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SOLUTIONS OF A MODIFIED FIFTH PAINLEVÉ EQUATION ARE MEROMORPHIC

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1. Introduction

This paper continues the recent studies by N. Steinmetz and the present authors, see [HL1], [St], [HL2], devoted to offering rigorous proofs for the meromorphic nature of the solutions of Painlevé differential equations. Combining the main results from the papers cited above, we have

Theorem A. *All local solutions of Painlevé's first, second, third (in a modified form) and fourth equations,*

$$w'' = z + 6w^2, \tag{1.1}$$

$$w'' = \alpha + zw + 2w^3, \tag{1.2}$$

$$ww'' = (w')^2 + \alpha w^3 + \gamma w^4 + \beta e^z w + \delta e^{2z}, \tag{1.3}$$

$$ww'' = \tfrac{1}{2}(w')^2 + \tfrac{3}{2}w^4 + 4zw^3 + 2(z^2-\alpha)w^2 + \beta, \tag{1.4}$$

where α, β, γ, $\delta \in \mathbb{C}$, can be analytically continued to single-valued meromorphic functions in the complex plane.

This paper is devoted to considering Painlevé's fifth equation

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2 w}(\alpha w^2 + \beta) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \tag{1.5}$$

from the same point of view. As in the previous situations included in Theorem A, see [HL1], [St] for more details, (1.5) also lacks a corresponding rigorous proof. Similarly as to Painlevé's third equation, the fixed singularity at $z = 0$ may prevent us from proving the meromorphic nature of all solutions of (1.5). To this end, consider (1.5) with $\alpha = \beta = \gamma = \delta = 0$, that is,

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z}. \tag{1.6}$$

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If a and C are non-zero complex numbers, then

$$w(z) := \left(\frac{1+Cz^a}{1-Cz^\alpha}\right)^2 \tag{1.7}$$

can be defined locally in a neighbourhood of any point $z_0 \in \mathbb{C} \setminus \{0\}$, but w cannot be continued to be single-valued meromorphic in $\mathbb{C}$ unless a is an integer. Moreover, (1.7) is a solution of (1.6). In fact, (1.6) may be rewritten as

$$2\frac{w''}{w'} + \frac{2}{z} - \frac{w'}{w} - 2\frac{w'}{w-1} = 0. \tag{1.8}$$

By a straightforward computation, for w given by (1.7),

$$\left(\frac{zw'}{w-1}\right)^2 \frac{1}{w} = a^2$$

is a constant, from which (1.8) follows at once. See also [K], p. 50.

To avoid a potential singularity at $z = 0$, we make use of the transformation $z = e^t$. After the transformation, we revert to writing z instead of t, and obtain a modified Painlevé's fifth equation

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 + (w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \gamma e^z w + \frac{\delta e^{2z} w(w+1)}{w-1}. \tag{1.9}$$

This enables us to prove

Theorem 1. *All local solutions of the modified Painlevé fifth equation* (1.9) *can be analytically continued to single-valued meromorphic functions in the complex plane.*

Concerning the proof below, we remark that the general pattern of our proof closely follows the corresponding reasoning in [HL2] for the modified third equation. In particular, we would like to thank Jeremy Schiff of Bar-Ilan University for drawing our attention to the method of [WTC] of truncated Painlevé series. This aided us in finding the function u in Case II below. The idea of finding a suitable function U as in (3.6), which can be proved to be bounded on an arc where w remains bounded away from a suitable constant c, as well as the essentials of the differential inequality technique used in Section 4, go back to the work of Steinmetz [St].

As in [HL1] and [HL2], we again had to rely, even more extensively than earlier, on the software Mathematica, Version 3.0, to check several non-trivial symbolic computations.

