\frac{r}{2}}(0)) - m(\lambda, B_r(0))$.

Define the homotopic mapping $H : [0,1] \times \Gamma \to I_\lambda^{m(\lambda,B_{r/2}(0))}$ by

$$H(t,x) = \eta(1, \bar{h}(t,x)),$$

where $\bar{h}$ is defined above. Because $\Gamma \subset I_\lambda^{m(\lambda,B_r(0))} \subset I_\lambda^{m(\lambda,B_{r/2}(0))-\bar{\varepsilon}}$, we have

$$\begin{aligned} H(0,x) &= \eta(1, \bar{h}(0,x)) = \eta(1,x) = x \quad \forall x \in \Gamma \\ H(1,x) &= \eta(1, \bar{h}(1,x)) = \eta(1, v^*) \in I_\lambda^{m(\lambda,B_{r/2}(0))}. \end{aligned}$$

Then Γ is contractible in $I_\lambda^{m(\lambda,B_{r/2}(0))}$. A contradiction!

Therefore, I_λ has at least cat$\Omega + 1$ distinct critical points. By maximum principle, we have that (1.1) has at least cat$\Omega + 1$ distinct positive solutions. Thus we complete the proof.

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