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THE INFINITY LAPLACIAN: EXAMPLES AND OBSERVATIONS

Petri Juutinen, Peter Lindqvist, and Juan J. Manfredi

Abstract. In this note, we discuss the infinity Laplace equation

$$\Delta_\infty u = 0$$

through specific examples. A related non-linear eigenvalue problem is also studied. We describe an interesting procedure of gluing solutions together. The examples show that the strong comparison principle and the principle of unique continuation are not valid.

1. Introduction

Often examples (including counterexamples) are decisive in the formation of a mathematical theory. In the fascinating case of the so called *infinity Laplacian*

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1.1}$$

the basic theory is still under development and so explicit examples are all the more expedient. The present note is a list of examples. Some of these are quite ordinary while others are constructed to serve a definite purpose. We will exhibit a rather simple, though intriguing, example, showing that *the strong comparison principle is not valid* for the equation

$$\Delta_\infty u = 0. \tag{1.2}$$

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This is at the same time *a counterexample to the principle of unique continuation.* Another counterexample to unique continuation has been constructed by Aronsson in [A2], who used the Cauchy-Kowalevskaya theorem. We will also see arcane phenomena hindering the smoothness of certain solutions. A substantial part is devoted to a non-linear eigenvalue problem, which is closely related to the operator Δ_∞.

Formally, the equation $\Delta_\infty u = 0$ arises as the limit of the Euler-Lagrange equations

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.3}$$

of the variational integral

$$\int |\nabla u|^p \, dx$$

as $p \to \infty$. Hence one may expect that its solutions would minimize the norm

$$\|\nabla u\|_\infty = \lim_{p\to\infty} \|\nabla u\|_p.$$

Indeed, its solutions, the so-called ∞-harmonic functions, provide the best Lipschitz extension of their boundary values, cf. [A1], [J]. The ∞-harmonic functions are only known to be of class $W^{1,\infty}_{loc}$ and their second derivatives, needed to evaluate $\Delta_\infty u$, do not always exist. To avoid this, the equation is nowadays understood *in the viscosity sense* [BDBM].

Definition. We say that the function $u \in C(\Omega)$ is ∞*-superharmonic* in Ω, if whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(i) $\varphi(x_0) = u(x_0)$, and
(ii) $\varphi(x) < u(x)$, when $x \neq x_0$,

then we have $\Delta_\infty \varphi(x_0) \leq 0$.

The ∞*-subharmonic functions* are defined analogously, the test-functions touching from above. A function is by definition ∞*-harmonic*, if it is both ∞-superharmonic and ∞-subharmonic. According to Jensen's uniqueness theorem, there is only one ∞-harmonic function with given continuous boundary values in a bounded domain, cf. [J]. An ∞-harmonic function can always be obtained as the locally uniform limit of solutions to equation (1.3) as $p \to \infty$. Our examples show that some fundamental properties are lost in the transition from a finite p to $p = \infty$.

Let us discuss the nonlinear eigenvalue problem with $p = \infty$. The equation

$$\max\left\{\Lambda - \frac{|\nabla u(x)|}{u(x)}, \Delta_\infty u(x)\right\} = 0 \tag{1.4}$$

was derived in [JLM]. It is tacitly understood that $u > 0$. The dichotomy means that, at each point x, the larger of the two quantities is zero. Again the equation has to be interpreted in the viscosity sense, cf. [JLM]. An *infinity ground state* u_∞ is a positive solution with zero boundary values in the given bounded domain Ω in $\mathbb{R}^n$:

$$u_\infty \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \qquad u_\infty > 0.$$

For no value of Λ other than

$$\Lambda = \frac{1}{\max\limits_{x \in \Omega} \operatorname{dist}(x, \partial\Omega)} \tag{1.5}$$

there exists a positive solution to (1.4). In other words, the "principal frequency" Λ is the reciprocal number of the radius of the largest inscribed ball in Ω. This was proved in [JLM] under the regularity assumption $\partial\Omega = \partial\overline{\Omega}$ later shown to be unnecessary by Juutinen [Ju].

