Papers on Analysis:
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A REMARK ON UNIQUENESS OF SOLUTIONS TO THE DIRICHLET PROBLEM

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1. Introduction

We consider free extremals, called F-extremals, of the variational integral

$$I_F(u,E)=\int_E F(x,\nabla u(x))\,dx\,,$$

where the kernel $F\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$, $n\geq 2$, satisfies the following assumptions for some constants $1<p<\infty$ and $0<\gamma\leq\delta<\infty$:

(1.1) $$\text{the mapping } x\mapsto F(x,\xi) \text{ is measurable for all } \xi\in\mathbf{R}^n;$$

for a.e. $x\in\mathbf{R}^n$

(1.2) $$\gamma w(x)|\xi|^p\leq F(x,\xi)\leq\delta w(x)|\xi|^p\,,\quad \xi\in\mathbf{R}^n\,,$$

(1.3) $$\text{the mapping } \xi\mapsto F(x,\xi) \text{ is convex and differentiable,}$$

and

(1.4) $$F(x,\lambda\xi)=|\lambda|^pF(x,\xi),\quad \lambda\in\mathbf{R}\,,\ \xi\in\mathbf{R}^n\,.$$

Here w is a p-admissible weight in the sense of [HKM]; for instance any Muckenhoupt's A_p-weight w will do.

The assumptions on the kernel F above are otherwise the same as those in [HKM, Ch. 5], but here we do not require the strict convexity of the kernel. The purpose of this simple note is to prove the uniqueness of the Dirichlet problem without employing the strict convexity of the kernel; cf. [HKM, p. 106]. Instead, the homogeneity assumption (1.4) will play a major role. We shall also give an example showing that the uniqueness result is lost if both homogeneity and strict convexity assumptions are dropped.

More precisely we prove:

Theorem A. *Let u and v be F-extremals in a bounded open set Ω. If u and v are continuous in $\overline{\Omega}$ and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Theorem **A** can be equivalently stated as follows:

Theorem B. *Let u and v be F-extremals in a bounded open set Ω, continuous in $\overline{\Omega}$. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

Recall that a function u is called an *F-extremal* in an open set Ω if $u \in H^{1,p}_{loc}(\Omega;\mu)$ and for each open $D \subset\subset \Omega$

$$I_F(u, D) \leq I_F(u + \varphi, D)$$

whenever $\varphi \in H^{1,p}_0(D;\mu)$. Here we use the notation of [HKM] for the weighted Sobolev spaces, i.e., $H^{1,p}(\Omega;\mu)$ is the completion of

$$\{\varphi \in C^\infty(\Omega) \colon ||\varphi||_{1,p} < \infty\}$$

with respect to the norm

$$||\varphi||_{1,p} = \left(\int_\Omega |\varphi|^p \, d\mu\right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p \, d\mu\right)^{1/p} ;$$

moreover $H^{1,p}_0(\Omega;\mu)$ is the closure of $C^\infty_0(\Omega)$ in $H^{1,p}(\Omega;\mu)$, and $H^{1,p}_{loc}(\Omega;\mu)$ is the corresponding local space; here μ is the measure

$$\mu(E) = \int_E w(x) \, dx \, .$$

It is rather easy to see that the strict monotonicity is not needed to prove the continuity Harnack's inequality for F-extremals; see [HKM, Ch. 3] or [S]. We shall employ these properties in what follows.

2. Proof

We prove first a weaker result than Theorem **A**.

2.1. Proposition. *Let $u, v \in H^{1,p}_{loc}(\Omega;\mu)$ be F-extremals in a bounded open set Ω and let $D \subset\subset \Omega$ be open. If $u = v$ on ∂D, then $u = v$ in D.*

Proof. If u and v do not coincide in the whole of D, we may, by relabeling and passing to a subdomain if necessary, assume that $v > u$ in D. We will show that this leads to a contradiction.

Our first task is to show that

$$F(x, \nabla u(x)) = F(x, \nabla v(x)) \quad \text{for a.e. } x \in D. \tag{2.2}$$

Suppose, on the contrary, that the set

$$E = \{x \in D \colon F(x, \nabla u(x)) \neq F(x, \nabla v(x))\}$$

is of positive measure.

