

Papers on Analysis

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On Lipschitz Continuity of Quasiconformal Mappings

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Abstract

We study the local growth of quasiconformal mappings in the plane. Estimates are given in terms of integral means of the pointwise angular dilatations. New sufficient conditions for a quasiconformal mapping f to be either Lipschitz or weakly Lipschitz continuous at a point are given.

1. INTRODUCTION

Let Ω be a domain in the complex plane \mathbb{C} . It is a classical result that every K -quasiconformal mapping $f : \Omega \rightarrow \mathbb{C}$ is locally Hölder continuous with exponent $1/K$, which is the best possible, see [1], namely,

$$(1.1) \quad |f(z) - f(z_0)| \leq C|z - z_0|^{1/K}.$$

If one allows the exponent in (1.1) to vary with respect to the point z_0 , one could get more precise information about the local behavior of f at z_0 . For instance, Belinskii [7, p. 37] showed that if a Beltrami coefficient μ is continuous, then the corresponding quasiconformal mapping f will be weakly Lipschitz continuous, that is, locally Hölder continuous with every exponent less than one. Iwaniec [12] extended this result to the case when μ has vanishing mean oscillation in the sense of Sarason [18]. Moreover, the weak Lipschitz continuity remains to hold for the higher-dimensional quasiconformal mappings if either the dilatation tensors or the matrix dilatations have vanishing mean oscillation [11].

In this paper we study the local behavior of plane quasiconformal mappings by employing the angular and the radial dilatations. One of our main results here is a local growth estimate for a K -quasiconformal mapping, which implies an estimate for the modulus of continuity in terms of the integral means of the angular dilatations. We will observe that the modulus of continuity of f in the form $O(|z - z_0| \log(1/|z - z_0|))$ at a given point z_0 does not necessarily follow from the continuity of μ , which implies $\mu \in \text{VMO}$, for the corresponding quasiconformal mapping f (see Section 4). This observation answers negatively

to a problem on the modulus of continuity, which was discussed during the XVIII-th Rolf Nevanlinna Colloquium held in Helsinki, 2000. Moreover, it is shown that the weak Lipschitz continuity of f at z_0 is the best possible under the assumption of the approximate continuity of μ (see Theorem 4.12).

Some of our estimates in the following are independent of the maximal dilatations K of the quasiconformal mappings. Then, in most cases, by a standard approximation procedure, we can extend the results for those homeomorphisms in the Sobolev class $W_{\text{loc}}^{1,1}(\Omega)$ for which the relevant quantities are bounded. For simplicity, we will not pursue this direction here.

We recall some basic definitions and notation. A sense-preserving topological embedding $f : \Omega \rightarrow \mathbb{C}$ is called quasiconformal if $f \in W_{\text{loc}}^{1,2}(\Omega)$ and if f satisfies the Beltrami equation

$$f_{\bar{z}} = \mu(z) f_z \quad \text{a.e. in } \Omega$$

for a complex-valued measurable function μ with $\|\mu\|_{\infty} < 1$, which is called the complex dilatation of f or the Beltrami coefficient of f . More specifically, f is called K -quasiconformal if $\|K_{\mu}\|_{\infty} \leq K$, where $K_{\mu}(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|)$.

For a given μ in Ω , we define the *angular dilatation* D_{μ,z_0} of μ at $z_0 \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$D_{\mu,z_0}(z) = \frac{\left|1 - \mu(z) \frac{\bar{z} - \bar{z}_0}{z - z_0}\right|^2}{1 - |\mu(z)|^2}$$

in the case when z_0 is finite, and by $D_{\mu,\infty}(z) = D_{\mu,0}(z)$ in the case when $z_0 = \infty$. For brevity, we will write $D_{\mu,0} = D_{\mu}$. The angular dilatation is a measurable function in Ω and satisfies the inequality

$$(1.2) \quad 1/K_{\mu}(z) \leq D_{\mu,z_0}(z) \leq K_{\mu}(z)$$

a.e. in Ω for each $z_0 \in \widehat{\mathbb{C}}$. The name of D_{μ,z_0} comes from the following important relation: if f is a quasiconformal mapping in Ω with complex dilatation μ and if we write $z = z_0 + re^{i\theta}$, then

$$(1.3) \quad \left| \frac{\partial f}{\partial \theta}(z) \right|^2 = r^2 D_{\mu,z_0}(z) J_f(z)$$

holds for almost all $z \in \Omega$, where J_f stands for the Jacobian of f . The *radial dilatation* of μ at z_0 is defined as $D_{-\mu,z_0}(z)$ and its name comes from the relation

$$(1.4) \quad \left| \frac{\partial f}{\partial r}(z) \right|^2 = D_{-\mu,z_0}(z) J_f(z) \quad \text{a.e. in } \Omega.$$

The idea of employing the directional dilatations for the study of quasiconformal mappings in the plane is due to Andreian Cazacu [3] and Reich and Walczak [17], and similar ideas are used in [19], [8] and [10].

2. GROWTH ESTIMATES

Given a Beltrami coefficient μ in the unit disk \mathbb{D} centered at the origin with $\|\mu\|_\infty \leq (K-1)/(K+1)$, we define $Q_\mu(r, R)$ by

$$Q_\mu(r, R) = \frac{1}{2\pi \log(R/r)} \iint_{r < |z| < R} \frac{D_\mu(z)}{|z|^2} dx dy$$

for $0 < r < R \leq 1$. This quantity can be regarded as the integral average of the angular dilatation $D_\mu(z)$ over the annulus $r < |z| < R$ with respect to the logarithmic measure. In particular, since $K^{-1} \leq D_\mu(z) \leq K$, we see that $K^{-1} \leq Q_\mu(r, R) \leq K$ holds.

2.1. Theorem. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ . Then*

$$|f(z)| \leq C \left(\frac{|z|}{R} \right)^{1/Q_\mu(|z|, R)}, \quad |z| < R,$$

for an absolute constant $0 < C \leq 64$.

