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BEHAVIOR OF QUASICONFORMAL MAPPINGS AT INFINITY

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1. Introduction

Based on the recent work [2], the behavior of quasiconformal mappings between proper Q-regular spaces that admit a Q-Poincaré inequality is well understood. Also see [1] for the case of domains in the euclidean space. However, very little is known if the volume growth at large is different from the local volume growth or when the Q-Poincaré inequality only holds at small scales. Both of these situations can happen, for example, when we deal with a complete manifold whose Ricci curvature is bounded from below by a negative number.

We consider the following setting. We are given two metric spaces X, Y which are both proper, meaning that closed balls are compact, pathwise connected, equipped with the path metric, and unbounded. We assume that X has bounded Q-geometry:

$$r^Q/C_\mu \le \mu\big(B(x,r)\big) \le C_\mu r^Q \tag{1}$$

for each $x \in X$ and all $0 < r < 1$, and

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \le C_Q \operatorname{diam}(B)\left(\frac{1}{\mu(B)}\int_B g^Q\,d\mu\right)^{1/Q} \tag{2}$$

whenever u is a continuous function on a ball $2B$ of diameter at most 2 and g is an upper gradient of u on $2B$. Here μ refers to the Q-dimensional Hausdorff measure on X and g is an upper gradient of u on $2B$ if

$$|u(x)-u(y)| \le \int_\gamma g\ ds$$

whenever γ is a rectifiable curve that joins x to y in $2B$. There will be two more standing assumptions on Y. We assume that for each $y \in Y$ there is a radius $r_y > 0$ so that

$$r^Q/C_\nu \le \nu\big(B(y,r)\big) \le C_\nu r^Q \tag{3}$$

for $0 < r < r_y$; here ν is the Q-dimensional Hausdorff measure. We further assume the connectivity property that, given $y \in Y$ and $r_0 < r < R$, each point in $\partial B(y,r)$ can be joined by a curve in $Y \setminus B(y, r/C_c)$ to some point in $Y \setminus B(y,R)$. Here the constants r_0, C_c are naturally required to be independent of y. The connectivity condition is essentially a requirement on the ends of the space.

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Recall that a homeomorphism $f : X \to Y$ is quasiconformal if

$$H_f(x) = \limsup_{r\to 0} H_f(x,r) \le H < \infty \tag{4}$$

for some fixed constant H and all $x \in X$. Here

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)},$$

and

$$L_f(x,r) := \sup\{d_Y\big(f(x), f(y)\big) : d_X(x,y) \le r\},$$

$$l_f(x,r) := \inf\{d_Y\big(f(x), f(y)\big) : d_X(x,y) \ge r\}.$$

The following theorem gives a growth estimate on quasicoformal mappings of X onto Y.

Theorem 1.1. *Let f be a quasiconformal mapping of X onto Y. Fix a point $x_0 \in X$. If $\nu(B(y,r)) \le C_0 r^\lambda$ for all $y \in Y$ and each $r \ge 1$ with some fixed $1 \le \lambda \le Q$, then*

$$d_Y\big(f(x), f(x_0)\big) \le C_1 \phi_\lambda\big(d_X(x,x_0)\big)$$

for all $x \in X$ with $d_X(x,x_0) \ge 1$, where

$$\phi_\lambda(t) = t^{(Q-1)/(Q-\lambda)}$$

when $\lambda < Q$ and

$$\phi_Q(t) = \exp(C_1 t).$$

Here C_1 depends on the constants associated with the data, and on diam$\big(f(B(x_0,1)\big)$.

These bounds on the distortion of the distances under f are essentially sharp, see the discussion at the end of the next section. Also, the connectivity assumption given before Theorem 1.1 cannot be omitted.

The question addressed in this note rose from the discussions the author had with Thierry Coulhon while visiting Université de Cergy-Pontoise in 1998. He would like to thank Thierry and the department there for the hospitality. Thanks are also due to the referee of this paper for constructive suggestions.

2. The proof of Theorem 1.1 and related comments

The proof of Theorem 1.1 is based on the following lemma and on the ideas in the proof of the global quasisymmetry of quasiconformal mappings between spaces with "globally" bounded Q-geometry in [2], [3].

