Papers on Analysis:
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ON THE CONCEPT OF THE WEAK JACOBIAN AND HESSIAN

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1. Introduction and Notation

Carl Gustav Jacob Jacobi (1804–1851), one of the prominent mathematical figures of his century, did not perhaps suspect that the determinants of differential matrices would enjoy rather special and high status today. No doubt, this has resulted from enormously creative (and exciting) expansion of the geometric theory of nonlinear PDEs and the calculus of variations [**M**1, **M**2, **BM**, **BZ**, **Š**1, **DM**, **W**1, **Š**2, **M**4, **RY**], geometric function theory (including mappings of bounded and unbounded distortion) [**R**2, **BI**, **R**3, **IM**, **IKO**1, **IKO**2, **AIKM**, **KKM**1, **KKM**2], continuum mechanics and nonlinear elasticity [**A**, **B**1, **B**2, **C**2, **GMS**, **M**7, **MS**, **MSS**, **MTS**, **T**1, **SS**], micro-structures of materials and crystals [**BJ**, **IVV**], and much more. These theories are largely concerned with the mappings

$$f = (f^1, f^2, ..., f^m) : \Omega \to \mathbb{R}^m$$

of an open region $\Omega \subset \mathbb{R}^n$ whose coordinate functions $f^1, f^2, ..., f^m$ belong to suitable Sobolev spaces. We call them Sobolev mappings. In particular, the linear differential map $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ (often called gradient) is defined at almost every point $x \in \Omega$. Using the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ this linear transformation is represented by the Jacobi matrix, again denoted by

$$Df(x) = \left[\frac{\partial f^i}{\partial x_j}\right] \in \mathbb{R}^{m\times n}, \quad \text{where} \quad i = 1, ..., m \quad \text{and} \quad j = 1, ..., n.$$

Observe that the notation $\mathbb{R}^{m\times n}$ is being used here for the space of all $m \times n$ matrices. To every pair (I, J) of ordered l-tuples $I : \ 1 \le i_1 < i_2 < ... < i_l \le m$ and $J : \ 1 \le j_1 < j_2 < ... < j_l \le n$ for $1 \le l \le \min\{m, n\}$, there corresponds an $l \times l$ -subdeterminant of $Df(x)$, commonly denoted by

$$\frac{\partial f^I}{\partial x_J} = \frac{\partial(f^{i_1}, ..., f^{i_l})}{\partial(x_{j_1}, ..., x_{j_l})}.$$

An important special case is $m = n = l$, for which we reserve three basic symbols:

$$\det [Df(x)] = J(x, f) = \frac{\partial(f^1, f^2, ..., f^n)}{\partial(x_1, x_2, ..., x_n)}.$$

When studying Jacobian determinants nothing deeper than integration by parts or Stokes' formula provide us with the starting point to important estimates. That is why the language of the exterior forms is best suited for dealing with Jacobian determinants. For example, we shall frequently write:

$$J(x,f)\,dx = df^1 \wedge df^2 \wedge \ldots \wedge df^n. \tag{1}$$

Subdeterminants of the differential matrix are none other than the coefficients of the wedge products:

$$df^I = df^{i_1} \wedge \ldots \wedge df^{i_l} = \sum_{1\le j_1<\ldots<j_l\le n} \frac{\partial(f^{i_1},\ldots,f^{i_l})}{\partial(x_{j_1},\ldots,x_{j_l})}\, dx_{j_1} \wedge \ldots \wedge dx_{j_l} = \sum \frac{\partial f^I}{\partial x_J}\, dx_J$$

corresponding to all l -tuples $I:\ 1 \le i_1 < \ldots < i_l \le m$.

Let $\Lambda^l = \wedge^l \mathbb{R}^n,\ l = 1,\ldots,n$, denote the space of all l -covectors in $\mathbb{R}^n$ (that is, the l -linear alternating forms), and set $\Lambda^0 = \wedge^0\mathbb{R}^n = \mathbb{R}$. A differential l -form on $\Omega \subset \mathbb{R}^n$ is simply a function or Schwartz distribution, with values in $\Lambda^l = \wedge^l\mathbb{R}^n$. Sobolev spaces of differential forms, denoted by $W^{k,p}_{loc}(\Omega,\Lambda^l),\ k = 1,2,\ldots,\ \ 1 \le p \le \infty$; will be used throughout this text. This means that all k^{th}-order derivatives of the coefficients of those forms are locally L^p-integrable. The class of locally Hölder continuous functions of exponent $0 \le \alpha < 1$ will be denoted by $C^\alpha_{loc}(\Omega)$, where $C^0_{loc}(\Omega) = C(\Omega)$ simply consists of continuous functions. As for the space $\mathcal{D}'(\Omega)$ of Schwartz distributions, we shall distinguish its subspaces $\mathcal{D}'_k(\Omega),\ k = 0,1,\ldots,$ which consist of the distributions of order at most k. Thus a linear functional $H : C^\infty_0(\Omega) \to \mathbb{R}$ belongs to $\mathcal{D}'_k(\Omega)$ if and only if for each compact $\Upsilon \subset \Omega$ there exists a number $C_\Upsilon \ge 0$ such that

$$|\ H[\phi]\ | \ \le\ C_\Upsilon \sup_{x\in\Upsilon} \sum_{|\alpha|\le k} |\ D^\alpha\phi(x)\ |$$

for all test functions $\phi \in C^\infty_0(\Omega)$ with support in Υ. Note, for the future discussion, that $\mathcal{D}'_0(\Omega)$ consists precisely of the Radon measures on Ω. Other relevant notation will remain standard or self-explanatory.
Besides the subdeterminants of a differential matrix, more general wedge products will also motivate our interest. To dignify the work of Jacobi we adopted the notation

$$\mathcal{J}(x,\Theta)\,dx = \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^m. \tag{2}$$

In this formula the m -tuple $\Theta = (\theta^1,\theta^2,\ldots,\theta^m)$ consists of closed (or exact) differential forms $\theta^i \in L^{p_i}(\Omega,\Lambda^{l_i})$ whose total degree, $l_1 + l_2 + \ldots + l_m$, equals the dimension n of the domain Ω. The reader may wish to know that the closed forms θ^i can always be expressed (at least locally) as $\theta^i = d\eta^i$ for some η^i of Sobolev class W^{1,p_i}_{loc}. In many instances these η^i will play the role parallel to that of the coordinate functions (0-forms) of Sobolev mappings just discussed. For the purpose of our analysis it will involve no loss of generality in assuming that all θ^i are actually exact in the entire domain Ω. Precisely we shall take

$$\theta^i = d\eta^i, \quad \text{with} \quad \eta^i \in W^{1,p_i}_{loc}(\Omega,\Lambda^{l_i-1}), \quad \text{for all } i = 1,2,\ldots,m.$$

We should indicate at this point that piecing together local estimates can routinely be effected in terms of a partition of unity. But detailed exposition of this procedure would take us too far afield.

2. Weak Extensions

A central theme in the theory of PDEs is the definition of the domain of a differential operator. One usually begins with a class of smooth functions. Then in case of linear PDEs the extension to larger classes goes through easily due to the concept of the formal adjoint operator. Of course, integration by parts is the key to this extension procedure. Such an extension is termed the closure of the operator. We emphasize that the definition of the closure depends upon the topology in both the domain space and the range space, see the seminal work of K.O. Friedrichs and further development by L. Hörmander [**F**, **H**1]. In this article we will set down weak extensions for some nonlinear operators. Bold face letters will be used to denote their new domains.

Perhaps most elegant and clear example is furnished by the Hessian in two real variables. We begin with a few elementary identities:

$$\begin{aligned} \mathcal{H}u &= u_{xx}u_{yy} - u_{xy}u_{xy} \\ &= (u_x u_{yy})_x - (u_x u_{xy})_y = (u_y u_{xx})_y - (u_y u_{xy})_x \\ &= \tfrac{1}{2}(uu_{xx})_{yy} + \tfrac{1}{2}(uu_{yy})_{xx} - (uu_{xy})_{xy} \\ &= (u_x u_y)_{xy} - \tfrac{1}{2}(u_x u_x)_{yy} - \tfrac{1}{2}(u_y u_y)_{xx} \end{aligned} \tag{3}$$

which are provisionally defined for $u \in C^\infty(\Omega)$. From these starting identities we shall extend $\mathcal{H}u$ to various Sobolev spaces. As a result the values of the Hessian operator will be Schwartz distributions of order 0,1,2 and 2, respectively. To this effect, we simply multiply the corresponding expression at (3) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate by parts to determine how Hessian acts on ϕ. This brings us to four different notions of the distributional Hessian.