In particular, because of (1.2), we immediately obtain

$$(2.2) \quad |f(z)| \leq C \left(\frac{|z|}{R} \right)^{1/P_\mu(|z|, R)}$$

for all $|z| < R$. Here

$$P_\mu(r, R) = \frac{1}{2\pi \log(R/r)} \iint_{r < |z| < R} \frac{K_\mu(z)}{|z|^2} dx dy.$$

We remark, in particular, that $\max\{1, Q_\mu(r, R)\} \leq P_\mu(r, R) \leq K$. Now the aforementioned Hölder continuity estimate

$$(2.3) \quad |f(z)| \leq C|z|^{1/K}, \quad z \in \mathbb{D},$$

is a simple consequence of (2.2).

We do not discuss here the sharp constant C in the case if we allow it to depend on $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. However, we make a few remarks here. One has the expectation that if $f : \mathbb{D} \rightarrow \mathbb{D}$ is a K -quasiconformal homeomorphism, normalized by $f(0) = 0$, then inequality (2.3) holds with $C = 16^{1-1/K}$. It is known that 16 is the smallest constant independent of K in this case, see, e.g., [14, p. 68]. Currently it is known that one can choose $C = 64^{1-1/K}$. More details are given in [21, p. 33].

The proof of Theorem 2.1 is based on the following statement which may be of independent interest.

2.4. Proposition. *Let $A = \{z : r < |z| < R\}$ and $f : A \rightarrow \mathbb{C}$ be a quasiconformal embedding with complex dilatation μ . Then*

$$(2.5) \quad \frac{1}{Q_\mu(r, R)} \leq \frac{\text{mod } f(A)}{\text{mod } A} \leq Q_{-\mu}(r, R).$$

We remark that the above result is equivalent to the following inequalities:

$$\begin{aligned} -\frac{1}{2\pi} \iint_A \frac{D_{-\mu}(z) - 1}{|z|^2} dx dy &\leq \text{mod } A - \text{mod } f(A) \\ &\leq \frac{\text{mod } f(A)}{\text{mod } A} \cdot \frac{1}{2\pi} \iint_A \frac{D_\mu(z) - 1}{|z|^2} dx dy. \end{aligned}$$

Proof. For simplicity, let us assume that $r = 1$ and set $m = \text{mod } A = \log R$. We first recall the following estimate due to [17]:

$$(2.6) \quad \text{mod } f(A) \geq \int_1^R \frac{dr}{r\psi(r)},$$

where

$$\psi(r) = \frac{1}{2\pi} \int_0^{2\pi} D_\mu(re^{i\theta}) d\theta.$$

A complete proof of (2.6) under somewhat weaker assumptions is also contained in [10]. Here we just sketch the proof of (2.6) for the reader's convenience. First we may assume that $f(A) = \{1 < |w| < R'\}$. Let γ_r be the circle $|z| = r$. Then

$$2\pi \leq \int_{\gamma_r} |d \arg f| \leq \int_0^{2\pi} \left| \frac{f_\theta(re^{i\theta})}{f(re^{i\theta})} \right| d\theta$$

for almost all $r \in (1, R)$. Noting (1.3) and making use of Schwarz' inequality, we get

$$\frac{2\pi}{r\psi(r)} \leq \int_0^{2\pi} J_f |f|^{-2} r d\theta.$$

If we integrate both sides of the above inequality with respect to the parameter r over $(1, R)$, we obtain the required estimate.

Next, the observation

$$\frac{1}{2\pi} \iint_A \frac{D_\mu(z)}{|z|^2} dx dy = \int_1^R \frac{\psi(r) dr}{r} = \int_0^m \psi(e^t) dt,$$

and the estimate

$$m^2 \leq \int_0^m \psi(e^t) dt \cdot \int_0^m \frac{dt}{\psi(e^t)},$$

which is a consequence of Schwarz' inequality, imply

$$(2.7) \quad \int_0^m \frac{dt}{\psi(e^t)} \geq \frac{m^2}{\frac{1}{2\pi} \iint_A \frac{D_\mu(z)}{|z|^2} dx dy} = \frac{m}{Q_\mu(1, R)}.$$

Combining inequalities (2.6) and (2.7), we obtain the right-hand side in (2.5). The left-hand side of inequality (2.5) can be handled similarly by using (1.4), cf. [17]. \square

The following auxiliary lemma will play an important role.

2.8. Lemma ([10]). *Let B be a ring domain in \mathbb{C} whose complement in $\widehat{\mathbb{C}}$ consists of the bounded component E_0 and the unbounded component E_1 . If $\text{mod } B > C_1$, we have the estimate*

$$\text{diam } E_0 \leq C_2 \text{dist}(E_0, E_1) e^{-\text{mod } B},$$

where C_1 and C_2 are positive absolute constants with $C_1 \leq 5 \log 2$ and $C_2 \leq 64$.

The statement is proved by a direct application of the well-known Teichmüller modulus theorem (see [14, p. 56]).

Proof of Theorem 2.1. For a given z with $|z| < R$, we set $m = \log(R/|z|)$ and let $A = \{w : |z| < |w| < R\}$. If $\text{mod } f(A) > 5 \log 2$, by Lemma 2.8, we obtain

$$(2.9) \quad |f(z)| \leq 64 \text{dist}(E_0, E_1) e^{-\text{mod } f(A)} \leq 64 e^{-\text{mod } f(A)}.$$

Otherwise, we see that

$$|f(z)| < 1 \leq e^{5 \log 2 - \text{mod } f(A)} = 32 e^{-\text{mod } f(A)},$$

and hence, (2.9) holds. On the other hand, by Proposition 2.4, we have

$$(2.10) \quad \text{mod } f(A) \geq \frac{m}{Q_\mu(|z|, R)}.$$

Combining (2.9) with (2.10), we arrive at the inequality

$$|f(z)| \leq 64 e^{-\log(R/|z|)/Q_\mu(|z|, R)} = 64 \left(\frac{|z|}{R} \right)^{1/Q_\mu(|z|, R)},$$

and thus, we complete the proof. \square

Since $D_{\pm\mu}(z) \leq K_\mu(z)$ a.e. in \mathbb{D} , we deduce from (2.5) an estimate for the change of moduli under quasiconformal mappings in the plane with pointwise dilatation $K_\mu(z)$ established by Belinskiĭ [6] (see also [14, p. 222]).