2. The special case $\gamma = \delta = 0$

We start by defining an auxiliary function

$$V := \frac{(w')^2}{w(w-1)^2} - 2\alpha w + \frac{2\beta}{w} + \frac{2\gamma e^z}{w-1} + \frac{2\delta e^{2z}}{w-1} + \frac{2\delta e^{2z}}{(w-1)^2}. \tag{2.1}$$

Differentiating, and making use of (1.9), we obtain

$$V' = \frac{2\gamma e^z}{w-1} + \frac{4\delta e^{2z}}{w-1} + \frac{4\delta e^{2z}}{(w-1)^2}. \tag{2.2}$$

Suppose now $\gamma = \delta = 0$. Then V' vanishes identically, and so V is a constant, say $V(z) \equiv c \in \mathbb{C}$. Hence,

$$(w')^2 = (w-1)^2(2\alpha w^2 + cw + 2\beta),$$

which may be written as

$$(v')^2 = v^2\big((c + 2\alpha + 2\beta) + (c + 4\alpha)v + 2\alpha v^2\big), \tag{2.3}$$

where $v = w-1$. If now $\alpha = c = 0$, then $w'/(w-1)$ has to be a constant, and so w is meromorphic. If $\alpha = 0$ and $c \neq 0$, then we may apply an additional transformation

$$u := 1 + \frac{\sigma}{v - \sigma}, \qquad \sigma \neq 0, \ -\frac{c + 2\beta}{c}$$

to obtain

$$(u')^2 = u^2(u-1)\big((c + 2\beta + c\sigma)u - (c + 2\beta)\big). \tag{2.4}$$

Therefore, using (2.3) or (2.4) and reasoning as in [HL2], Section 4, we conclude that w has to be meromorphic.

3. The proof in the general case

We now proceed to follow the pattern applied earlier in [HL1] and [HL2], and going in fact to the classical paper [P] by Painlevé, assuming that $\gamma \neq 0$ whenever $\delta = 0$. Hence, starting with a local solution $w(z)$ of (1.9) in a neighbourhood of z_0, consider the largest open disk $B(z_0, R)$ centred at z_0 of radius $R > 0$ such that $w(z)$ can be analytically continued to a single-valued meromorphic function in $B(z_0, R)$. If $R < \infty$, there is a point $a \in \partial B(z_0, R)$ such that $w(z)$ cannot be continued beyond a along the line segment $\mathit{\Gamma} := [z_0, a)$, modified slightly, if necessary, to avoid zeros, poles and one-points of $w(z)$.

Case I. In this case, we assume that there exist a constant $M > 1$ and a sequence $\{z_n\}$ with $z_n \to a$ on the (possibly modified) line segment $\mathit{\Gamma}$ such that

$$\frac{1}{M} \leq |w(z_n)| \leq M, \quad \frac{1}{M} \leq |w(z_n) - 1| \quad \text{and} \quad |w'(z_n)| \leq M.$$

Writing now (1.9) as the pair of first order differential equations

$$\begin{cases} w' = g \\ g' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) g^2 + (w-1)^2 \left(\alpha w + \frac{\beta}{w}\right) + \gamma e^z w + \delta e^{2z} \frac{w(w+1)}{w-1}, \end{cases} \tag{3.1}$$

we may apply the standard Cauchy estimate reasoning [H], Hilfssatz 2.2, to conclude that w must be analytic around a.

Before proceeding, we remark that we postpone the case when w and $1/w$ are both bounded on Γ. Observe, however, that if $1/(w-1)$ is bounded on Γ as well, then V' is bounded on Γ, hence V too. By (2.1), w' is bounded on Γ and so this reduces to Case I. Therefore, the case to be postponed is when w and $1/w$ are bounded on Γ while $1/(w-1)$ is unbounded.

So, from now on, for the time being, we assume that at least one of w or $1/w$ is unbounded on Γ. Actually, we may assume that there exists a sequence $\{z_n\}$ on Γ, $z_n \to a$, such that $w(z_n) \to \infty$. In fact, if $w(z_n) \to 0$ on a sequence $\{z_n\}$, $z_n \to a$, define $\zeta := 1/w$ to obtain

$$\zeta'' = \left(\frac{1}{2\zeta} + \frac{1}{\zeta - 1}\right)(\zeta')^2 + (\zeta - 1)^2\left(-\frac{\alpha}{\zeta} - \beta\zeta\right) - \gamma e^z\zeta + \delta e^{2z}\frac{\zeta(\zeta+1)}{\zeta - 1}.$$

This means that ζ satisfies (1.9) with $(\alpha, \beta, \gamma, \delta)$ replaced by $(-\beta, -\alpha, -\gamma, \delta)$ and $\zeta(z_n) \to \infty$.