Formally, equation (1.4) is the limit of the equations

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = 0 \tag{1.6}$$

as $p \to \infty$ (λ depends on p). These are the Euler-Lagrange equations of the minimization problem for the non-linear Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}.$$

This procedure provides an existence proof for infinity ground states. The so obtained u_∞ is called *variational*. To the best of our knowledge the uniqueness is an open problem. We do not know, for sure, if the variational infinity ground state is unique. Neither do we know, whether equation (1.4) admits of non-variational solutions. This has the effect that in a symmetric domain one cannot exclude the possibility of an (extra) unsymmetrical infinity ground state. However, a *local* uniqueness result has been proved in [JLM].

To illuminate the dichotomy we consider the example of a ball $\{x \in \mathbb{R}^n : |x| < 1\}$. The infinity ground state is

$$u_\infty(x) = 1 - |x|$$

and $\Lambda = 1$. It is unique in this case (multiplication by constants discarded). Notice that $\Delta_\infty u_\infty = 0$ (even in the classical sense) when $x \neq 0$. At the origin

$$|\nabla \log u_\infty(0)| = 1, \qquad \Delta_\infty u_\infty(0) < 0$$

in the viscosity sense. At this point alone Λ is determined.

In general Λ cannot be detected where the solution is smooth. The following result is illuminating.

Lemma. (The Λ-Lemma) *If u_∞ has continuous second derivatives at the point x_0 in Ω, then*

$$\Lambda < |\nabla \log u_\infty(x_0)| \qquad \text{and} \qquad \Delta_\infty u_\infty(x_0) = 0.$$

Proof. (Reductio ad absurdum) We abbreviate, writing u for u_∞. Suppose now that $\Lambda = |\nabla \log u(x_0)|$ while $\Lambda \le |\nabla \log u(x)|$ near x_0. In other words, the quantity

$$\frac{|\nabla u|^2}{u^2} = \frac{\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2}{u^2}$$

attains its minimum at x_0. Hence

$$\frac{\partial}{\partial x_j}\left(\frac{|\nabla u|^2}{u^2}\right) = 0, \quad j = 1, 2, \ldots, n.$$

This yields

$$u \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_k \partial x_j} = \frac{\partial u}{\partial x_j} |\nabla u|^2, \quad j = 1, 2, \ldots, n.$$

Multiplying by $\frac{\partial u}{\partial x_j}$ and adding up the equations we arrive at

$$u \Delta_\infty u = |\nabla u|^4 = \Lambda^4 u^4$$

at the point x_0. This contradicts the equation, which requires that $\Delta_\infty u \le 0$. Thus the antithesis was false and, consequently, $\Lambda < |\nabla \log u(x_0)|$. Then $\Delta_\infty u(x_0) = 0$ according to the equation. This proves also the second assertion. □

The distance function

$$\delta(x) = \operatorname{dist}(x, \partial\Omega)$$

is closely related to the matter. We have the minimization property

$$\Lambda = \frac{\|\nabla \delta\|_\infty}{\|\delta\|_\infty} = \frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty} = \inf_{\varphi \in C_0^\infty(\Omega)} \frac{\|\nabla \varphi\|_\infty}{\|\varphi\|_\infty}, \tag{1.7}$$

the maximal norms being taken over Ω. Both u_∞ and δ solve the same min-max problem. This property is rather immediate for a variational u_∞, while the Harnack inequality

$$|\nabla \log u| \le |\nabla \log \delta|,$$

valid a.e. for non-negative ∞-superharmonic functions u, seems to be called for when one works directly with equation (1.4). An important consequence of (1.7) is that the normalization

$$\|u_\infty\|_\infty = \|\delta\|_\infty = \Lambda^{-1}$$

enables us to conclude

$$0 < u_\infty \le \delta \tag{1.8}$$

pointwise in Ω. Moreover, u_∞ can attain its largest value only at a point where the distance to the boundary is maximal.