DEFINITION 1 (*Hessian in the plane*). The identities at (3) give rise, respectively, to the following nonlinear differential operators:

i) zero order Hessian $\quad \mathcal{H}_0 : \mathbf{W}^{2,2}_{loc}(\Omega) \to L^1_{loc}(\Omega) \subset \mathcal{D}'_0(\Omega)$

ii) first order Hessian $\quad \mathcal{H}_1 : \mathbf{W}^{2,4/3}_{loc}(\Omega) \to \mathcal{D}'_1(\Omega)$

iii) second order Hessian $\quad \mathcal{H}_2 : \mathbf{W}^{2,1}_{loc}(\Omega) \to \mathcal{D}'_2(\Omega)$

iv) very weak Hessian $\quad \mathcal{H}^*_2 : \mathbf{W}^{1,2}_{loc}(\Omega) \to \mathcal{D}'_2(\Omega)$

Note the inclusions:

$$\mathbf{W}^{2,2}_{loc}(\Omega) \subset \mathbf{W}^{2,4/3}_{loc}(\Omega) \subset \mathbf{W}^{2,1}_{loc}(\Omega) \subset \mathbf{W}^{1,2}_{loc}(\Omega). \tag{4}$$

These domains have been determined by using Sobolev imbedding theorems and Hölder's inequality to ensure that upon integration by parts the resulting integrals will converge. We illustrate more explicitly the case $\mathcal{H}_2$. By the definition, $\mathcal{H}_2$ acts on a test function $\phi \in C_0^\infty(\Omega)$ as

$$(\mathcal{H}_2 u)[\phi] = \frac{1}{2}\int_\Omega u\,(u_{xx}\,\phi_{yy} + u_{yy}\,\phi_{xx} - 2u_{xy}\,\phi_{xy})\;dx\,dy.$$

To justify existence of this integral we need only observe that the second order derivatives of u are locally integrable and, consequently, u is continuous. Other explicit formulas are presented in Sections 4 and 5.

Hessian arises in a large variety of geometric problems (Gaussian curvature) and nonlinear elliptic PDEs. By way of an example, let us take on stage a genuine nonlinear elliptic type equation

$$\mathfrak{A}(x, y,\;\; u_x, u_y,\;\; u_{xx}, u_{xy}, u_{yy}) = 0, \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

This includes the familiar equations of gas dynamics for subsonic flows, minimal surfaces or the p-harmonic equation $div\,(|\nabla u|^{p-2}\nabla u) = 0$. In any case the complex gradient $f(z) = u_x - iu_y,\; z = x+iy$, solves an elliptic first order system of PDEs and, therefore, is a mapping of bounded distortion. For instance, when u is p-harmonic we arrive at the quasilinear system which reduces to one complex equation for f;

$$\frac{\partial f}{\partial \overline{z}} = \frac{2-p}{2p}\left(\frac{\overline{f}}{f}\,\frac{\partial f}{\partial z} + \frac{f}{\overline{f}}\,\overline{\frac{\partial f}{\partial z}}\right).$$

This in turn yields the distortion inequality

$$\left|\frac{\partial f}{\partial \overline{z}}\right| \le \left|1 - \frac{2}{p}\right|\;\left|\frac{\partial f}{\partial z}\right|.$$

Within this wider context the Jacobian of f (Hessian of u) becomes an absolutely essential part of this theory. In quasiconformal analysis one usually assumes that f lies in the Sobolev class $W^{1,2}_{loc}(\Omega)$. Having disposed of the notion of the very weak Hessian, one could formulate and try to investigate the distributional Jacobian for $f \in \mathbf{L}^2_{loc}(\Omega)$. But we shall not pursue these matters here because such very weak Jacobian is lacking weak continuity properties. A simple example is dealt with in Section 3.
Writing nonlinear differential expressions in a suitable form for the purpose to weaken the regularity hypotheses is an art. There are methods galore, especially if we admit nonlinear classes of functions or certain measures as new domains of those operators. For instance, plurisubharmonic functions in the complex space $\mathbb{C}^n$ can be used to investigate Hessian without getting into the second derivatives. In this article, however, our concern with nonlinear domains of the Jacobian and Hessian type operators will be restricted to rather general remarks.

We now bring the discussion back to a Sobolev mapping $f = (f^1, f^2, ..., f^n) : \Omega \to \mathbb{R}^n$. In contrast to the very weak Hessian case the trick with the integration by parts will not entirely eliminate differentiation of the coordinate functions. Still we have something to gain; namely, we can lower the required degree of integrability of $df^1, df^2, ..., df^n$. The natural domain for the Jacobian operator consists of mappings

whose coordinate functions f^i lay in $W^{1,p_i}_{loc}(\Omega)$, where the Sobolev exponents $1 \leq p_i \leq \infty$, are Hölder conjugate, $\frac{1}{p_1} + ... + \frac{1}{p_n} = 1$. With the aid of Hadamard's inequality $|J(x,f)| \leq |df^1|\ |df^2|...|df^n|$ this natural exponents ensure that $J(x,f)$ is a locally integrable function.
However, we can do slightly better once we integrate by parts

$$\int_\Omega \phi(x)\ J(x,f)\ dx = -\int_\Omega f^i\ df^1 \wedge ... \wedge df^{i-1} \wedge d\phi \wedge df^{i+1} \wedge ... \wedge df^n \tag{5}$$

for all test functions $\phi \in C_0^\infty(\Omega)$ and each index $i = 1, 2, ..., n$. Just as distributional Hessian was defined one may take this identity as the starting point for a definition of the weak Jacobian. This time, the integral in the right hand side converges if the coordinate functions f^i belong to $W^{1,s_i}_{loc}(\Omega)$. We need only assume that the exponents $1 \leq s_1, s_2, ..., s_n < \infty$ satisfy the so-called Sobolev relation

$$\frac{1}{s_1} + \frac{1}{s_2} + ... + \frac{1}{s_n} = 1 + \frac{1}{n}. \tag{6}$$

Note that such Sobolev conjugate tuples always contain an exponent $1 \leq s_i < n$, and this index i is exactly the one we shall select to validate the definition below. Note that different choices of such indices yield the same value of the integral in the right hand side of (5). The following definition expands on the idea of J. Ball [**B**1]

DEFINITION 2 *(Weak Jacobian).* Under the above terms, the weak Jacobian $\mathcal{J}_f$ is a distribution of order 1. Its value on a test function $\phi \in C_0^\infty(\Omega)$ is defined at the right hand side of Equation (5).

It is immediately obvious that the value $\mathcal{J}_f[\phi]$ on a test function $\phi \in C_0^\infty(\Omega)$ is controlled by

$$|\mathcal{J}_f[\phi]| \leq \| \nabla\phi \|_\infty \| df^1 \|_{s_1} \cdots \| df^{i-1} \|_{s_{i-1}} \| f^i - f_0^i \|_{\frac{ns_i}{n-s_i}} \| df^{i+1} \|_{s_{i+1}} \cdots \| df^n \|_{s_n}, \tag{7}$$

where f_0^i can be any constant. All norms here are taken over a relatively compact subdomain $\Omega' \subset \Omega$ containing the support of ϕ. We wish to arrange that Ω' be smooth. At this stage one may appeal to Poincarè - Sobolev Lemma to obtain

$$|\,\mathcal{J}_f[\phi]\,| \leq C_{\Omega'}(s_1, ..., s_n)\ \| \nabla\phi \|_\infty \| df^1 \|_{s_1} \| df^2 \|_{s_2} \cdots \| df^n \|_{s_n}.$$

It is relatively simple to derive analogous estimates for the difference of two Jacobians. We then conclude, among other things, with the following result

THEOREM 1 *(Weak compactness). The nonlinear differential operator*

$$\mathcal{J} : \mathbf{W}^{1,s_1}_{loc}(\Omega) \times \mathbf{W}^{1,s_2}_{loc}(\Omega) \times ... \times \mathbf{W}^{1,s_n}_{loc}(\Omega) \longrightarrow \mathcal{D}_1'(\Omega)$$

is weakly compact, whenever the Sobolev exponents satisfy

(8) $$\frac{1}{s_1} + \frac{1}{s_2} + ... + \frac{1}{s_n} < 1 + \frac{1}{n}.$$

In other words, for each $\phi \in C_0^\infty(\Omega)$ we have

(9) $$\lim_{\nu \to \infty} \mathcal{J}_{f_\nu}[\phi] = \mathcal{J}_f[\phi]$$

whenever the mappings $f_\nu = (f_\nu^1, f_\nu^2, ..., f_\nu^n)$ converge locally to $f = (f^1, f^2, ..., f^n)$ with respect to the weak topology in $\mathbf{W}_{loc}^{1,s_1}(\Omega) \times \mathbf{W}_{loc}^{1,s_2}(\Omega) \times ... \times \mathbf{W}_{loc}^{1,s_n}(\Omega)$. Probably, this fact was first brought to light in 1953 by R. Caccioppoli [**C**1] for two dimensional case. Then it was profitably extended to higher dimensions in [**M**1, **M**2, **R**1, **B**1]. Weak continuity of the Jacobian operator follows from the inequality (9) and compactness of the Sobolev imbedding $W_{loc}^{1,s_i}(\Omega) \subset L_{loc}^{s}(\Omega)$, for $1 \leq s < \frac{ns_i}{n-s_i}$. A far reaching generalization of the weak continuity (biting convergence) is due to K. Zhang [**Z**1].