2.11. Corollary. *Under the assumption same as in Proposition 2.4*

$$(2.12) \quad \frac{1}{P_\mu(r, R)} \leq \frac{\text{mod } f(A)}{\text{mod } A} \leq P_\mu(r, R).$$

If f is K -quasiconformal mapping, then (2.12) yields

$$\frac{\text{mod } A}{K} \leq \text{mod } f(A) \leq K \text{mod } A$$

and we recognize the classical Grötzsch inequality for annuli (see, e.g., [14, p. 38]).

2.13. Corollary. *Under the assumptions of Proposition 2.4*

$$\begin{aligned} -\frac{1}{2\pi} \iint_A \frac{D_{-\mu}(z) - 1}{|z|^2} dx dy &\leq \text{mod } A - \text{mod } f(A) \\ &\leq \frac{1}{2\pi} \iint_A \frac{D_\mu(z) - 1}{|z|^2} dx dy. \end{aligned}$$

Proof. Set $m = \text{mod } A$, $m' = \text{mod } f(A)$ and $Q = Q_\mu(r, R)$. We then have to verify the inequality $m - m' \leq mQ - m$, only. The left-hand side of (2.5) implies $m'/m \geq 1/Q$. The trivial inequality $(Q - 1)^2 \geq 0$ yields $1/Q \geq 2 - Q$. Hence, one can get $m'/m \geq 2 - Q$, which is nothing but the required inequality. \square

In particular, we have the following simple form due to Belinskiĭ [6] (see also [14, p. 222]).

2.14. Corollary. *Under the assumptions of Proposition 2.4 the following inequality holds:*

$$|\text{mod } A - \text{mod } f(A)| \leq \frac{1}{2\pi} \iint_A \frac{K_\mu(z) - 1}{|z|^2} dx dy.$$

Similar growth estimates for homeomorphic solutions to the degenerate Beltrami equation were recently obtained by Chen [9].

3. CONDITIONS FOR LIPSCHITZ CONTINUITY

We will say that a function f is Lipschitz continuous at z_0 if $|f(z) - f(z_0)| \leq C|z - z_0|$ holds in a neighborhood of z_0 for a constant C . In this section, as applications of the previous results, we give a sufficient condition for a quasiconformal mapping to be Lipschitz continuous at a given point.

Suppose that $f(z)$ is a quasiconformal mapping in \mathbb{D} normalized by $f(0) = 0$. It is known, e.g., [14, p. 232], that the condition

$$(3.1) \quad \iint_{|z| < R} \frac{K_\mu(z) - 1}{|z|^2} dx dy < +\infty$$

for some $R > 0$ implies that f is conformally differentiable at the origin, that is, the limit $\lim_{z \rightarrow 0} f(z)/z$ exists and differs from 0 and ∞ . The existence of the limit $\lim_{z \rightarrow 0} |f(z)|/|z|$ under the hypothesis (3.1) was first proved by Teichmüller [20] and Wittich [22]. The existence of $\lim_{z \rightarrow 0} \arg(f(z)/z)$ under the assumption of (3.1) is due to Belinskiĭ (see, e.g., [7, p. 54]) and Lehto [13]. So, this result is referred to as the Teichmüller-Wittich-Belinskiĭ-Lehto regularity theorem. Under the weaker assumption than that of (3.1) we have the following statement.

3.2. Theorem. *Suppose that $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, is a quasiconformal embedding in \mathbb{D} with complex dilatation μ satisfying the assumption*

$$(3.3) \quad \sup_{0 < r < 1} \iint_{r < |z| < 1} \frac{D_\mu(z) - 1}{|z|^2} dx dy < +\infty.$$

Then

$$(3.4) \quad |f(z)| \leq C|z|$$

holds in \mathbb{D} with some constant C depending only on the value of the above supremum, and therefore, the mapping f is Lipschitz continuous at the origin.

3.5. Remark. 1. If the improper integral converges to a number including $-\infty$, we can state (3.3) simply by

$$\iint_{|z|<1} \frac{D_\mu(z) - 1}{|z|^2} dx dy < +\infty.$$

2. The statement of Theorem 3.2 can also be obtained by a careful observation of the proof of Theorem 3.1 in [17].

For the later convenience, we introduce the quantity

$$T_\mu(r, R) = \frac{1}{2\pi} \iint_{r<|z|<R} \frac{D_\mu(z) - 1}{|z|^2} dx dy$$

for $0 < r < R$. Note that the obvious relation $T_\mu(r, R) = (Q_\mu(r, R) - 1) \log(R/r)$ holds. Then condition (3.3) means that $\limsup_{r \rightarrow 0} T_\mu(r, 1) < +\infty$. This can also be viewed as a Dini type condition. Precisely, (3.3) holds if and only if

$$(3.6) \quad \limsup_{r \rightarrow 0} \int_r^1 \frac{\omega_\mu(t) dt}{t} < +\infty,$$

where

$$\omega_\mu(t) = \frac{1}{\pi t^2} \iint_{|z|<t} (D_\mu(z) - 1) dx dy.$$

The quantity $\omega_\mu(t)$ can be considered as the integral average of $D_\mu(z) - 1$ over the disk $|z| < t$ with respect to the plane Lebesgue measure. In particular, $1/K \leq \omega_\mu(t) + 1 \leq K$ if $K_\mu(z) \leq K$ a.e. in \mathbb{D} . As above, (3.6) can be expressed by $\int_0^1 \omega_\mu(t) t^{-1} dt < +\infty$ in the case when the improper integral is convergent. Since the relation

$$(3.7) \quad T_\mu(r, R) = \frac{\omega_\mu(R) - \omega_\mu(r)}{2} + \int_r^R \frac{\omega_\mu(t) dt}{t}$$

holds, we see that Dini type condition (3.6) is equivalent to (3.3). The formula in (3.7) can be proved in a more general setting.