2. Analytic expressions

The following functions

$$a\sqrt{x_1^2+\cdots+x_k^2}+b \qquad (1\le k\le n)$$
$$\langle \bar{a},x\rangle + b \qquad (\bar{a}\in\mathbb{R}^n)$$
$$a_1|x_1|^{4/3}+\cdots+a_n|x_n|^{4/3}+b \qquad (a_1^3+\cdots+a_n^3=0)$$

are ∞-harmonic. The last two functions are ∞-harmonic in the whole n-dimensional space, whereas the first one is a solution outside the set $x_1^2+\cdots+x_k^2=0$. All the angles in spherical coordinates, like

$$\arctan\frac{x_2}{x_1}, \quad \arctan\frac{x_3}{\sqrt{x_1^2+x_2^2}},$$

can be added to the list. Superpositions of expressions in disjoint variables are possible.[1] For example

$$3\sqrt{x_1^2+x_2^2}+7\sqrt{x_3^2+x_4^2}+c(\,|x_5|^{4/3}-|x_6|^{4/3})$$

is ∞-harmonic outside the hyperplanes, where the square roots vanish. The function is ∞-subharmonic in the whole $\mathbb{R}^6$.

The interesting example

$$|x|^{4/3}-|y|^{4/3}$$

in two variables belongs to a celebrated family of solutions, the "quasi-radial" solutions found by Aronsson [A3]. It is ∞-harmonic in the entire xy-plane. It is not smooth on the coordinate axes. The regularity class is

$$C^{1,1/3}_{loc}\cap W^{2,\frac{3}{2}-\varepsilon}_{loc}$$

for each $\varepsilon>0$. (The 3^{rd} derivatives are not Sobolev derivatives.) It has been conjectured that, at least in the plane, every ∞-harmonic function is of Hölder class $C^{1,\alpha}_{loc}$ with $\alpha=\frac{1}{3}$. Even the continuity of the gradient seems to be an open problem, see [CEG].

[1] This seemingly innocent property has surprising consequences. It implies that the regularity of those ∞-harmonic functions in $\mathbb{R}^{n+1}$ that have non-vanishing gradient cannot be any better than that of all ∞-harmonic functions in $\mathbb{R}^n$. This is certainly not the case for the p-Laplace equation.

3. The distance function

We will describe a class of domains for which the function

$$\delta(x) = \operatorname{dist}(x, \partial\Omega)$$

representing the distance from the point x to the boundary of the bounded domain Ω, is an infinity ground state. The distance δ is always ∞-superharmonic. Moreover, δ is ∞-harmonic outside the *ridge* of Ω, cf. [A1]. The set of points where δ is not differentiable will do as our definition of the ridge. See [F] for a more refined distinction.

3.1. Proposition. *Let $R = \Lambda^{-1}$ be the radius of the largest inscribed ball. Suppose that the distance function δ is differentiable at all other points except at those at which it attains its maximum R. Then δ is an infinity ground state.*

Proof. Since δ is ∞-superharmonic and $|\nabla\delta| = 1 \geq \Lambda\delta$ a.e. in Ω, we easily obtain that

$$\max\left\{\Lambda - \frac{|\nabla\delta(x)|}{\delta(x)}, \Delta_\infty\delta(x)\right\} \leq 0$$

in the viscosity sense. This is, in fact, true in any domain.

To show that δ is also a subsolution of (1.4), we first notice that since δ is ∞-harmonic outside the ridge of Ω, it suffices to check that

$$\frac{|\nabla\varphi(x_0)|}{\varphi(x_0)} \leq \Lambda$$

for every smooth test-function φ touching δ from above at a point $x_0 \in \Omega$ in which $\delta = R$. But this must be the case because