An exactly similar consideration pertains in proving weak continuity properties of wedge products of differential forms. The purely formal approach to the meaning of the distribution $\mathcal{J}_\Theta$ proceeds by use of Stokes' formula. Suppose that the components of the m-tuple $\Theta = (\theta^1, \theta^2, ..., \theta^m)$ are closed forms. More precisely,

(10) $$\Theta = (\theta^1, ..., \theta^m) \in L_{loc}^{s_1}(\Omega, \Lambda^{l_1}) \times ... \times L_{loc}^{s_m}(\Omega, \Lambda^{l_m}),$$

where $d\theta^1 = ... = d\theta^m = 0$. As remarked in Section 1, we can always assume that $\theta^i = d\eta^i$ for some forms $\eta^i \in W_{loc}^{1,s_i}(\Omega, \Lambda^{l_i - 1})$. For the interested reader let us also indicate that using L^p -Hodge theory we can control the W_{loc}^{1,s_i} -norms of η^i by means of the $L_{loc}^{s_i}$ -norms of θ^i. The following formula is classical if all η^i are smooth

(11) $$\int_\Omega \phi(x)\, \mathcal{J}(x, \Theta)\, dx = -\int_\Omega \theta^1 \wedge ... \wedge \theta^{i-1} \wedge (d\phi \wedge \eta^i) \wedge \theta^{i+1} \wedge ... \wedge \theta^m$$

for each test function $\phi \in C_0^\infty(\Omega)$. By virtue of the imbedding theorems, the right hand side can be extended satisfactorily to the case in which the Sobolev exponents, $1 \leq s_1, s_2, ..., s_m < \infty$, satisfy

(12) $$\frac{1}{s_1} + \frac{1}{s_2} + ... + \frac{1}{s_m} \leq 1 + \frac{1}{n}, \quad \text{and} \quad 1 \leq s_i < n, \quad \text{for some} \quad i = 1, .., m.$$

As one might expect, different choices of the forms η^i (closed forms can be added to them) will yield the same integrals in the right hand side of (11). Accordingly, we make the following

DEFINITION 3 *(Weak wedge product).* The weak wedge product, denoted by $\mathcal{J}_\Theta \in \mathcal{D}'(\Omega)$, operates on test functions $\phi \in C_0^\infty(\Omega)$ by the rule given at the right hand side of (11).

We have to add one remark concerning Hadamard's type inequality:

$$|\theta^1 \wedge \theta^2 \wedge ... \wedge \theta^m| \leq \; C\,|\theta^1|\,|\theta^2|...|\theta^m|.$$

Rather surprisingly, the sharp constant $C = C(l_1, l_2, ..., l_m)$ not always equals 1. But it does if $l_1 = l_2 = ... = l_m = 1$, see [**IKKS**] for sharp constants.

Now if $s_1, s_2, ..., s_m$ satisfy strict inequality at (12) then the weak compactness of the wedge product operator

$$\mathcal{J} : \mathbf{L}^{s_1}_{loc}(\Omega, \Lambda^{l_1}) \times ... \times \mathbf{L}^{s_m}_{loc}(\Omega, \Lambda^{l_m}) \longrightarrow \mathcal{D}'_1(\Omega, \Lambda^l),$$

will indeed continue to hold when restricted to closed forms. Further details and generalizations can be found in [**RRT**, **IL**, **I**1, **GIM**, **G**1, **CG**, **IV**].

The above weak compactness property of the Jacobian, Hessian and wedge products is often useful in the discussion of the existence of minima of variational integrals [**B**1] or limit theorems in quasiconformal geometry [**GI**1, **IKO**2]. In this category of useful applications we must certainly include recent advances in the theory of mappings with unbounded distortion [**IM**, **IKMS**].

Concerning weak wedge products, a case of special interest is that in which the closed forms θ^i, $\;i = 1, ..., m,\;$ are composed from a mapping $f = (f^1, f^2, ..., f^n)$: $\Omega \to \mathbb{R}^n$ by the rule

$$\theta^i = df^{I_i} = df^{i|1} \wedge df^{i|2} \wedge ... \wedge df^{i|l_i}, \qquad i = 1, 2, ..., m,$$

where all the ordered l_i -tuples $I_i : 1 \leq i|1 < i|2 < ... < i|l_i \leq n$ constitute a partition of the set $\{1, 2, ..., n\}$. The general underlying idea here is that we can take lower order minors of the differential matrix $Df(x)$ in different L^p -spaces, as needed for a specific problem [**IL**, **IM**, **GIOV**, **I**4]. Before leaving this brief exposition mention should be made of seminal ideas about div-curl product and the principle of compensated compactness in [**T**2, **M**5, **M**6], with far reaching extensions in [**CLMS**], see also [**I**3].

3. Hessian in the complex plane revisited

The two dimensional case is of sufficient interest to call for closer examination. We then identify $\mathbb{R}^2$ with the complex plane $\mathbb{C} = \{z = x + iy; \;\; (x, y) \in \mathbb{R}^2\}$. The Cauchy-Riemann derivatives $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ are the basic differential operators in $\mathbb{C}$. Now Hessian of a real valued function u is just the Jacobian determinant of the complex gradient $f(z) = 2\,u_{\overline{z}} = u_x + iu_y$. The point-wise Hessian

$$\mathcal{H}(z, u) = u_{xx}\,u_{yy} \;-u_{xy}\,u_{xy} \;=\; |f_z|^2 \;-\; |f_{\overline{z}}|^2 \;= 4\,(|u_{z\overline{z}}|^2 \;-\; |u_{\overline{z}\overline{z}}|^2)$$

is usually considered in the Sobolev space $\mathbf{W}^{2,2}_{loc}(\Omega)$ as a regular distribution. Using the notation of exterior forms we find that $\mathcal{H}$ operates on test functions $\phi \in C_0^\infty(\Omega)$ by the rule

$$(\mathcal{H}_0 u)[\phi] = \int_\Omega \phi\; du_x \wedge du_y = 2i \int_\Omega \phi\; du_z \wedge du_{\overline{z}}.$$

Our first extension of $\mathcal{H}_0$ is defined for $u \in \mathbf{W}^{2,4/3}_{loc}(\Omega)$ by the equation

$$\mathcal{H}_1 u = 4(u_{\overline{z}} u_{z\overline{z}})_z - 4(u_{\overline{z}} u_{zz})_{\overline{z}}.$$

Hence $\mathcal{H}_1 u$ is a distribution of order 1. Accordingly, it acts on ϕ as

$$(\mathcal{H}_1 u)[\phi] = \int_\Omega u_x \, du_y \wedge d\phi = 2i \int_\Omega d\phi \wedge u_{\overline{z}} du_z.$$

We have two second order distributions. The first one

$$\mathcal{H}_2 u = 4 \, (u \, u_{z\overline{z}})_{z\overline{z}} - 2 \, (u \, u_{zz})_{\overline{z}\overline{z}} - 2 \, (u \, u_{\overline{z}\overline{z}})_{zz}$$

is defined for each $u \in \mathbf{W}^{2,1}_{loc}(\Omega)$ by the integral formula

$$(\mathcal{H}_2 u)[\phi] = \frac{1}{2} \int_\Omega u \, (du_x \wedge d\phi_y - du_y \wedge d\phi_x) = 2 \, \Im \left(\int_\Omega u \, du_z \wedge d\phi_{\overline{z}} \right).$$

Our main interest in this section lies in the second type extension of $\mathcal{H}_1$, called very weak Hessian

$$\mathcal{H}^*_2 u = 2 \, (u_z u_z)_{\overline{z}\overline{z}} + 2 \, (u_{\overline{z}} u_{\overline{z}})_{zz} - 4 \, (u_z u_{\overline{z}})_{z\overline{z}}.$$

The largest Sobolev space in which $\mathcal{H}^*_2$ makes sense is of course $\mathbf{W}^{1,2}_{loc}(\Omega)$. We again use the language of exterior forms to express its action on test functions

$$(\mathcal{H}^*_2 u)[\phi] = \frac{1}{2} \int_\Omega du \wedge \, (u_y d\phi_x - u_x d\phi_y) = 2 \, \Im \left(\int_\Omega u_z \, du \wedge d\phi_{\overline{z}} \right).$$

In order to employ weak extensions of a nonlinear differential operator with confidence one must carefully investigate the question of continuity under the weak topology in the extended domains, like for linear operators. Being so, this will put a new complexion on the weak Jacobian or Hessian only if we can relate them with the underlying point-wise formulas. Our detailed account of what can go wrong will be confined to the case of the very weak Hessian $\mathcal{H}^*_2 : \mathbf{W}^{1,2}_{loc}(\Omega) \mapsto \mathcal{D}'_2(\Omega)$. We aim to show that in such a large domain as $\mathbf{W}^{1,2}_{loc}(\Omega)$ the operator $\mathcal{H}^*_2$ fails to be weakly continuous.