Lemma 2.1. *Let E, $F \subset X$ be continua, both of diameter no less than one, and of distance at least one, and let u be continuous with $u = 0$ in E and $u = 1$ in F, and g an upper gradient of u. Then*

$$\int_X g^Q \, d\mu \ge C d_X(E,F)^{1-Q}.$$

Proof. Pick $x \in E$ and $y \in F$ with $d_X(x,y) = d(E,F)$. If $u_{B(x,1)} \geq 1/3$ or if $u_{B(y,1)} \leq 2/3$, then (the proof of) Theorem 5.9 in [3] shows that

$$\int_X g^Q \, d\mu \geq \delta(X) > 0,$$

and the claim follows. Otherwise, we join x to y by a curve γ of length no more than $2d_X(x,y)$ and select k balls $B_1, B_2, ..., B_k$, all of radius one and with centers on γ so that $k \leq 4d_X(x,y)$ and $\mu(B_i \cap B_{i+1}) \geq \delta_0(C_{Q,\mu}) > 0$. We may assume that $B_1 = B(x,1)$ and $B_k = B(y,1)$. By comparing averages and using the triangle, Poincaré (inequality (2)) and Hölder inequalities we conclude that

$$\begin{aligned} 1/3 \leq \sum_1^{k-1} |u_{B_{i+1}} - u_{B_i}| &\leq C \sum_1^k \left(\int_{B_i} g^Q \, d\mu \right)^{1/Q} \\ &\leq C k^{(Q-1)/Q} \left(\int_X g^Q \, d\mu \right)^{1/Q} . \end{aligned}$$

Here C depends on C_μ, C_Q, δ_0. The claim follows.

The above estimate may seem weak, but it turns out to be sharp. Consider, for example, the planar space $X = \{(x,y) : \ |y| \leq 1\}$ which has bounded 2-geometry and whose volume growth is linear at large scales. It is easy to check that the exponent $1-Q$ (-1 in this case) in Lemma 2.1 is sharp for X. Assuming that X satisfies (1) for all radii does not help either: consider the planar space $A = \{(x,y) : \ 0 \leq x, \ |y| \leq u(x)\}$ with $u(x) = 1$ if the integer part of x is even and $u(x) = x$ otherwise, and set X to be the closure of A in the plane. We leave it to the reader to check that the exponent in Lemma 2.1 is again sharp.

Let us now sketch the proof of Theorem 1.1. Fix x with $d(x,x_0) \geq 1$. We may assume that $d\big(f(x), f(x_0)\big) \geq C_c\big(r_0 + 2\mathrm{diam}(f(B(x_0,1)))\big) + 2$ where C_c, r_0 are the constants in the connectivity assumption given before Theorem 1.1.

Notice first that the quasiconformality and the local volume estimate (3) guarantee that

$$\mathrm{diam}\big(f(B(z,r)\big)^Q/C \leq \nu\big(f(B(z,r)\big) \leq C\mathrm{diam}\big(f(B(z,r)\big)^Q \tag{5}$$

with $C = C(H, C_\nu)$ whenever $z \in X$ and $0 < r < r_z$, where r_z may depend on f, z. Considering such balls and their images, the arguments in [2], [3] give us a collection of balls B_i and sets $\frac{1}{5}B_i \subset V_i \subset B_i$ so that (5) holds for each ball B_i in our collection, $\nu\big(f(B_i)\big) \leq C_1 \nu\big(f(V_i)\big)$, $\mathrm{diam}\big(f(B_i)\big) \leq \min\{1/10, r_0/10\}$, $X \subset \cup_i V_i$, and so that no point in X belongs to more than two of the sets V_i. Here C_1 only depends on the data. We define

$$\rho(z) = \sum_i \frac{\mathrm{diam}\big(f(B_i)\big)}{\mathrm{diam}(B_i)} v_\lambda\big(f(B_i)\big) \chi_{f^{-1}(\overline{B}(f(x_0), d(f(x), f(x_0)))}(z)\, \chi_{2B_i}(z), \tag{6}$$

where v_λ will be given momentarily. The factor $\chi_{f^{-1}(\overline{B}(f(x_0), d(f(x), f(x_0)))}(z)$ in (6) is included to simplify the definitions of the terms v_λ; notice that the balls B_i cover all of X. If $\lambda < Q$, we set

$$v_\lambda\big(f(B_i)\big) = 1/d\big(f(x), f(x_0)\big),$$

and we define

$$v_Q\big(f(B_i)\big) = \frac{1}{d\big(f(B_i), f(x_0)\big)\log d\big(f(x), f(x_0)\big)}$$

for those B_i for which $f(B_i) \cap B\big(f(x_0), \operatorname{diam}(f(B(x_0,1)))\big) = \emptyset$ and we set

$$v_Q\big(f(B_i)\big) = 0$$

for the remaining balls B_i. Reasoning as in [2] it is then easy to check using our initial assumption on $d\big(f(x), f(x_0)\big)$ that