3.8. Lemma. For a given Lebesgue integrable function φ in the disk $|z| < R$, the function η_φ defined by

$$\eta_\varphi(t) = \frac{1}{\pi t^2} \iint_{|z|<t} \varphi(z) dx dy, \quad 0 < t \leq R,$$

satisfies the relation

$$\frac{1}{2\pi} \iint_{r<|z|<R} \frac{\varphi(z)}{|z|^2} dx dy = \frac{\eta_\varphi(R) - \eta_\varphi(r)}{2} + \int_r^R \frac{\eta_\varphi(t) dt}{t}.$$

Proof. Set $\alpha(t) = \pi t^2 \eta_\varphi(t)$. This function can be written in the form $\alpha(t) = \int_0^t \psi(r) dr$, where

$$\psi(r) = r \int_0^{2\pi} \varphi(re^{i\theta}) d\theta.$$

Note that $\psi \in L^1([0, R])$ by Fubini's theorem. In particular, the function $\alpha(t)$ is absolutely continuous on $[0, R]$ and $\alpha'(t) = \psi(t)$ for a.a. t . We now compute

$$\begin{aligned} \iint_{r < |z| < R} \frac{\varphi(z)}{|z|^2} dx dy &= \int_r^R \frac{\psi(t) dt}{t^2} = \left[\frac{\alpha(t)}{t^2} \right]_r^R + \int_r^R \frac{2\alpha(t)}{t^3} dt \\ &= \pi(\eta_\varphi(R) - \eta_\varphi(r)) + 2\pi \int_r^R \frac{\eta_\varphi(t)}{t} dt, \end{aligned}$$

and thus the required identity follows. \square

Theorem 3.2 can be derived from the following more general statement.

3.9. Theorem. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ . Then the function f satisfies the inequality*

$$|f(z)| \leq \frac{C}{R} |z| e^{T_\mu(|z|, R)}$$

for $|z| < R \leq 1$, where $C \leq 64$ is an absolute constant.

Proof. Fix z with $0 < |z| < R$ and let $m = \log(R/|z|)$ and $A = \{w : |z| < |w| < R\}$. By Corollary 2.14, we have $\text{mod } f(A) \geq m - T_\mu(|z|, R)$. As in the proof of Theorem 2.1, we now obtain $|f(z)| \leq 64 e^{-m+T_\mu(|z|, R)} = 64 |z| e^{T_\mu(|z|, R)} / R$. \square

We now combine (3.7) with the above theorem to obtain the following statement.

3.10. Corollary. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a K -quasiconformal embedding with complex dilatation μ . Then the function f satisfies the inequality*

$$|f(z)| \leq C |z| \exp \left(\int_{|z|}^1 \frac{\omega_\mu(t) dt}{t} \right)$$

for $|z| < 1$, where $C = C(K)$ is a constant depending on K only.

The following result produces examples of quasiconformal mappings which satisfy (3.3) but do not necessarily satisfy (3.1).

3.11. Proposition. *Let ρ be an arbitrary non-negative real-valued measurable function in a neighborhood U of the origin with $\|\rho\|_\infty < 1$. Then there exists a quasiconformal embedding $f : U \rightarrow \mathbb{C}$, $f(0) = 0$, which is Lipschitz continuous at the origin, with complex dilatation μ satisfying*

$$|\mu(z)| = \rho(z) \quad \text{and} \quad D_\mu(z) = 1 \quad \text{a.e. in } U$$

Proof. Given $\rho(z)$ we define a Beltrami coefficient $\mu(z)$ a.e. in U which takes one of the following two values

$$(3.12) \quad \left(\rho^2(z) \pm i \rho(z) \sqrt{1 - \rho^2(z)} \right) \frac{z}{\bar{z}}$$

at each point $z \in U$ in a measurable way. The straightforward computation shows that $|\mu(z)| = \rho(z)$ and $D_\mu(z) = 1$ for almost all $z \in U$. By the measurable Riemann mapping theorem, there exists a quasiconformal mapping $f : U \rightarrow \mathbb{C}$, $f(0) = 0$, with complex dilatation μ . Since (3.3) trivially holds, by Theorem 3.2, the mapping f is Lipschitz continuous at the origin. \square

We remark that if $\rho(z) = \rho(|z|)$ for $z \in \mathbb{D}$ and $\rho = 0$ for $z \in \mathbb{C} \setminus \mathbb{D}$, then the quasiconformal mapping $f : \mathbb{C} \rightarrow \mathbb{C}$, fixing 0 and 1 with complex dilatation μ of the form (3.12) is explicitly given by the formula

$$f(z) = ze^{i\theta(|z|)}, \quad \theta(r) = \int_r^1 \frac{2\rho(t)}{\sqrt{1-\rho^2(t)}} \frac{dt}{t}$$

and represents a radial rotation. For example, if we set $\rho(z) = \varepsilon/\sqrt{4+\varepsilon^2}$, $\varepsilon \geq 0$, then we obtain the well-known spiral mapping

$$f(z) = ze^{i\varepsilon \log(1/|z|)},$$

transforming the rays $re^{i\theta}$, $0 < r < 1$, into logarithmic spirals which wind about the origin infinitely many times. Note, in particular, that $\lim_{z \rightarrow 0} \arg(f(z)/z)$ does not exist, and therefore the mapping f is not conformally differentiable at the origin, nevertheless $|f(z)| = |z|$ holds. This example was observed also by Reich and Walczak [17] in the same context.