EXAMPLE 1 (The failure of continuity). *There is a sequence $\{u_k\}$ converging weakly to zero in the space $W^{1,2}(\Omega)$ such that*

$$\mathcal{H}^*_2 u_k \not\rightharpoonup 0 \quad in \;\; \mathcal{D}'(\Omega). \tag{13}$$

Proof. We shall use both complex and polar coordinates in the unit disk $\Omega = \{z = re^{i\alpha};\ 0 \le r \le 1,\ 0 \le \alpha < 2\pi\}$. The functions in question are defined by

$$u_k(z) = \frac{2}{k} r^2 \cos\,(2k\,\alpha) = \frac{1}{k} \left(\frac{z^{k+1}}{\overline{z}^{k-1}} + \frac{\overline{z}^{k+1}}{z^{k-1}} \right) \tag{14}$$

for $k = 1, 2, \ldots$. A short sketch proof of (13) runs somewhat as follows. It is clear that $u_k \to 0$ uniformly in Ω. Elementary computation shows that

$$\frac{\partial u_k}{\partial z} = \left(1 + \frac{1}{k}\right) \frac{z^k}{\overline{z}^{k-1}} + \left(\frac{1}{k} - 1\right) \frac{\overline{z}^{k+1}}{z^k}.$$

It is relatively simple to infer from here that $\{u_k\}$ converges to zero weakly in $W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. Although the second gradient of each u_k lies in $L^\infty(\Omega)$ the sequence $\{\nabla^2 u_k\}$ is not bounded in $L^\infty(\Omega)$, neither in the space $L^1(\Omega)$. The latter can be seen from the following elementary formulas

$$\frac{\partial^2 u_k}{\partial z \partial \overline{z}} = \left(\frac{1}{k} - k\right) \left(\frac{z^k}{\overline{z}^k} + \frac{\overline{z}^k}{z^k}\right),$$

$$\frac{\partial^2 u_k}{\partial z^2} = \left(k+1\right) \frac{z^{k-1}}{\overline{z}^{k-1}} + \left(k-1\right) \frac{\overline{z}^{k+1}}{z^{k+1}}.$$

Notice that the following integrals over every disk $B = B(0,r)$ are unbounded

$$\frac{1}{\pi\, r^2} \int_B \frac{\overline{z}^k}{z^k} \, \frac{\partial^2 u_k}{\partial z \partial \overline{z}} \; dx\, dy = \frac{1}{k} - k, \qquad k = 1, 2, \dots .$$

We therefore reason that the sequence $\{u_k\}$ is not bounded in any $W^{2,1}(U)$, whenever U contains the origin. It is perhaps worth computing the point-wise Hessian to find that it is strictly negative almost everywhere

$$\mathcal{H}(z, u_k) = \frac{8}{k^2} - 24 + \left(\frac{4}{k^2} - 4\right) \left(\frac{z^{2k}}{\overline{z}^{2k}} + \frac{\overline{z}^{2k}}{z^2 k}\right) \tag{15}$$

$$= 16\, k^{-2}(1 - k^2)\, \cos^2(2k\, \alpha) \; - 16 \quad \leq -16.$$

It remains to observe that, for fixed $k = 1, 2, \dots$, the point-wise Hessian of u_k coincides (as a distribution) with the very weak Hessian. Precisely, for each test function $\phi \in C_0^\infty(\Omega)$, we have

$$(\mathcal{H}_2^* u_k)[\phi] = \int_\Omega \phi(z) \mathcal{H}(z, u_k) \; dx\, dy \;\; \longrightarrow \;\; -24 \int_\Omega \phi(z) \; dx\, dy$$

by the first equation at (15). Thus the limit of the sequence $\{\mathcal{H}_2^* u_k\} \subset \mathcal{D}'(\Omega)$ is a constant distribution equal to -24 in Ω.

This example illustrates clearly that even uniform bounds of the first order gradients are insufficient for the weak continuity conclusion. It is this failure of continuity that qualifies $\mathcal{H}_2^*$ for the name very weak Hessian. Some degree of integrability of the second order derivatives are really necessary if we are to consider a theory of Hessian in the widest reasonable sense.

Interesting Question. Is it possible to make up Example 1 so that $\mathcal{H}(z, u_k) \geq 1$

almost everywhere for each $k = 1, 2, \ldots$?

4. Three Dimensional Analogues

Guessing the differential identities that might lead to the definition of weak Hessian in three and higher dimensions is much harder than in dimension two. The reader patient with (lengthy though elementary) computation may wish to verify the following identity

$$3\,\mathcal{H}(x,u) = 3\begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix} = \tag{16}$$

$$\left(\begin{vmatrix} u_{yy} & u_{yz} \\ u_{zy} & u_{zz} \end{vmatrix} u\right)_{xx} + \left(\begin{vmatrix} u_{xx} & u_{xz} \\ u_{zx} & u_{zz} \end{vmatrix} u\right)_{yy} + \left(\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} u\right)_{zz}$$

$$-2\left(\begin{vmatrix} u_{xy} & u_{xz} \\ u_{zy} & u_{zz} \end{vmatrix} u\right)_{xy} - 2\left(\begin{vmatrix} u_{yz} & u_{yx} \\ u_{xz} & u_{xx} \end{vmatrix} u\right)_{yz} - 2\left(\begin{vmatrix} u_{zx} & u_{zy} \\ u_{yx} & u_{yy} \end{vmatrix} u\right)_{zx}$$

One major advantage of this identity is that, upon integration by parts (twice), we arrive at an integrand which depends quadratically on the second order derivatives of u. By contrast, the Hessian determinant is a cubic polynomial in $\nabla^2 u$. Take notice that our new expression is a distribution of order 2 (second order Hessian), whenever $u \in W^{2,2}_{loc}(\Omega) \subset W^{1,6}_{loc}(\Omega) \subset C^{1/2}_{loc}(\Omega)$.

More sophisticated calculation is required to obtain the integrands which depend linearly on $\nabla^2 u$. But this is possible only at the expense of involving quadratic terms with respect to the first gradient. By way of a curiosity, if $u \in W^{2,\,9/5}_{loc}(\Omega) \subset W^{1,\,9/2}_{loc}(\Omega)$, then

$$-6\,\mathcal{H}(x,u) = -6\begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix} = \tag{17}$$

$$\begin{gathered}
(u_y^2\, u_{zz} + u_z^2\, u_{yy} - 2u_y u_z\, u_{yz})_{xx} + \\
(u_z^2\, u_{xx} + u_x^2\, u_{zz} - 2u_z u_x\, u_{zx})_{yy} + \\
(u_x^2\, u_{yy} + u_y^2\, u_{xx} - 2u_x u_y\, u_{xy})_{zz} + \\
2\,(u_z u_x\, u_{yz} + u_z u_y\, u_{zx} - u_x u_y\, u_{zz} - u_z^2\, u_{xy})_{xy} + \\
2\,(u_x u_z\, u_{xy} + u_x u_y\, u_{zx} - u_y u_z\, u_{xx} - u_x^2\, u_{yz})_{yz} + \\
2\,(u_y u_x\, u_{yz} + u_y u_z\, u_{xy} - u_z u_x\, u_{yy} - u_y^2\, u_{zx})_{zx}
\end{gathered}$$

the right hand side being a distribution of second order, called *very weak Hessian.*

This alternative formula owes its discovery to a more efficient calculation to be introduced in the next section. One may also ask at this stage if some amount of juggling with the integration by parts would result in a complete absence of the second gradient of u, as for the very weak Hessian in dimension 2. This is impossible for $n = 3$, and seemingly in higher dimensions as well.

5. Distributional Hessian in $\mathbb{R}^n$

Hessian of u is none other than the Jacobian determinant of the gradient field $f = (u_{x_1}, u_{x_2}, ...u_{x_n})$. Following the definition of distributional Jacobian, we find that for the distributional Hessian it is required that u be in the Sobolev class $\mathbf{W}_{loc}^{2,\frac{n^2}{n+1}}(\Omega)$. It is worth while noting the inclusions

$$W_{loc}^{2,\frac{n^2}{n+1}}(\Omega) \subset W_{loc}^{1,n^2}(\Omega) \subset C_{loc}^{\alpha}(\Omega)\ , \quad \text{where } \alpha = \tfrac{n-1}{n}.$$

We regard $\mathcal{J}_f$ as being the first order distributional Hessian of u.

DEFINITION 4 *(First order Hessian).* For each $u \in W_{loc}^{2,\frac{n^2}{n+1}}(\Omega)$ the distribution $\mathcal{H}_1 u \in \mathcal{D}_1'(\Omega)$ is defined to equal $\mathcal{J}_{(\nabla u)}$. Accordingly, it acts on a given test function $\phi \in C_0^\infty(\Omega)$ by the rule

$$(\mathcal{H}_1 u)[\phi] = \mathcal{J}_f[\phi] = -\int_\Omega du_{x_1} \wedge ... \wedge du_{x_{i-1}} \wedge u_{x_i} d\phi \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n}. \tag{18}$$

As before, this integral is independent of the choice of the index $i = 1, 2, ..., n$. The gradient structure of f will be reflected in further integration by parts leading to less restrictive classes of functions. Before making any integration, we must establish the following identity.

LEMMA 2. *Let $\Gamma : \Omega \mapsto \Lambda^1 = \wedge^1\mathbb{R}^n$ be a differential 1-form of Sobolev class $W_{loc}^{1,n-1}(\Omega, \Lambda^1)$, where $\Omega \subset \mathbb{R}^n$. Then*

$$\sum_{i=1}^n (-1)^i \frac{\partial}{\partial x_i}\left(\Gamma_{x_1} \wedge ... \wedge \widehat{\Gamma}_{x_i} \wedge ... \wedge \Gamma_{x_n}\right) = 0 \tag{19}$$

as a distribution.