$$\int_\gamma \rho \; ds \geq \delta > 0 \tag{7}$$

for each rectifiable curve that joins the set $E = \overline{B}(x_0, 1)$ to the set

$$f^{-1}\big(Y \setminus B(f(x_0), d(f(x), f(x_0))/C_c)\big).$$

Here $\delta > 0$ is a constant only depending on the data. Notice that, by the connectivity assumption on Y and the fact that f and f^{-1} map bounded sets to bounded sets (recall here that closed balls are compact), the set $f^{-1}\big(Y \setminus B(f(x_0), d(f(x), f(x_0))/C_c)\big)$ contains a continuum F of diameter at least one and with $x \in F$. It then follows from Lemma 2.1 and Proposition 2.17 in [3], according to which estimates as in Lemma 2.1 can be used for functions satisfying (7), that

$$\int_X \rho^Q \; d\mu \geq C d(x_0, x)^{1-Q}. \tag{8}$$

The desired growth estimate on f thus follows if we can prove a suitable upper bound on the integral of ρ^Q. To this end, observe first that the functions $\chi_{2B_i}(z)$ in equation (6) can – as far as upper bounds on this integral are concerned – be replaced with $\chi_{\frac{1}{5}B_i}$ by using a maximal function argument, see the page 67 in [2]. Then, the volume growth condition (1) and inequality (5) yield the estimate

$$\int_X \rho^Q \; d\mu \leq C \sum_i^{*} \nu\big(f(B_i)\big)\, v_\lambda\big(f(B_i)\big),$$

where the sum is taken over those indices i for which

$$f\big(\frac{1}{5}B_i\big) \cap \overline{B}\big(f(x_0), d(f(x), f(x_0))\big) \neq \emptyset,$$

and the bounded overlap property of the sets V_i together with the doubling-type estimate $\nu\big(f(B_i)\big) \leq C_1 \nu\big(f(V_i)\big)$ reduce us to an estimate in Y. Taking the assumption $\nu\big(B(y,r)\big) \leq C r^\lambda$ for $r \geq 1$ and the explicit formula for $v_\lambda\big(f(B_i)\big)$ into account gives us the estimate

$$\int_X \rho^Q \; d\mu \leq h_\lambda\big(d(f(x), f(x_0)\big) \tag{9}$$

with $h_\lambda(t) = C t^{\lambda - Q}$ for $\lambda < Q$ and $h_Q(t) = C\big(\log d(f(x), f(x_0)\big)^{1-Q}$. The claim follows by combining (8) with (9).

Let us briefly discuss the sharpness of the growth rates in Theorem 1.1. Set $X = \{(x,y) : \; 0 \leq x \text{ and } 0 \leq y \leq \pi\}$. Then X equipped with the Lebesgue area satisfies the assumptions of Theorem 1.1 with $Q = 2$, and this also holds for $Y_\lambda = f_\lambda(X) \subset \mathbf{R}^2$, $1 < \lambda < 2$, when

$$f_\lambda(z) = z|z|^{(\lambda-1)/(2-\lambda)}$$

and for $Y_2 = f(X) \subset \mathbf{R}^2$ with $f(z) = e^z$. Thus the indicated rates cannot be improved on. One can give similar examples in higher dimensions.

We close this note by commenting on the necessity of the connectivity assumption we have posed on Y. First of all, the reader familiar with the lingo of quasiconformal mappings notices that this is a weak version of the second condition in the concept of linear local connectivity; the first condition of this concept automatically holds in our setting because we assumed that the metrics are path metrics. To see that we need to assume that points outside large balls centered at $f(x_0)$ can be joined to some point far away from $f(x_0)$ without essentially entering the balls in question, consider, for example, the following setting. Let $X = \{(x, y) : \ 0 \le y \le \pi\}$. Our space Y will be the union of X and closed rectangles in the upper half plane, parallel to the coordinate axes, based on the intervals $[2^j, 2^j + 1]$ and of height 2^{2j}, $j = 1, 2, \cdots$. It is easy to construct a quasiconformal mapping f of X onto Y so that f is the identity on the real axis and f maps the points $\left(2^j + \frac{1}{2}, \pi\right)$ to points on the tops of the corresponding rectangles. Moreover, X, Y satisfy the assumptions (except for the connectivity assumption on Y) of Theorem 1.1 with $\nu\big(B(y, r)\big) \le Cr \log r$ for large $r \ge 1$ when ν is the Lebesgue measure. Similar examples can be constructed for other volume growth bounds. In higher dimensions, one can make constructions of this type by quasiconformally erecting cylindrical towers on the top of a horizontal cylinder, see e.g. [4] for the existence of such maps.

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