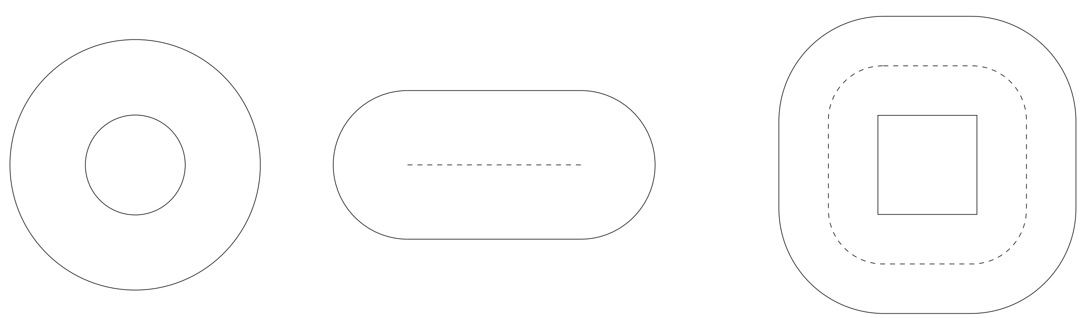
$$\varphi(x) \geq \delta(x) \geq R - |x - x_0|$$

for every $x \in \Omega$, and thus

$$|\nabla\varphi(x_0)| \leq 1 = \Lambda\varphi(x_0).$$

□

An example of a domain satisfying the above condition is the annulus $r < |x| < R$, including the punctured ball $0 < |x| < R$. A ball is another example.

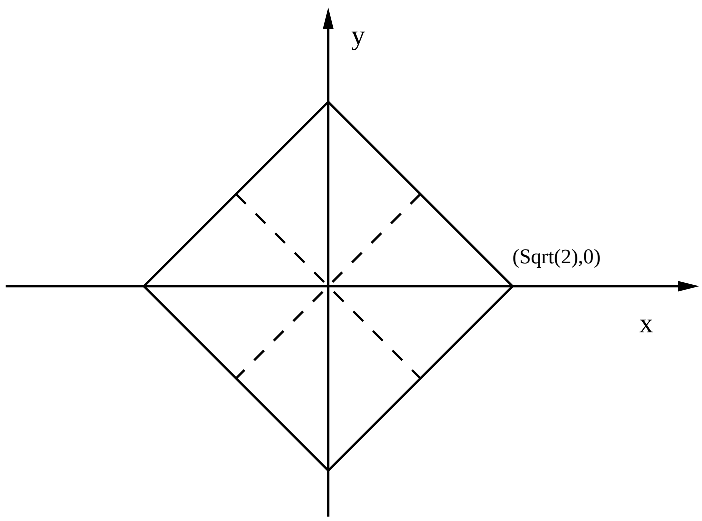


Often Ω can be defined as the set of points at a distance $< R$ from a given curve γ. Of course, γ has to meet some requirements. If the curve is a line segment, we get a "stadium" (in two dimensions). In general, the resulting domain need not be smooth as exemplified by the "squarish annulus" above on the right.

We do not know if there are other solutions than $u_\infty = \delta$ in these domains, the favorable exception being the ball, where we know the uniqueness.

4. The square

This example is an important building block. Let Q denote the square $|x|+|y| < \sqrt{2}$.



Then $\Lambda = 1$. By comparison[2] we obtain

$$1 - \sqrt{x^2+y^2} \le V_\infty(x,y) \le u_\infty(x,y) \le 1 - \frac{|x|+|y|}{\sqrt{2}} \tag{4.1}$$

for any properly normalized infinity ground state u_∞. The last function is the distance $\delta = \delta(x,y)$ and the first represents the largest inscribed cone. As usual, V_∞ is the ∞-harmonic function in the punctured square $0 < |x|+|y| < \sqrt{2}$ with boundary values 0 on ∂Q and 1 at the midpoint (the origin). The function V_∞ is unique by Jensen's theorem. It has all the symmetries of the square. However, we do not know that about u_∞.

From (4.1) we can read off a useful fact. Along the segments of the normals $y = \pm x$ the functions coincide:

$$V_\infty = u_\infty = \delta \qquad \text{when } y = \pm x. \tag{4.2}$$

This is needed for the gluing in the next section.

[2]It follows from equation (1.4) that u_∞ is ∞-superharmonic.