The hat marks that Γ_{x_i} has been omitted. The distributional meaning of (19) is that for each 1 -form $\Xi \in C_0^\infty(\Omega, \Lambda^1)$ we have

$$\sum_{i=1}^n \int_\Omega \Gamma_{x_1} \wedge ... \wedge \Gamma_{x_{i-1}} \wedge \Xi_{x_i} \wedge \Gamma_{x_{i+1}} \wedge ... \wedge \Gamma_{x_n} \ = 0.$$

Here and subsequently the x_i -derivative of a differential form $\Xi = \chi^1 dx_1 + ... + \chi^n dx_n$ is again a differential form, partial differentiation is simply made component-wise.

$$\Xi_{x_i} = \frac{\partial \Xi}{\partial x_i} = \frac{\partial \chi^1}{\partial x_i}\, dx_1 + \frac{\partial \chi^2}{\partial x_i}\, dx_2 \ + ... + \ \frac{\partial \chi^n}{\partial x_i}\, dx_n.$$

Proof. We shall have established the lemma if we prove (19) for $\Gamma \in C^\infty(\Omega, \Lambda^1)$, since smooth 1-forms are dense in $W^{1,n-1}_{loc}(\Omega, \Lambda^1)$. To this end we differentiate in (19) by applying the usual product rule. Caution must be exercised because the exterior multiplication of 1-forms is anti-commutative. This will produce the second order derivatives $\Gamma_{x_k x_l}$, with $k \neq l$. For a given pair $1 \leq k < l \leq n$ there will emerge exactly two wedge products that contain $\Gamma_{x_k x_l}$ as a factor. These products are hidden in two terms of the sum at (19), namely

$$(-1)^k \frac{\partial}{\partial x_k}\left(\Gamma_{x_1} \wedge ... \wedge \widehat{\Gamma}_{x_k} \wedge ... \wedge \Gamma_{x_n}\right) + (-1)^l \frac{\partial}{\partial x_l}\left(\Gamma_{x_1} \wedge ... \wedge \widehat{\Gamma}_{x_l} \wedge ... \wedge \Gamma_{x_n}\right).$$

The first term gives

$$(-1)^k \Gamma_{x_1} \wedge ... \wedge \widehat{\Gamma}_{x_k} \wedge ... \wedge \Gamma_{x_{l-1}} \wedge \frac{\partial}{\partial x_k}\Gamma_{x_l} \wedge \Gamma_{x_{l+1}} \wedge ... \wedge \Gamma_{x_n}$$

while the second term gives another wedge product

$$(-1)^l \Gamma_{x_1} \wedge ... \wedge \Gamma_{x_{k-1}} \wedge \frac{\partial}{\partial x_l}\Gamma_{x_k} \wedge \Gamma_{x_{k+1}} \wedge ... \wedge \widehat{\Gamma}_{x_l} \wedge ... \wedge \Gamma_{x_n}.$$

As $\frac{\partial}{\partial x_k}\Gamma_{x_l} = \frac{\partial}{\partial x_l}\Gamma_{x_k}$, the anticommutation of exterior multiplication shows that these two latter products cancel out, completing the proof of the lemma. □

We now apply Lemma 1 to $\Gamma = du$, provisionally assuming that $u \in C^\infty(\Omega)$. Note the commutation formula $(du)_{x_i} = du_{x_i}$, thus we have

$$\sum_{i=1}^n \int_\Omega du_{x_1} \wedge ... \wedge du_{x_{i-1}} \wedge \Xi_{x_i} \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n} = 0$$

which we apply to $\Xi = \phi\, du$. Our computation results in the following integral identity

(20)
$$\int_\Omega \phi(x)\, \mathcal{H}(x,u)\, dx = -\frac{1}{n}\sum_{i=1}^n \int_\Omega \phi_{x_i}\left(du_{x_1} \wedge ... \wedge du_{x_{i-1}} \wedge du \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n}\right).$$

From here there are two ways to continue integration by parts, both facilitate the search for the distributions of the second order. At the expense of involving higher order derivatives of the test function we may pass the exterior differentiation on u to ϕ_{x_i}. This is legitimate since $du_{x_1} \wedge .... \widehat{du_{x_i}} ... \wedge du_{x_n}$ is a closed form. Continuing in this way yields

$$\int_\Omega \phi(x)\mathcal{H}(x,u)\,dx = \frac{1}{n}\sum_{i=1}^n \int_\Omega u \;\; du_{x_1}\wedge ...\wedge du_{x_{i-1}}\wedge d\phi_{x_i}\wedge du_{x_{i+1}}\wedge ...\wedge du_{x_n}.$$

While this formula remains valid for all functions u of class $W^{2,n}_{loc}(\Omega)$, the integrals in the right hand side actually converge for $u \in W^{2,n-1}_{loc}(\Omega) \subset C^{\alpha}_{loc}(\Omega)$, where $\alpha = \frac{n-2}{n-1}$. It is in this way that we are led to the following definition

DEFINITION 5 *(Second order Hessian).* The nonlinear differential operator

$$\mathcal{H}_2 : \mathbf{W}^{2,n-1}_{loc}(\Omega) \mapsto \mathcal{D}'_2(\Omega)$$

is defined by the equation

$$(\mathcal{H}_2 u)[\phi] = \frac{1}{n}\sum_{i=1}^n \int_\Omega u \;\; du_{x_1}\wedge ...\wedge du_{x_{i-1}}\wedge d\phi_{x_i}\wedge du_{x_{i+1}}\wedge ...\wedge du_{x_n}. \tag{21}$$

Another way to integrate by parts at (20) is by passing the exterior differentiation from one of the selected factors, say du_{x_j} with $j \neq i$, to ϕ_{x_i}. This will create the following n-linear forms with respect to the variables $(\nabla u, \nabla^2 u)$:

$$u_{x_j}\; d\phi_{x_i}\wedge du\wedge du_{x_1}\wedge ...\widehat{du}_{x_i}...\widehat{du}_{x_j}...\wedge du_{x_n}.$$

The point to make here is that two factors in this form depend (linearly) on the first gradient of u whereas the remaining $(n-2)$ -factors depend (also linearly) on the second gradient. We have made somewhat lengthy though elementary computation to express the result in a symmetric form

$$2n(n-1)\int_\Omega \phi(x)\;\mathcal{H}(x,u)\;dx = \tag{22}$$

$$\sum_{1\le i\neq j\le n} \varepsilon_{ij}\int_\Omega (u_{x_i}d\phi_{x_j} - u_{x_j}d\phi_{x_i})\wedge du\wedge(du_{x_1}\wedge ...\widehat{du}_{x_i}...\widehat{du}_{x_j}...\wedge du_{x_n}).$$

The symbol ε_{ij} equals 1 provided $i-j$ is positive and even, or negative and odd. Otherwise ε_{ij} equals -1. Imbedding theorems reveal that the integrals in the right hand side converge whenever u lies in $W^{2,\frac{n^2}{n+2}}_{loc}(\Omega)$. We state it as:

DEFINITION 6 *(Very weak Hessian).* Formula (22) defines (in the usual fashion) the very weak Hessian in n -dimensions,

$$\mathcal{H}^*_2 :\; \mathbf{W}^{2,\frac{n^2}{n+2}}_{loc}(\Omega) \longrightarrow \mathcal{D}'_2(\Omega). \tag{23}$$

In contrast to the case $n=2$, we did not succeed in eliminating the second order derivatives of u. Nevertheless, we did gain some degree of the required integrability of the second gradient. The domains of the n-dimensional Hessian operators $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_2^*$ are coupled, respectively, in the following chain of inclusions

$$\mathbf{W}^{2,n}_{loc}(\Omega) \subset \mathbf{W}^{2,\frac{n^2}{n+1}}_{loc}(\Omega) \subset \mathbf{W}^{2,n-1}_{loc}(\Omega) \subset \mathbf{W}^{2,\frac{n^2}{n+2}}_{loc}(\Omega).$$

They all consist of continuous functions of Hölder's class $C^{\alpha}_{loc}(\Omega)$, with $\alpha = 1-\frac{2}{n} \geq 0$, at least. We emphasize that each of these operators is an extension to the preceding one. Perhaps nothing more can be extracted from the integration by parts.

In many instances distributional Jacobian is used to justify properties of point-wise differential inequalities. Just to mention the distortion inequality in the geometric function theory [**IM**, **IKO**1, **IKMS**]. Let us emphasize explicitly that a point-wise Jacobian, even if it happens to be locally integrable, need not coincide with its distributional counterpart; some measures can emerge [**M**11, **M**12]. It is therefore important to develop some technique for computing the point-wise determinants given their weak forms. Next section deals with these questions.