Let f be a quasiconformal self-mapping of \mathbb{C} . Since f has locally integrable partial derivatives, the image Jordan curve of the circle $|z| = r$ under f , which is usually called a quasicircle, is rectifiable for almost all $r > 0$. However, we cannot say that the specific curve $f(\partial\mathbb{D})$ is rectifiable in general, see e.g., [14, p. 104] and [16, p. 249]. The following statement provides a sufficient condition for the quasicircle to be rectifiable.

3.13. Theorem. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a quasiconformal mapping with complex dilatation μ and Γ be a rectifiable curve in \mathbb{C} . If there are positive constants R and M such that*

$$\iint_{|z-z_0|<R} \frac{D_{\mu,z_0}(z) - 1}{|z - z_0|^2} dx dy \leq M$$

holds for each $z_0 \in \Gamma$, then f is Lipschitz continuous on Γ and therefore, the image curve $f(\Gamma)$ is rectifiable, too.

Proof. Given $z_0 \in \Gamma$, we consider the quasiconformal mapping F defined by $F(z) = f(z + z_0) - f(z_0)$. This mapping preserves the origin and has the complex dilatation $\nu(z) = \mu(z + z_0)$. In order to apply Theorem 3.2, we first calculate the angular dilatation D_ν . By definition, we see that

$$D_\nu(z) = \frac{|1 - \mu(z + z_0) \frac{\bar{z}}{z}|^2}{1 - |\mu(z + z_0)|^2} = D_{\mu,z_0}(z + z_0),$$

and therefore, we have

$$\begin{aligned} \iint_{|z|<R} \frac{D_\nu(z) - 1}{|z|^2} dx dy &= \iint_{|z|<R} \frac{D_{\mu,z_0}(z + z_0) - 1}{|z|^2} dx dy \\ &= \iint_{|z-z_0|<R} \frac{D_{\mu,z_0}(z) - 1}{|z - z_0|^2} dx dy \leq M. \end{aligned}$$

By Theorem 3.2, $|F(z)|/|z| = |f(z + z_0) - f(z_0)|/|z|$ is bounded in $|z| \leq R$ uniformly with respect to $z_0 \in \Gamma$. Hence, the mapping f is Lipschitz continuous on Γ . \square

More general conditions for the rectifiability of the image curve can be deduced from Theorem 3.2 by decomposing f into a suitable quasiconformal mapping and an affine mapping. We will illustrate this procedure in the next section. For related results on the rectifiability of quasicircles, see, for instance, [2], [5], [15], [4] and the references therein.

4. WEAK LIPSCHITZ CONTINUITY

We will say that a quasiconformal mapping f is weakly Lipschitz continuous at z_0 if for every $0 < \alpha < 1$ there is a constant $C > 0$ such that

$$(4.1) \quad |f(z) - f(z_0)| \leq C|z - z_0|^\alpha$$

holds if $|z - z_0| < \delta$, where $\delta > 0$ is a sufficiently small number.

4.2. Theorem. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ satisfying*

$$(4.3) \quad \limsup_{t \rightarrow 0} \frac{1}{\pi t^2} \iint_{|z|<t} (D_\mu(z) - 1) dx dy \leq 0.$$

Then f is weakly Lipschitz continuous at 0.

Proof. It suffices to show that

$$(4.4) \quad \log^+ \frac{|f(z)|}{|z|} = o\left(\log \frac{1}{|z|}\right), \quad z \rightarrow 0,$$

where $\log^+ x = \max\{\log x, 0\}$. Condition (4.3) means that $\limsup_{t \rightarrow 0} \omega_\mu(t) \leq 0$, and thus,

$$\limsup_{z \rightarrow 0} \frac{1}{\log(1/|z|)} \int_{|z|}^1 \frac{\omega_\mu(t) dt}{t} \leq 0.$$

Now, from Corollary 3.10, the conclusion follows. \square

Note that this result is qualitatively best possible as we will see in Theorem 4.12 below. Corollary 3.10 can be considered to give more precise information about the behavior of $f(z)/z$ at the origin in terms of the local quantity $\omega_\mu(t)$ defined by the complex dilatation.

As might be expected in view of the above proof, if the limit in (4.3) is negative, we may obtain even a stronger consequence.

4.5. Theorem. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ . Suppose that*

$$(4.6) \quad \limsup_{t \rightarrow 0} \frac{1}{\pi t^2} \iint_{|z| < t} D_\mu(z) dx dy \leq \frac{1}{\alpha}$$

holds for a constant $\alpha > 0$. Then f is weakly Hölder continuous with exponent α , that is, for each $0 < \alpha' < \alpha$, there is a constant C such that

$$|f(z)| \leq C|z|^{\alpha'}, \quad z \in \mathbb{D}.$$

Proof. By assumption, there is a non-negative function $\varepsilon(t)$ with $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ such that $\omega_\mu(t) \leq \alpha^{-1} - 1 + \varepsilon(t)$ holds for $t > 0$. By (3.7), we have

$$(Q_\mu(r, 1) - 1) \log(1/r) = \int_r^1 \frac{\omega_\mu(t) dt}{t} + O(1) \leq (\alpha^{-1} - 1) \log(1/r) + o(\log(1/r))$$

as $r \rightarrow 0$. This implies that

$$Q_\mu(r, 1) \leq \frac{1}{\alpha} + o(1)$$

as $r \rightarrow 0$. We can now apply Theorem 2.1 to get the conclusion. \square

The following statement produces examples of quasiconformal mappings which satisfy (4.3) but do not necessarily satisfy the stronger assumption

$$\lim_{t \rightarrow 0} \frac{1}{t^2} \iint_{|z| < t} (K_\mu(z) - 1) dx dy = 0.$$

4.7. Proposition. *Let $\rho(z)$ be an arbitrary non-negative real-valued measurable function in \mathbb{D} such that $\rho(z) \leq (K - 1)/(K + 1)$ a.e. in \mathbb{D} . Then the quasiconformal embedding $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, with complex dilatation*

$$\mu(z) = \rho(z) \frac{z}{\bar{z}}$$

satisfies (4.6) with $\alpha = K$, in particular, f is weakly Hölder continuous with exponent K at the origin.