Along the diagonals (= the coordinate axes):

$$V_\infty \le u_\infty < \delta, \qquad \text{when } x = 0 \text{ or } y = 0. \tag{4.3}$$

The strict inequality (4.3) was proved by direct testing in the viscosity sense in [JLM], but the fact that $V_\infty < \delta$ on the diagonals (except at the origin) comes more directly from a plain comparison with solutions of the type

$$\pm(x - x_0)^{4/3} \mp (y - y_0)^{4/3}.$$

The normals through the midpoint (the lines $y = \pm x$) divide the square Q in four subsquares. In the west subsquare we have

$$\frac{(x+\sqrt{2})^2 - y^2}{2} \le V_\infty(x, y) \le \frac{(x+\sqrt{2})^{4/3} - y^{4/3}}{2^{2/3}}. \tag{4.4}$$

Here $|x+1/\sqrt{2}|+|y| \le 1/\sqrt{2}$. There are similar expressions in the other subsquares. The lower bound is ∞-subharmonic in its subsquare and the upper bound is ∞-harmonic. Thus it is sufficient to compare the boundary values, using (4.2). Notice that the upper bound in (4.4) is less than δ on the diagonal $-\sqrt{2} < x < 0$, $y = 0$.

The potential function V_∞ is the uniform limit of the corresponding p-harmonic functions V_p as $p \to \infty$. According to a theorem of J. Lewis [L], each V_p is real analytic in the punctured square $0 < |x| + |y| < \sqrt{2}$. This is not the case for V_∞. This is a striking collapse of regularity. Indeed, V_∞ is not even of class C^2. To see this, notice that V_∞ inherits the symmetry

$$V_\infty(x, y) = V_\infty(x, -y)$$

from V_p. For symmetry it is essential to know that V_∞ is unique. (The corresponding piece of information is missing for u_∞.) To proceed, assume that V_∞ has continuous second derivatives on the segment $0 < x < \sqrt{2}$ of the x-axis. This leads to a contradiction. By symmetry

$$\frac{\partial V_\infty}{\partial y}(x, 0) = 0 \qquad \text{for all } 0 < x < \sqrt{2}$$

so that

$$\Delta_\infty V_\infty = \left(\frac{\partial V_\infty}{\partial x}\right)^2 \frac{\partial^2 V_\infty}{\partial x^2} = 0$$

on the x-axis. Hence

$$\frac{d}{dx}\left(\frac{\partial V_\infty(x, 0)}{\partial x}\right)^3 = 0,$$

which implies that $V_\infty(x, 0)$ is the linear function

$$V_\infty(x, 0) = \frac{\sqrt{2} - x}{\sqrt{2}} = \delta(x, 0).$$

This contradicts (4.3). Thus there has to be at least one point (on the diagonal) where V_∞ is not of class C^2.

The corresponding consideration is possible for a *variational* infinity ground state u_∞. In the general case, because the uniqueness question is unsettled, we have to proceed without relying upon symmetry. The proof is indirect (reductio ad absurdum). Assume that u_∞ belongs to $C^2(Q \setminus \{0\})$. According to our Λ-lemma u_∞ is ∞-harmonic. It has the same boundary values as V_∞. By Jensen's uniqueness theorem $u_\infty = V_\infty$. But we have already established that V_∞ is not of class C^2. The antithesis was false.

It stands to reason that the functions u_∞ and V_∞ coincide in the square, but we do not have a valid proof.

To this we may add that it is not the corners that cause the lack of regularity. The same effect occurs in the ellipse.