6. How to get back the point-wise determinant

We begin with the prevalent example of a Sobolev mapping in the unit ball $\mathbf{B} \subset \mathbb{R}^n$

$$f(x) = \frac{x}{|x|} = \nabla |x|.$$

Clearly, f belongs to $\mathbf{W}^{1,p}(\mathbf{B}, \mathbb{R}^n)$ for all $1 \leq p < n$. Its point-wise Jacobian (Hessian of $u(x) = |x|$) is equal to zero a.e., whereas the distributional Jacobian is a scalar multiple of the Dirac delta

$$\mathcal{J}_f = \mathcal{H}_1 u = \mathcal{H}_2 u = \mathcal{H}_2^* u = \tfrac{\omega_{n-1}}{n}\,\delta, \qquad \text{while } \; J(x,f) = \det\left[u_{x_i x_j}\right] = 0,$$

where the factor $\frac{\omega_{n-1}}{n}$ is none other than the volume of the unit ball in $\mathbb{R}^n$.

As we all know, every distribution $H \in \mathcal{D}'(\Omega)$ can be approximated by smooth functions, usually via regularization procedure. Perhaps a brief description of the regularization of distributions is in order. Let $\Phi_\epsilon(x) = \epsilon^n\, \Phi(\frac{x}{\epsilon})$, $\epsilon > 0$ be the standard mollifiers supported in the balls $B(0,\,\epsilon) = \{x;\; |x| < \epsilon\}$. The regularization of H is the family $\{H_\epsilon\}_{\epsilon>0}$ of smooth functions defined on the sets $\Omega_\epsilon = \{x \in \Omega\,;\quad \operatorname{dist}(x, \partial\Omega)\; > \epsilon\}$ by the rule

$$H_\epsilon(x) = (H \star \Phi_\epsilon)(x) = H\left[\Phi_\epsilon(x- \quad)\right].$$

Here H acts on a test function $y \to \Phi_\epsilon(x-y)$ in y variable. This function is clearly supported in the ball $B(x,\,\epsilon) \subset \Omega$, for a fixed x. Recall that $H \star \Phi_\epsilon \to H$ in $\mathcal{D}'(\Omega)$

when $\epsilon \to 0$. In case of the Dirac delta, this procedure leads us back to the mollifiers $\Phi_\epsilon = (\delta \star \Phi_\epsilon) \to 0$, almost everywhere. More generally, given a Radon measure $H \in \mathcal{D}'_0(\Omega)$, at almost every $x \in \Omega$ we have $\lim_{\epsilon\to 0}(H \star \Phi_\epsilon)(x) = H^{reg}(x)$, where the regular part of a distribution in $D'_0(\Omega)$ is the absolutely continuous part of the measure it represents. Being so, the function $\lim_{\epsilon\to 0} (H \star \Phi_\epsilon)$ is always locally integrable.
Returning to our example $f(x) = \frac{x}{|x|}$, we find that $\lim_{\epsilon\to 0} \mathcal{J}_f \star \Phi_\epsilon = 0$, almost everywhere. And this is not by chance. In [**IM**] we have shown that

THEOREM 2 (Regularization of the weak Jacobian). *For almost every $a \in \Omega$ it holds*

$$\lim_{\epsilon\to 0} (\mathcal{J}_f \star \Phi_\epsilon)(a) = J(a, f) , \qquad whenever \;\; f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega, \mathbb{R}^n). \tag{24}$$

Remark. The reader is urged to check that the distributional Jacobian remains well defined (by the same integral formula at (7)) for $f \in W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$, provided the cofactors of Df lay in $L^q_{loc}(\Omega)$, with the exponent $q = \frac{n^2-n}{n^2-n-1} < \frac{n}{n-1}$. It turns out [**IKO**2] that (24) holds under these weaker hypotheses as well. An analogous result for the wedge products is in fact true.

THEOREM 3 (Regularization of weak wedge products). *Given an m-tuple $\Theta = (\theta^1, ..., \theta^m) \in L^{s_1}_{loc}(\Omega, \Lambda^{l_1}) \times ... \times L^{s_m}_{loc}(\Omega, \Lambda^{l_m})$, where $d\theta^1 = ... = d\theta^m = 0$ and the exponents satisfy the Sobolev relation $\frac{1}{s_1} + \cdots + \frac{1}{s_m} = 1 + \frac{1}{n}$. Then*

$$\lim_{\epsilon\to 0} (\mathcal{J}_\Theta \star \Phi_\epsilon)(a) = J(a, \Theta), \qquad for\ almost\ every\ \ a \in \Omega. \tag{25}$$

The proof of this result is much the same as that for Theorem 2, and is based upon ideas found in [**M**9]. We leave details to the reader due to lack of space. Instead we shall pay more attention to analogous regularization of the weak Hessian, as this case will require some interesting new ingredients.

THEOREM 4 (Regularization of the weak Hessian). *If $u \in W^{2,n-1}_{loc}(\Omega)$, then at almost every $a \in \Omega$ we have*

$$\lim_{\epsilon\to 0} (\mathcal{H}_2 u \star \Phi_\epsilon)(a) = det \left[\frac{\partial^2 u(a)}{\partial x_i \partial x_j} \right] . \tag{26}$$

But the proof must wait until we collect and improve a few facts concerning local approximation of Sobolev functions.

7. Taylor's Approximation of Sobolev Functions

Let $u \in W^{k,p}(\Omega)$, $k = 1, 2...$, $1 \leq p < \infty$, and $\Omega \subset \mathbb{R}^n$. We are interested in a polynomial approximation to $u(x)$ on balls $B = B(x,r) \subset \Omega$. In the foregoing formulas, we always assume that the center of the ball is a Lebesgue point of all partial derivatives $D^\alpha u$, with the multi-index $|\alpha| \leq k$. To each such point there corresponds unique k^{th} order polynomial $P = P_{k,a}(x)$, so that the values of $D^\alpha P$, $|\alpha| \leq k$, match the values of $D^\alpha u$ at $x = a$. This is what we call the k^{th} Taylor polynomial about $a \in \Omega$ for u. Precisely, we have

$$P_{k,a}(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha u(a) \ (x-a)^\alpha. \tag{27}$$

The average error over the ball $B(a,r)$ that results when u is approximated by $P_{k,a}$ is described in the following theorem.

THEOREM 5 (Calderón-Zygmund [**CZ**]). *Given* $u \in W^{k,p}(\Omega)$, with $k = 1, 2, ...,$ *and* $1 \leq p < \infty$. *For almost every* $a \in \Omega$ *and all balls* $B = B(a,r) \subset \Omega$, *we have*

$$\left(\frac{1}{|B|} \int_B |u(x) - P_{k,a}(x)|^q \, dx \right)^{\frac{!}{q}} = o(r^k) \tag{28}$$

as r *approaches zero. The exponent* q *is subjected to the following conditions*:

Case 1. If $1 \leq p \leq \frac{n}{k}$, *then* q *can be any positive number not greater than* $\frac{np}{n-kp}$.

Case 2. If $p > \frac{n}{k}$, *then* $q = \infty$ *and we are reduced to a uniform approximation*

$$\sup_{x \in B(a,r)} |u(x) - P_{k,a}(x)| = o(r^k). \tag{29}$$

Here and in the sequel, we work only with the continuous representatives of the Sobolev function in $W^{k,p}(\Omega)$, whenever $p > \frac{n}{k}$ or $p = 1$ if $k = n$.

Remark. In the original paper [**CZ**, Theorem 12] $P_{k,a}$ are not explicitly identified as being Taylor's polynomials of u . Nevertheless, it follows from Theorem 13 in [**CZ**] that these $P_{k,\alpha}$ ought to be the Taylor polynomials, at least for almost every Lebesgue point a of $\{D^\alpha u\}_{|\alpha| \leq k}$.

By virtue of the imbedding $W^{n,1}(\Omega) \subset C(\Omega)$ one may ask whether local uniform approximations are still possible in the borderline case of $k = n$ and $p = 1$. Recall that the k^{th} -order gradient of u, denoted by $\nabla^k u = \{D^\alpha u\}_{|\alpha|=k}$, is the list of all partial derivatives of order k. The following result seems to be missing in the literature.

THEOREM 6 (Uniform approximation). *Let u be a continuous function in $W^{n,1}(\Omega)$ and let $a \in \Omega$ be a Lebesgue point of its n^{th}-order gradient. Then to every ball $B = B(a,r) \subset \Omega$ there corresponds a polynomial*

$$P_{n,B}(x) = \sum_{|\alpha| \le n-1} c_\alpha (x-a)^\alpha + \sum_{|\alpha|=n} \tfrac{1}{\alpha!} D^\alpha u(a)\,(x-a)^\alpha \tag{30}$$

such that

$$\sup_{x \in B} |\, u(x) - P_{n,B}(x)\,| = o(r^n), \qquad \text{as } r \text{ approaches zero.} \tag{31}$$

As a point of emphasize, our polynomials $P_{n,B}(x)$ depend not only on the point $a \in \Omega$ but also on the radii of the balls. Though the n^{th} order coefficients $\frac{1}{\alpha!} D^\alpha u(a)$, $|\alpha| = n$, are still free from the radii of the balls.