Proof. Given $\rho(z)$, we see that $D_\mu(z) = 1/K_\mu(z) \leq 1/K$ a.e. in \mathbb{D} and thus the conclusion follows from Theorem 4.5. \square

In particular, if we set $\rho(z) = (K - 1)/(K + 1)$, where $K \geq 1$ is a constant, then we see that

$$f(z) = z|z|^{K-1}.$$

Then μ fails to be continuous at 0 and it does not have vanishing mean oscillation, however, f is Lipschitz continuous at the origin.

By using the composition procedure with affine mappings, we now deduce more general statement from the previous theorem.

4.8. Theorem. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, be a K -quasiconformal mapping. Suppose that for some $a \in \mathbb{D}$ the complex dilatation μ of f satisfies*

$$(4.9) \quad \limsup_{t \rightarrow 0} \frac{1}{\pi t^2} \iint_{|w| < t} (D_{\mu_a}(w) - 1) dudv \leq 0,$$

where

$$\mu_a = \frac{\mu - a}{1 - \bar{a}\mu} \circ L^{-1}$$

and L is the affine mapping defined by $L(z) = z + a\bar{z}$. Then f is weakly Lipschitz continuous at the origin.

Proof. Let us consider the quasiconformal mapping $F(w) = f \circ L^{-1}(w)$ whose complex dilatation agrees with μ_a . By Theorem 4.2, assumption (4.3) implies F to be weakly Lipschitz continuous at the origin, which is equivalent to the condition that

$$\log^+ \frac{|F(w)|}{|w|} = o\left(\log \frac{1}{|w|}\right)$$

as $w \rightarrow 0$. Hence

$$\log^+ \frac{|f(z)|}{|L(z)|} = o\left(\log \frac{1}{|L(z)|}\right)$$

as $z \rightarrow 0$, which is equivalent to (4.4). Thus, f is weakly Lipschitz continuous at the origin. \square

Note that the above number a necessarily satisfies $|a| \leq (K - 1)/(K + 1)$.

4.10. Corollary. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a quasiconformal embedding whose complex dilatation μ is approximately continuous at the origin. Then f is weakly Lipschitz continuous at the origin.*

Proof. The approximate continuity of μ at the origin means that there exists a point $a \in \mathbb{D}$ such that

$$\frac{1}{\pi t^2} \iint_{|z| < t} |\mu(z) - a| dx dy \rightarrow 0$$

as $t \rightarrow 0$. The chain of elementary inequalities

$$\begin{aligned} \frac{1}{\pi t^2} \iint_{|w| < t} (D_{\mu_a}(w) - 1) dudv &\leq \frac{1}{\pi t^2} \iint_{|w| < t} (K_F(w) - 1) dudv \\ &\leq \frac{C}{\pi t^2} \iint_{|z| < t/(1-|a|)} |\mu(z) - a| dx dy \end{aligned}$$

implies (4.9), where C is a constant depending on K only. Thus, the required weak Lipschitz continuity of f at the origin follows from Theorem 4.8. \square

4.11. Remark. If the complex dilatation is approximately continuous uniformly on a compact set E , then the mapping f is weakly Lipschitz continuous uniformly on E , that is, for each $0 < \alpha < 1$ there exists a constant $C = C(E) > 0$ such that $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^\alpha$ holds whenever $z_1, z_2 \in E$.

The example of the quasiconformal radial stretching

$$f(z) = z \left(\log \frac{1}{|z|} \right)^2$$

defined on a neighborhood of the origin, whose complex dilatation

$$\mu(z) = \frac{1}{1 - \log(1/|z|)} \frac{z}{\bar{z}}$$

is *continuous* at the origin, tells us that the weak Lipschitz continuity cannot be replaced by the stronger estimate $|f(z)| \leq C|z| \log(1/|z|)$ in Corollary 4.10. Moreover, we cannot say anymore than (4.1) in the following sense.

4.12. Theorem. *Let $h : (0, \delta] \rightarrow \mathbb{R}$ be a continuous function such that*

- (i) $h(t) = o(\log(1/t))$ as $t \rightarrow 0$, and
- (ii) $h(t) \rightarrow \infty$ as $t \rightarrow 0$.

Then there exists a quasiconformal mapping $f : \mathbb{D} \rightarrow \mathbb{D}$ with $f(0) = 0$ such that

$$\limsup_{z \rightarrow 0} \frac{\log^+(|f(z)|/|z|)}{h(|z|)} > 0,$$

while the complex dilatation μ is continuous in \mathbb{D} , vanishes at the origin, and therefore, satisfies (4.3).

Before proving the theorem we will introduce standard radial stretchings. Let $M : (0, \delta) \rightarrow [1, +\infty)$ be a bounded measurable function. For convenience, we extend $M(t)$ to be identically 1 for $t \geq \delta$. We consider the two homeomorphisms f_M and g_M of the complex plane onto itself defined by

$$(4.13) \quad f_M(z) = \frac{z}{|z|} \exp \left(\int_1^{|z|} \frac{dt}{tM(t)} \right) \quad \text{and} \quad g_M(z) = \frac{z}{|z|} \exp \left(\int_1^{|z|} \frac{M(t)dt}{t} \right).$$

Then the complex dilatations of f_M , g_M are given by μ_M , $-\mu_M$, respectively, where

$$\mu_M(z) = -\frac{M(|z|) - 1}{M(|z|) + 1} \cdot \frac{z}{\bar{z}}.$$

Therefore, we have $D_{\mu_M}(z) = M(|z|)$, $D_{-\mu_M}(z) = 1/M(|z|)$ and $K_{\mu_M}(z) = K_{-\mu_M}(z) = M(|z|)$ for almost all z .