Write $\vartheta = \frac{1}{2}u + \frac{1}{2}v$ and observe that for a.e. x the mapping $\xi \mapsto F(x,\xi)^{1/p}$ is a norm in $\mathbf{R}^n$; see [K1, 3.1] or [HKM, 5.23]. Moreover, since $t \mapsto t^p$ is strictly convex for $t \geq 0$, we have

$$\begin{aligned} F(x, \nabla\vartheta(x)) &\leq \left(\frac{1}{2}F(x,\nabla u(x))^{1/p} + \frac{1}{2}F(x,\nabla v(x))^{1/p}\right)^p \\ &\leq \frac{1}{2}F(x,\nabla u(x)) + \frac{1}{2}F(x,\nabla v(x)) \end{aligned}$$

a.e. in D; moreover the last inequality is strict a.e. in E, whence by integrating

$$\begin{aligned} \int_D F(x,\nabla\vartheta(x))\,dx &= \int_{D\setminus E} F(x,\nabla\vartheta(x))\,dx + \int_E F(x,\nabla\vartheta(x))\,dx \\ &< \frac{1}{2}\int_D F(x,\nabla u(x))\,dx + \frac{1}{2}\int_D F(x,\nabla v(x))\,dx \\ &= \int_D F(x,\nabla u(x))\,dx\,, \end{aligned}$$

since both u and v are F-extremals. This contradicts the F-extremality of u, and (2.2) follows.

Next let $f\colon \mathbf{R} \to \mathbf{R}$ be a convex increasing C^2 function. Then $f \circ v$ is a sub-F-extremal in D, i.e.

$$I_F(f\circ v, D) \leq I_F(f\circ v - \varphi, D) \tag{2.3}$$

for each nonnegative $\varphi \in H_0^{1,p}(D;\mu)$. This was proven in [HKM], but unfortunately the proof there uses the uniqueness result we are proving right here. So we shall use the proof given in [K2].

Observe, that (2.3) is equivalent to requiring that

$$\int_D \nabla_\xi F(x,\nabla(f\circ v))\cdot\nabla\varphi\,dx \leq 0 \tag{2.4}$$

for each nonnegative $\varphi \in C_0^\infty(D)$ (cf. [HKM, 5.13]). To prove (2.4) we use

$$\psi = f'(v)^{p-1}\varphi$$

as a test function in the Euler equation of v. So we have by using the convexity of f and the structure assumption (1.2) that

$$\begin{aligned} 0 &= \int_D \nabla_\xi F(x,\nabla v)\cdot\nabla\psi\,dx \\ &= (p-1)\int_D f''(v)f'(v)^{p-2}\varphi\nabla_\xi F(x,\nabla v)\cdot\nabla v\,dx \\ &\quad + \int_D f'(v)^{p-1}\nabla_\xi F(x,\nabla v)\cdot\nabla\varphi\,dx \\ &\geq \int_D f'(v)^{p-1}\nabla_\xi F(x,\nabla v)\cdot\nabla\varphi\,dx \\ &= \int_D \nabla_\xi F(x,\nabla(f\circ v))\cdot\nabla\varphi\,dx\,, \end{aligned}$$

since by the homogeneity assumption (1.4) it holds that

$$\nabla_\xi F(x,\lambda\xi) = \lambda^{p-1}\nabla_\xi F(x,\xi) \qquad \text{for } \lambda \ge 0\,.$$

Hence (2.3) follows.

Our next step is to show that

(2.5) $f\circ u$ is an F-extremal in the open set

$$V = \{x \in D\colon f\circ u(x) < f\circ v(x)\}$$

whenever $f\colon \mathbf{R}\to\mathbf{R}$ is a convex increasing C^2 function.

Observe that f need not be strictly increasing in the above discussion.

To prove (2.5), let $\eta \in C_0^\infty(V)$. Then

$$\delta = \min_{\operatorname{spt}\eta}(f\circ v - f\circ u) > 0$$

so that for all $\varepsilon\in\mathbf{R}$ with

$$|\varepsilon| \le \frac{\delta}{\sup|\eta|}$$

it holds that

$$f\circ u + \varepsilon\eta \le f\circ v \ \text{ in } V\,,$$

whence by (2.3)

$$I_F(f\circ v, V) \le I_F(f\circ u + \varepsilon\eta, V)\,.$$

Now using (2.2) we infer that

$$\begin{aligned} I_F(f\circ u, V) &= \int_V F(x,\nabla(f\circ u))\,dx = \int_V f'(u)^p F(x,\nabla u)\,dx \\ &\le \int_V f'(v)^p F(x,\nabla v)\,dx = I_F(f\circ v, V) \\ &\le I_F(f\circ u + \varepsilon\eta, V)\,. \end{aligned}$$

Hence $f\circ u$ is and F-extremal in V since by the convexity of F

$$\begin{aligned} I_F\big(f\circ u + \varepsilon\eta, V\big) &= I_F\big((1-\varepsilon)f\circ u + \varepsilon(f\circ u+\eta), V\big) \\ &\le (1-\varepsilon)I_F\big(f\circ u, V\big) + \varepsilon I_F\big(f\circ u + \eta, V\big)\,; \end{aligned}$$

so (2.5) follows.