Proof. For notational convenience two different symbols will be employed to denote the integral mean of a function $u \in L^1(B)$, over a ball $B = B(a,r) \subset \Omega$. Namely

$$u_B = \frac{1}{|B|} \int_B u(y)\, dy.$$

For each integer $k \ge 0$ we consider a k^{th}-order polynomial

$$P_k(x) = \frac{1}{|B|} \int_B P_k(x,y)\, dy,$$

where

$$P_k(x,y) = \sum_{|\alpha| \le k-1} \frac{1}{\alpha!} D^\alpha u(y)\,(x-y)^\alpha \; + \sum_{|\alpha|=k} \frac{1}{\alpha!} \big(D^\alpha u\big)_B\,(x-y)^\alpha$$

for $k = 1, 2, ..., n$, and we set $P_0(x) \equiv u_B$. If u lies in the Sobolev space $W^{1,1}(\Omega)$ then its zero order error (when approximated by the constant polynomials) is given explicitly at almost every point by the familiar inequality

$$|\, u(x) - u_B \,| \le C_n \int_B \frac{|\nabla u(y)|\, dy}{|x-y|^{n-1}}$$

see, for example, [**GT** , Lemma 7.16]. At this point we shall appeal to the higher order counterpart of this estimate derived in [**BH**]. Accordingly, for all functions $u \in W^{k,1}(\Omega)$, we have

$$|\, u(x) - P_k(x)\,| \;\le\; C_n \int_B \frac{\left|\nabla^k u(y) - (\nabla^k u)_B\right|\, dy}{|x-y|^{n-k}}$$

for almost every $x \in B = B(a,r) \subset \Omega$. What will be significant to us is the case when $u \in W^{n,1}(\Omega)$. Once we choose the continuous representative of u the following inequality holds for all $x \in B = B(a,r) \subset \Omega$

$$|\, u(x) - P_n(x)| \;\le\; C_n \int_B |\, \nabla^n u(y) - (\nabla^n u)_B|\, dy.$$

From now on, we assume that the ball $B = B(a,r)$ is centered at the Lebesgue point of $\nabla^n u = \{D^\alpha u\}_{|\alpha|=n}$. Thus $|u(x) - P_n(x)| = o(r^n)$, uniformly in $x \in B$. To complete the proof of Theorem 6, it only remains to modify the polynomial P_n so that the n^{th} -order term $\frac{1}{\alpha!}(D^\alpha u)_B\ (x-y)^\alpha$ will be replaced by $\frac{1}{\alpha!}D^\alpha u(a)\ (x-a)^\alpha$. This replacement is legitimate since $[(D^\alpha u)_B - D^\alpha u(a)](x-a)^\alpha = o(r^n)$, again uniformly in $x \in B$, as a was chosen to be the Lebesgue point of $D^\alpha u$ for all $|\alpha| = n$. The proof of Theorem 6 is complete. □

8. Proof of the Regularization Theorem

We shall draw on an idea from [**M**9]. Let us reveal at once that the points $a \in \Omega$ for which we are going to validate Equation (26) are just the L^{n-1} -continuity points of the second gradient $\nabla^2 u = \{D^\alpha u\}_{|\alpha|=2}$. This means that for small balls $B_\epsilon = B(a,\epsilon) \subset \Omega$ centered at such points, we have

$$\lim_{\epsilon\to 0} \frac{1}{|B_\epsilon|} \int_{B_\epsilon} \left| \nabla^2 u(x) - \nabla^2 u(a) \right|^{n-1}\ dx = 0.$$

Of course, this remains valid if we replace ∇^2 by any second order operator.
Fix one of those points $a \in \Omega$. For each sufficiently small ball $B_\epsilon = B(a,\epsilon)$ we can find a quadratic polynomial

$$P_\epsilon(x) = \sum_{|\alpha|\le 1} c_\alpha (x-a)^\alpha \ + \ \sum_{|\alpha|=2} \frac{1}{\alpha!}\ D^\alpha u(a)\ (x-a)^\alpha$$

such that

$$\sup_{x\in B_\epsilon} |u(x) - P_\epsilon(x)| = o(\epsilon^2).$$

Observe that the second order terms in this polynomial are independent of ϵ . This is immediate from Theorem 6 if $n=2$ and Theorem 5 (*Case* 2) if $n>2$. By virtue of the definitions given above, we convolve $\mathcal{H}_2 u$ with the mollifier $\Phi = \Phi_\epsilon$ by the rule

$$(\mathcal{H}_2 u \ \star\ \Phi_\epsilon)\,(a) = (\mathcal{H}_2 u)[\Phi_\epsilon(a-\ )]$$

$$= \frac{1}{n}\sum_{i=1}^n \int_\Omega u\ \ du_{x_1}\wedge ... \wedge du_{x_{i-1}} \wedge d\Phi_{x_i}(a-x) \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n} = \mathfrak{B}_\epsilon \ + \ \mathfrak{C}_\epsilon,$$

where we have decomposed the integral into two parts:

$$\mathfrak{B}_\epsilon = \frac{1}{n}\sum_{i=1}^n \int_\Omega (u - P_\epsilon)\ du_{x_1}\wedge ... \wedge du_{x_{i-1}} \wedge d\Phi_{x_i}(a-x) \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n},$$

$$\mathfrak{C}_\epsilon \ = \ \frac{1}{n}\sum_{i=1}^n \int_\Omega P_\epsilon\ \ du_{x_1}\wedge ... \wedge du_{x_{i-1}} \wedge d\Phi_{x_i}(a-x) \wedge du_{x_{i+1}} \wedge ... \wedge du_{x_n}.$$

The first part is easily shown to converge to zero. Indeed, we have

$$|\mathfrak{B}_\epsilon| \ \le\ \|\, u - P_\epsilon \,\|_{L^\infty(B_\epsilon)}\ \|\, \nabla^2 \Phi_\epsilon \,\|_{L^\infty(B_\epsilon)}\ \|\, \nabla^2 u \,\|^{n-1}_{L^{n-1}(B_\epsilon)}\ .$$

When ϵ approaches zero the first factor is of the magnitude $o(\epsilon^2)$, the second $O(\epsilon^{-2})$ and the third $O(1)$. Thus $\lim_{\epsilon\to 0} \mathfrak{B}_\epsilon = 0$, as claimed.

Concerning the term $\mathfrak{C}_\epsilon$, we notice that the integrand therein possesses enough degree of regularity to integrate by parts twice. This time the procedure will be exactly backwards to the one we have used to define $\mathcal{H}_2 u$. Thus, we carry back the exterior differential in $d\Phi_{x_i}$ to u by using Stokes' formula. In order to pass the x_i -differentiation from $\Phi = \Phi_\epsilon$ to du we employ Lemma 2 again

$$\mathfrak{C}_\epsilon = \frac{1}{n}\sum_{i=1}^{n}\int_\Omega \Phi_\epsilon(a-x)\ \ du_{x_1}\wedge ... \wedge du_{x_{i-1}}\wedge d(P_\epsilon)_{x_i}\wedge du_{x_{i+1}}\wedge ... \wedge du_{x_n}.$$

Next observe that $d(P_\epsilon)_{x_i}$ is a constant covector equal to $du_{x_i}(a),$ as the second order gradient of P_ϵ matches the values of $\nabla^2 u$ at a . Therefore, we can take the constant term $d(P_\epsilon)_{x_i}$ outside the integral sign. What we obtain upon this operation is the following n -covector

$$\mathfrak{C}_\epsilon\ dx = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\ du_{x_i}(a)\wedge\Big[\Phi_\epsilon\ \star(du_{x_1}\wedge ... \widehat{du}_{x_i} ... \wedge du_{x_n})\Big]\ (a).$$

Note that the convolution of Φ_ϵ with the function $(du_{x_1}\wedge ... \widehat{du}_{x_i} ... \wedge du_{x_n}) \in L^1(\Omega,\ \Lambda^{n-1})$ is being employed here. Finally, as $a\in\Omega$ is the Lebesgue point of this function we conclude with the desired equation

$$(\lim_{\epsilon\to 0}\ \mathfrak{C}_\epsilon)\ dx = \frac{1}{n}\sum_{i-1}^{n}(-1)^{i-1}du_{x_i}(a)\wedge[du_{x_1}\wedge ... \widehat{du}_{x_i} ... \wedge du_{x_n}](a) = \mathcal{H}(a,u)\ dx$$

completing the proof of Theorem 4. □

The regularization theorems can be turned around so as to yield local integrability of the point-wise determinants and wedge products.

COROLLARY 1. *Suppose that* $f,$ Θ *and* u *are given as in Theorems 2, 3 and 4, respectively. If the distributions* $\mathcal{J}_f,$ $\mathcal{J}_\Theta$ *and* $\mathcal{H}_2 u$ *have order zero (signed Radon measures) then the corresponding point-wise determinants* $J(x,f),$ $J(x,\Theta)$ *and* $H(x,u)$ *coincide with the absolutely continuous parts of those measures. Consequently, these point-wise determinants are locally integrable on* Ω.

Proof. This corollary is readily inferred from two facts already observed. First, for every distribution $H\in\mathcal{D}'_0(\Omega)$ its regular part $H^{reg}(x)$ equals (point-wise almost everywhere) $\lim_{\epsilon\to 0}(H\star\Phi_\epsilon)(x)$. Second, the regular part of every distribution of order zero is represented by a locally integrable function, whence the name *regular.* □

To be realistic, we have practically no way of verifying the hypotheses of this corollary without getting involved with integration by parts; a vicious circle inevitably would occur. Nevertheless, Corollary 1 has significant theoretical value, as attested to in our last section.