4.14. Lemma. *Let h be as in Theorem 4.12. Then there exists a function $g : (0, \delta] \rightarrow \mathbb{R}$ with properties (a), (b), (c) and either (d) or (d') below:*

- (a) g is differentiable, strictly decreasing and satisfies $\lim_{t \rightarrow 0} g(t) = +\infty$.
- (b) g' is absolutely continuous on the interval $[\delta_1, \delta]$ for each $0 < \delta_1 < \delta$.
- (c) $g''(t) = o(1/t^2)$ as $t \rightarrow 0$.
- (d) $g(t_n) = h(t_n)$ for some sequence t_n in $(0, \delta]$ with $t_n \rightarrow 0$.
- (d') $g(t) \leq h(t) + O(1)$ as $t \rightarrow 0$.

Note that condition (c) implies that $g'(t) = o(1/t)$ and $g(t) = o(\log 1/t)$ as $t \rightarrow 0$.

Proof of Lemma 4.14. Set $H(x) = h(\delta e^{-x}) - h(\delta) + 1$ for $x \geq 0$. Since $h(t) = o(\log 1/t)$, we note that $H(x) = o(x)$ as $x \rightarrow +\infty$. For a given $y > 0$, we define $H^*(y) = \max\{x \geq 0 : H(x) \leq y\}$. One can then see that $H^*(y) \rightarrow +\infty$ while $y/H^*(y) \rightarrow 0$ as $y \rightarrow +\infty$. Set $y_0 = \min\{H(x) : x \geq 0\}$ and $x_0 = 0$. Let y_n , $n = 1, 2, \dots$, be a monotone sequence of positive numbers such that $y_1 > y_0$ and $y_n \rightarrow \infty$ as $n \rightarrow \infty$. Also let $x_n = H^*(y_n)$. It is easy to see that $H(x_n) = y_n$ for $n \geq 1$, $H(x) \geq y_n$ for $x \geq x_n$, $x_0 < x_1 < x_2 < \dots$ and $x_n \rightarrow +\infty$ as $n \rightarrow \infty$. Choosing a suitable sequence, we can make the following two conditions valid:

- (1) $x_{n+1} - x_n \rightarrow +\infty$ if $n \rightarrow \infty$,
- (2) $p_n := (y_{n+1} - y_n)/(x_{n+1} - x_n)$ decreases to 0 as $n \rightarrow \infty$.

First, we give a function g satisfying (a), (b), (c) and (d). Define the preliminary function G_0 by

$$(4.15) \quad G_0(x) = y_n + p_n(x - x_n) + \frac{p_{n+1} - p_n}{2(x_{n+1} - x_n)}(x - x_n)^2, \quad x \in [x_n, x_{n+1}),$$

for $n = 0, 1, \dots$. Then, $G_0(x_n) = y_n$, $G'_0(x_n) = p_n$ and $G'_0(x_{n+1} - 0) = p_{n+1}$. This function has the gap

$$D_n := G_0(x_{n+1}) - \lim_{x \rightarrow x_{n+1}-} G_0(x) = \frac{p_n - p_{n+1}}{2}(x_{n+1} - x_n)$$

at the point x_{n+1} . So, we will fill it up by adding a suitable term to G_0 . To this end, we consider an auxiliary function $\varphi : [0, 1] \rightarrow [0, 1]$ with the following properties:

- (i) $\varphi \in C^2[0, 1]$,
- (ii) $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi(1) = 1$, and $\varphi'(1) = 0$, and
- (iii) $0 \leq \varphi'(x) \leq C_1$ and $|\varphi''(x)| \leq C_2$ for $x \in [0, 1]$, where C_1 and C_2 are constants.

For example, we can take $\varphi(x) = \sin^2(\pi x/2)$.

We now define the continuous function $G : [0, +\infty) \rightarrow \mathbb{R}$ by

$$G(x) = G_0(x) + D_n \varphi\left(\frac{x - x_n}{x_{n+1} - x_n}\right) \quad x \in [x_n, x_{n+1}),$$

for $n = 0, 1, \dots$. Furthermore, G has the locally absolutely continuous derivative

$$G'(x) = p_n \frac{x_{n+1} - x}{x_{n+1} - x_n} + p_{n+1} \frac{x - x_n}{x_{n+1} - x_n} + \frac{p_n - p_{n+1}}{2} \varphi'\left(\frac{x - x_n}{x_{n+1} - x_n}\right)$$

on $[x_n, x_{n+1}]$. Note here that $G'(x) \geq 0$ and $G'(x) \rightarrow 0$ as $x \rightarrow +\infty$ by (2). From the expression

$$G''(x) = -\frac{p_{n+1} - p_n}{x_{n+1} - x_n} \left[1 - \frac{1}{2} \varphi''\left(\frac{x - x_n}{x_{n+1} - x_n}\right) \right], \quad x \in (x_n, x_{n+1}),$$

we obtain $G''(x) \rightarrow 0$ as $x \rightarrow +\infty$. We now set $g(t) = G(\log(\delta/t))$ for $0 < t \leq \delta$. Then, $g'(t) = -G'(\log(\delta/t))/t < 0$ and $g''(t) = (G''(\log(\delta/t)) - G'(\log(\delta/t)))/t^2$, and thus, properties (a), (b) and (c) follow. Let $t_n = \delta e^{-x_n}$. Then $g(t_n) = G(x_n) = H(x_n) = h(t_n)$, and (d) has been shown.