Now we are ready to conclude the proof. Fix $x_0\in D$ and let U be the x_0-component of the open set

$$\{x\in D\colon v(x) > u(x_0)\}\,.$$

Write

$$f(t) = \begin{cases} \big(t-u(x_0)\big)^3, & \text{if } t > u(x_0) \\ 0\,, & \text{if } t \le u(x_0)\,. \end{cases}$$

Then $f \circ u$ is an F-extremal in U by (2.5). Since $f \circ u \geq 0$ in U, it follows from the minimum principle (Harnack's inequality), see [HKM, 6.5 and 5.18], that

$$f \circ u = 0 \quad \text{in } U\,.$$

This means that each point $x_0 \in D$ is a local maximum point of u; therefore u is constant in D and thus

$$0 = F(x, \nabla u(x)) = F(x, \nabla v(x))$$

for every $x \in D$. Thus v is also constant in D. This is however absurd, for $v > u$ in D and they coincide on the boundary. This contradicion completes the proof of Proposition 2.1. □

Proof of Theorem B. Assume, on the contrary, that there is $x_0 \in \Omega$ with

$$u(x_0) < v(x_0)\,.$$

Then for $0 < \varepsilon < v(x_0) - u(x_0)$ there is a domain $D \subset\subset \Omega$ containing x_0 so that $u + \varepsilon = v$ on ∂D. Thus by Proposition 2.1 $u + \varepsilon = v$ in D, which is a contradiction since

$$v(x_0) = u(x_0) + \varepsilon < u(x_0) + v(x_0) - u(x_0) = v(x_0)\,.$$

□

Remark. The above proof also shows that Theorem B is a consequence of Theorem A. Since the converse is trivial we have proven the equivalent statements of Therems A and B.

Example. We show that both strict convexity and homogeneity cannot be dropped without losing the uniqueness feature. The function

$$F(x, \xi) = \max(2\xi_n - 1 + 2^{-p}, |\xi_1|^p, |\xi_2|^p, \ldots, |\xi_n|^p)$$

defines a kernel which satisfies (1.1) and (1.2), and the convexity part of (1.3) with $w = 1$. The example below could easily be modified to expose the non-uniquenes phenomenon for differentiable kernels too, but for simplicity we provide the example without smoothing.

Let Ω be the unit cube $\Omega = (0,1)^n$. Let $u(x) = x_n$. Then u is an F-extremal in Ω: since the gradient $\nabla_\xi F(x, \nabla u(x))$ exists we have by the convexity of F that

$$\begin{aligned} 0 &= \int_\Omega (0, \ldots, 0, 2) \cdot \nabla\varphi(x)\, dx = \int_\Omega \nabla_\xi F(x, \nabla u(x)) \cdot \nabla\varphi(x)\, dx \\ &\leq \int_\Omega F(x, \nabla u(x) + \nabla\varphi(x))\, dx - \int_\Omega F(x, \nabla u(x))\, dx \end{aligned}$$

whenever $\varphi \in C_0^\infty(\Omega)$.

Let then $\varepsilon \in (0, \frac{1}{2})$ be so that

$$(1+\varepsilon)^p < 1 + 2^{-p} + \varepsilon$$

and write

$$v(x) = \varepsilon \min_{j=1,\ldots,n} (x_j, 1 - x_j).$$

Then $u + v = u$ on $\partial\Omega$ and

$$\nabla_\xi F(x, \nabla(u+v)(x)) = (0, 0, \ldots, 0, 2).$$

So

$$\begin{aligned} 0 &= \int_\Omega \nabla_\xi F(x, \nabla(u+v)(x)) \cdot \nabla\varphi(x)\, dx \\ &\leq \int_\Omega F(x, \nabla(u+v)(x) + \nabla\varphi(x))\, dx - \int_\Omega F(x, \nabla(u+v)(x))\, dx \end{aligned}$$

whenever $\varphi \in C_0^\infty(\Omega)$; hence $u + v$ is also an F-extremal, so that the uniqueness fails.

References

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