9. Weak Limits and Orientation

Surprisingly, in passing to the weak limit there can be a change in the sign of positive determinants. This phenomenon was first observed in [**BM**] and further refined in [**IM**] as follows:

EXAMPLE 2. *There exist Sobolev mappings* $f_\nu : \Omega \mapsto \mathbb{R}^n$, $\nu = 1, 2, \dots$, *with the following properties:*

- $$\sup_{\nu \geq 1} \int_\Omega \mathfrak{F}\left(|Df_\nu(x)|^n\right) dx \ < \ \infty$$

whenever $\mathfrak{F} : R_+ \mapsto R_+$ *is a continuous function satisfying*

$$\int_0^\infty \frac{\mathfrak{F}(t)\ dt}{t^2} \ < \ \infty. \tag{32}$$

- *The mappings* f_ν *are orientation preserving. Indeed, we have*

$$J(x, f_\nu) \ = \ det\,[Df_\nu(x)] \ \geq \ 1$$

for almost every $x \in \Omega$ *and for all* $\nu = 1, 2, \dots$.

- *The sequence* $\{f_\nu\}$ *converges uniformly to* $f(x) = (x_1, \dots, x_{n-1}, -x_n)$. *This is an orientation reversing map with*

$$J(x, f) = det\,[Df(x)] \equiv \ -1.$$

Note that the sequence $\{f_\nu\}$ is bounded in every Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$, with $p < n$.
Let us also reveal that in this example the weak Jacobians $\mathcal{J}_{f_\nu}$ are indeed distributions of order zero (signed Radon measures). If they were positive the above anomaly with the change of orientation would not have happened, by Corollary 1.
Mathematical formulation of the nonlinear elasticity and continuum mechanics [**A**, **B**1, **C**2] is based on the principle that the point-wise Jacobian of a deformation of an elastic body is nonnegative, or at lest does not change sign. The same assumption is made in the theory of mappings with finite distortion [**IM**, **IKO**1]. This does not mean that the mappings in question have automatically nonnegative distributional Jacobian. It, therefore, becomes very natural to ask for simple conditions that are sufficient in order that $J(x, f)$ shall be integrable and obey the rule of integration by parts. Various such criteria have been discussed in [**IS**, **BFS**, **G**2, **G**3, **G**4, **GI**2, **KZ**, **LZ**, **M**3, **M**8, **M**9, **M**10, **W**2] . We shall briefly outline the most general recent result in [**GIOV**].

10. A Short Look at L^1-Integrability

Throughout this section the Sobolev mappings $f = (f^1, f^2, \dots, f^n) : \Omega \mapsto \mathbb{R}^n$, will have nonnegative Jacobian determinant. We shall take for our analysis the category of Orlicz spaces $L^{\mathcal{P}}(\Omega)$, where $\mathcal{P} : [0, \infty) \mapsto [0, \infty)$ is continuously increasing from $\mathcal{P}(0) = 0$ to $\mathcal{P}(\infty) = \lim_{t\to\infty} \mathcal{P}(t) = \infty$ and smooth on $(0, \infty)$. Note that no assumption regarding convexity of $\mathcal{P}$ is made here. However, Example 2 suggests that we should stay close to the Lebesgue space $L^1(\Omega)$. The exact grows condition on $\mathcal{P}$ that will be required is just opposite to (32), namely

$$\int_1^\infty \frac{\mathcal{P}(t)\, dt}{t^2} = \infty. \tag{33}$$

Cases of special interest include the the following logarithmic scale of Orlicz functions

$$\mathcal{P}(t) = t, \qquad \mathcal{P}(t) = \frac{t}{\ln(e+t)}, \qquad \mathcal{P}(t) = \frac{t}{\ln(e+t)\ \ln\ln(e^e+t)},$$

$$\mathcal{P}(t) = \frac{t}{\ln(e+t)\ \ln\ln(e^e+t)\ \dots\ \ln\ln\dots\ln(e^{e^{\cdot^{\cdot}}}+t)}.$$

In what follows we actually need some other technical assumptions on $\mathcal{P}$ but we shall not bother the reader about those really minor details [**GIOV**]. Before jumping to a conclusion, we indicate that the qualitative analysis of mappings with unbounded distortion naturally begins with the assumption that $|Df|^n \in L^{\mathcal{P}}(\Omega)$ [**IM**], see also [**IKO**1, **IKMS**]. This hypothesis is legitimate if we work with the so-called *outer distortion* function. However, to formulate the theory in terms of the *inner distortion* we must work under slightly less restrictive hypotheses. Perhaps the best example to illustrate is the compactness principle for mappings with unbounded distortion. It relies on certain estimates in terms of the (n-1)×(n-1) -minors. At the beginning one only knows that $|D^{\#}f|^{\frac{n}{n-1}} \in L^{\mathcal{P}}(\Omega)$, where $D^{\#}f$ denotes the cofactor matrix of Df. Then, the following result comes to the rescue [**IKO**2].

THEOREM 7. *Let* $f = (f^1, f^2, ..., f^n) \in W^{1,n-1}_{loc}(\Omega,\ \mathbb{R}^n)$ *have nonnegative Jacobian determinant,* $J(x,f) \geq 0$, *and its* (n-1)× (n-1) *-minors satisfy*

$$\left|\frac{\partial(f^1,\dots\widehat{f^i}\dots,f^n)}{\partial(x_1,\dots\widehat{x_j}\dots,x_n)}\right|^{\frac{n}{n-1}} \in L^{\mathcal{P}}(\Omega) \qquad for \quad i,\, j = 1,2,...,n. \tag{34}$$

Then $J(x,f)\, dx = df^1 \wedge df^2 \wedge \dots \wedge df^n$ *is locally integrable and obeys the rule of integration by parts.*

In addition to this result we have an elegant and powerful local estimate.

$$\left(\int_B df^1 \wedge df^2 \wedge \dots \wedge df^n\right)^{n-1} \leq \frac{C(n)}{|B|} \prod_{i=1}^{n} \left(\int_{2B} |df^1 \wedge \dots\ \widehat{df^i} \dots \wedge df^n|\right) \tag{35}$$

for every concentric balls $B = B(a,\, r) \subset B(a,\, 2r) = 2B$. As a matter of fact we have slightly stronger isoperimetric type inequality

$$\left(\int_B df^1 \wedge df^2 \wedge \dots \wedge df^n\right)^{n-1} \leq C(n) \prod_{i=1}^{n} \left(\int_{\partial B} |df^1 \wedge \dots\ \widehat{df^i} \dots \wedge df^n|\right) \tag{36}$$

for almost every radius $0 < r \leq \operatorname{dist}(a,\, \partial\Omega)$. Sobolev mappings with L^p-integrable cofactors were introduced to nonlinear elasticity by J. Ball [**B**1] and then studied by V. Šverák [**Š**1] and P. Hajłasz [**H**2].

We shall end this article with one more requisite that proves useful when dealing with the compactness questions in the geometric function theory. It concerns the mappings $f : \Omega \mapsto \mathbb{R}^n$ of Sobolev-Orlicz class $W^{1,\mathcal{Q}}(\Omega, \mathbb{R}^n)$, where $\mathcal{Q}(t) = \mathcal{P}(t^n)$ and $\mathcal{P}$ satisfies the divergence condition at (33).

COROLLARY 2. *The class of orientation preserving mappings in the Sobolev-Orlicz space* $W^{1,\mathcal{Q}}(\Omega, \mathbb{R}^n)$ *is weakly closed.*

Proof. Consider a sequence $\{f_\nu\}$ of mappings with nonnegative Jacobian, and weakly converging to f in $W^{1,\mathcal{Q}}(\Omega, \mathbb{R}^n)$. It is not clear a priori whether $J(x, f)$ is also nonnegative, though at the end this is the case . Our proof runs as follows. By virtue of Theorem 1, we see that $\mathcal{J}_{f_\nu} \to \mathcal{J}_f$ in $\mathcal{D}'(\Omega)$. Thus it suffices to show that each distribution $\mathcal{J}_{f_\nu}$ is nonnegative, as this will imply the same holds for $\mathcal{J}_f$. It will then imply, by Corollary 1, that $J(x, f)$ is nonnegative and locally integrable. At this point we appeal to Theorem 7, for the hypothesis simply notice that $|D^{\#}f|^{\frac{n}{n-1}} \le |Df|^n \in L^{\mathcal{P}}(\Omega)$. This theorem tells us not only that $J(x, f_\nu)$ are locally integrable but also provides with the key to nonnegativity of the induced distributions

$$\mathcal{J}_{f_\nu}[\phi] = \int_\Omega \phi(x)\ J(x, f_\nu)\ dx \ge 0$$

as desired. By way of digression, the distribution $\mathcal{J}_f$ has no singular part. □

Other related papers not mentioned here are: [**BCO**, **BK**, **GV**, **GISS**, **HLMZ**, **I**2, **IO**, **L**, **MM**, **MZ**, **McM**, **S**, **Z**2].

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