Next, we construct a function g satisfying (a), (b), (c) and (d'). First, we change the definition of p_n by setting $p_n = (y_n - y_{n-1})/(x_{n+1} - x_n)$, and then, in the definition of G_0 above, we replace y_n by y_{n-1} , where we set $y_{-1} = y_0$, and define g in the exactly same way as above. Properties (a), (b) and (c) now follow similarly. Also, in this case, we see that $G_0(x) \leq y_n \leq H(x)$ for $x \in [x_n, x_{n+1})$. If D_n is bounded, we obtain the estimate $G(x) = G_0(x) + O(1) \leq H(x) + O(1)$, and hence, we confirm property (d'). Otherwise, taking a positive constant C , we set $E_n = \max\{D_n - C, 0\}$ for each n . Note that $E_n > 0$ for infinitely many n , in this case. Then, we modify the above definition of G by

$$G(x) = G_0(x) + \min\{D_n, C\} \varphi\left(\frac{x - x_n}{x_{n+1} - x_n}\right) - \sum_{k=0}^{n-1} E_k, \quad x \in [x_n, x_{n+1}),$$

for $n = 0, 1, \dots$ and set $g(t) = G(\log(\delta/t))$. Then, by definition, $G(x) \leq G_0(x) + O(1)$ as $x \rightarrow +\infty$. Since $G(x_{n+1}) - G(x_n) \geq C$ if $E_n > 0$, we can see that $G(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. Now properties (a), (b), (c) and (d') are easy to see for this function g . \square

Proof of Theorem 4.12. Let g be a function satisfying properties (a), (b), (c) and (d) in Lemma 4.14. Since $\varepsilon(t) = -tg'(t)$ is non-negative and converges to zero as $t \rightarrow 0$, we can choose a small $0 < \delta_1 \leq \delta$ so that $M(t) = 1/(1 - \varepsilon(t))$ is bounded in $0 < t < \delta_1$. Now we extend M by setting $M(t) = M(\delta_1)$ for $t \geq \delta_1$. Then the radial stretching f_M has continuous complex dilatation μ_M . We also note that f_M has the form $f_M(z) = Cz \exp(g(|z|))$ for $|z| < \delta_1$, where C is a constant. Let t_n be the sequence in property (d). Then

$$\limsup_{z \rightarrow 0} \frac{\log^+(|f(z)|/|z|)}{h(|z|)} \geq \limsup_{n \rightarrow \infty} \frac{g(t_n) + \log C}{h(t_n)} = 1 > 0.$$

Thus, the assertions of the theorem have been proved. \square

Now we show that (3.1) is the qualitatively best possible condition for a function f to be Lipschitz continuous at the origin. The same is true for (3.3). Recall that (3.3) is equivalent to Dini type condition (3.6).

4.16. Theorem. *Let $h : (0, \delta] \rightarrow \mathbb{R}$ be a continuous function satisfying $h(t) \rightarrow +\infty$ if $t \rightarrow 0$. Then there exist a quasiconformal radial stretching $f : \mathbb{D} \rightarrow \mathbb{D}$, $f(0) = 0$, with complex dilatation μ and a constant C such that $D_\mu = K_\mu$ and that*

$$\int_r^1 \frac{\omega_\mu(t) dt}{t} \leq h(r) + C$$

for $r > 0$ while f fails to be Lipschitz continuous at the origin.

Proof. The proof is somewhat similar to that of the previous theorem. First, we can assume that $h(t) = o(\log(1/t))$ as $t \rightarrow 0$. Let g be a function

satisfying (a), (b), (c) and (d') in Lemma 4.14. If we set $\varepsilon(t) = -tg'(t)$, by (b) we then have $\varepsilon(t) = o(1)$ and $t\varepsilon'(t) = -tg'(t) - t^2g''(t) = o(1)$ as $t \rightarrow 0$. In particular, we can choose a sufficiently small $\delta_1 > 0$ so that $\varepsilon(t) < 1/2$ and $|t\varepsilon'(t)| < 1$ for $t \in (0, \delta_1)$. We now define $M(t)$ by $M(t) = 1 + \varepsilon(t) + t\varepsilon'(t)/2$ for $t \in (0, \delta_1)$ and by $M(t) = 1$ for $t \geq \delta_1$. Then M is a bounded measurable function with $1 \leq M \leq 2$. Let us consider the function f_M given by (4.13), which has complex dilatation $\mu = \mu_M$. We compute

$$\omega_\mu(t) = \frac{2}{t^2} \int_0^t (M(s) - 1) s ds = \frac{1}{t^2} \int_0^t (s^2 \varepsilon(s))' ds = \varepsilon(t)$$

for $0 < t < \delta_1$, and by (d') we obtain

$$\begin{aligned} \int_r^1 \frac{\omega_\mu(t) dt}{t} &= \int_r^{\delta_1} \frac{\varepsilon(t) dt}{t} + O(1) = - \int_r^{\delta_1} g'(t) dt + O(1) \\ &= g(r) + O(1) \leq h(r) + O(1) \end{aligned}$$

as $r \rightarrow 0$. On the other hand, since $1/M(t) \leq 1 - (M(t) - 1)/2$, we see that

$$\begin{aligned} \int_r^1 \frac{dt}{tM(t)} &\leq -\log r - \frac{1}{2} \int_r^{\delta_1} \frac{\varepsilon(t) dt}{t} - \frac{1}{4} \int_r^{\delta_1} \varepsilon'(t) dt \\ &= -\log r + \frac{g(\delta_1) - g(r)}{2} - \frac{\varepsilon(\delta_1) - \varepsilon(r)}{4} \end{aligned}$$

for sufficiently small $r > 0$, and therefore, $\log |f_M(z)/z| \geq g(|z|)/2 + O(1) \rightarrow \infty$ as $z \rightarrow 0$. \square

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