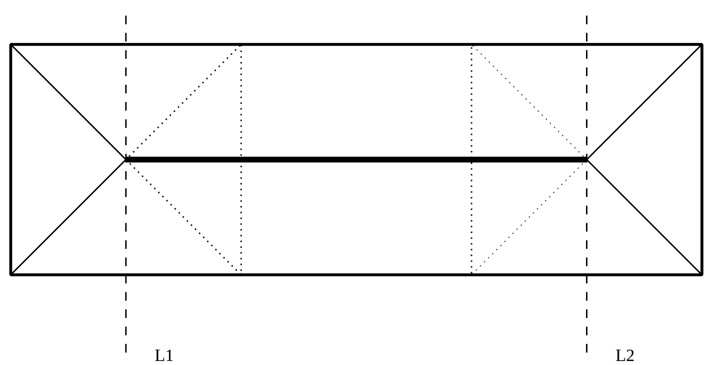
5. The rectangle

The counterexample to the strong comparison principle comes from our investigations in a rectangle. We will find two ∞-harmonic functions, say u and v, in a certain domain Ω with the properties

(1) $u \le v$ in Ω,
(2) $u = v$ in a subdomain of Ω,
(3) $u < v$ somewhere in Ω.

Such a behavior is out of the question for the p-Laplace equation (1.3) in the plane, cf. [M], but it is an open problem in higher dimensions. We may add that u and v are linear in the set of coincidence.

Our rectangle R consists of three pieces: two halves of the square Q (placed at opposite ends) and a rectangle (placed between the halves of the square). The gluing is along the vertical lines L_1 and L_2.



Let X denote the high ridge, i.e., the closed line segment in the middle between the lines L_1 and L_2. This is the thick line in the figure. It is precisely the set of points where the distance δ achieves its maximum. Let V_∞ be the ∞-harmonic function in $R \setminus X$ with boundary values 0 on ∂R and 1 on X.

Actually, V_∞ can be glued of three known functions. $V_\infty = \delta$ between the lines L_1 and L_2. To the right of the right line L_2, V_∞ equals to the potential function V^Q for the square (described in Section 4). A similar gluing is to the left. It is clear that the so obtained function is ∞-harmonic off the segments of the gluing lines. Notice that it is continuous, because of (4.2). We have to verify that we have a solution also at L_1 and L_2. The concept of viscosity solutions is well-suited for a situation like this where the functions actually are overlapping. First, suppose that φ is a test-function touching from above at the point (x_0, y_0) on L_2. Since the glued function is pointwise

$$\max\{\delta, V^Q\}$$

to the left of L_2 (see (4.1)), $\varphi \geq V^Q$. Thus φ will do as test-function for the square case and hence we know that

$$\Delta_\infty \varphi(x_0, y_0) \geq 0.$$

This is what is required. Second, if φ touches from below at (x_0, y_0), then $\varphi \leq \delta$ and $\varphi(x_0, y_0) = \delta(x_0, y_0)$. Thus φ will do as test-function for δ, which is known to be ∞-superharmonic. Thus

$$\Delta_\infty \varphi(x_0, y_0) \leq 0$$

in this case.

This proves that we have constructed an ∞-harmonic function. By Jensen's uniqueness theorem the function is V_∞. To violate the strong comparison principle we take the lower half of the rectangle R as our domain Ω. Define $u = V_\infty$ (the restriction of the function constructed above) and v as the linear function that coincides with δ between the lines L_1 and L_2 (but not outside). This is the desired counterexample. A fortiori it demonstrates that the continuation of a solution to the infinity harmonic equation is not unique.

The same construction can be used to glue the infinity ground state u_∞ of the rectangle together of three pieces: the two halves of the infinity ground state of the appropriate square and a piece of the distance function.

Remark. Very recently Eero Saksman has pointed out that a simpler example is obtained, if the square in the previous construction is replaced by a disc. In this case it is the two halves of the disc and the rectangle, placed between the halves of the disc, that are glued together along the diameter. The domain is the "stadium" mentioned in Section 3. The advantage is that the solutions are explicitly known since they are the distance functions for the disc and stadium respectively.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O.Box 35 (MaD), FIN-40351 Jyväskylä, Finland

E-mail address: peanju@maths.jyu.fi

Department of Mathematics, Norwegian University of Science and Technology, N-7034 Trondheim, Norway

E-mail address: lqvist@math.ntnu.no

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

E-mail address: manfredi@pitt.edu