Case II. Suppose first that $\alpha \neq 0$. We now define $v := 1/w$ and u by $v' = A + uv$ with $A^2 = 2\alpha$. By a straightforward computation,

$$\left\{\begin{array}{l} v' = A + uv \\ -u' = \alpha + \beta(1-v)^2 + \gamma e^z + \frac{1}{2}u^2 + \frac{2\alpha + 2Au - u^2v + \delta e^{2z}(1+v)}{1-v}. \end{array}\right. \tag{3.2}$$

In addition to $w(z_n) \to \infty$, so that $v(z_n) \to 0$, we assume in this case that $u(z_n)$ remains bounded as $n \to \infty$. Then (3.2) readily implies that $w(z)$ continues meromorphically over the point a.

Suppose next that $\alpha = 0$, and define $w = 1/v^2$. ¿From $u := v'$ we conclude

$$\left\{\begin{array}{l} v' = u \\ u' = -\frac{2u^2v}{1-v^2} - \frac{v}{2}(1-v^2)^2 - \frac{1}{2}\gamma e^z v - \frac{1}{2}\delta e^{2z} v\frac{1+v^2}{1-v^2}. \end{array}\right. \tag{3.3}$$

As $v(z_n) \to 0$, the sequences $v(z_n)$ and $1/\big(1 - v(z_n)^2\big)$ are both bounded. If, moreover, $u(z_n) = v'(z_n)$ remains bounded as $n \to \infty$, then w permits an analytic continuation over the point a again.

Case III. We now assume that $w(z_n) \to \infty$ while $V(z_n)$ remains bounded, see (2.1).

(a) Assume first that $\alpha \neq 0$, and define $v := 1/w$, hence $v(z_n) \to 0$. Take now A such that $A^2 = 2\alpha$ and define u by $v' = A + uv$ as in Case II. Moreover, define

$$\varphi := V - \left(2\beta v + \frac{2\gamma e^z v}{1-v} + 2\delta e^{2z}\left(\frac{v}{1-v} + \frac{v^2}{(1-v)^2}\right)\right) \tag{3.4}$$

so that $\varphi(z_n)$ remains bounded as $n \to \infty$. Substituting (2.1) into (3.4), we obtain

$$\varphi v(1-v)^2 = (uv + A)^2 - 2\alpha + 4\alpha v - 2\alpha v^2.$$

Therefore, we must have

$$(uv + A)^2 - 2\alpha = uv(uv + 2A) \to 0$$

on z_n. If $uv \to 0$ on z_n, or even on a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, then

$$\varphi(1-v)^2 - 4\alpha + 2\alpha v$$

remains bounded on $\{z_n\}$ as $n \to \infty$. Since

$$\varphi(1-v)^2 - 4\alpha + 2\alpha v = u(uv + 2A) = u\big(2A + o(1)\big)$$

on z_n, it must be that $u(z_n)$ is bounded on $\{z_n\}$. But then this reduces to Case II.

Therefore, assuming that uv is bounded away from zero on $\{z_n\}$, we must have $uv \to -2A$ on $\{z_n\}$ as $n \to \infty$. We now define

$$h := u + \frac{2A}{v} = \frac{uv + 2A}{v} = \frac{\varphi(1-v)^2 - 4\alpha + 2\alpha v}{uv}.$$

Obviously, $h(z_n)$ remains bounded as $n \to \infty$. Moreover,

$$\begin{cases} v' = A + uv = A + v\left(h - \frac{2A}{v}\right) = -A + hv \\ h' = -\alpha - \beta(1-v) - \gamma e^z - \frac{1}{2}h^2 - \frac{2\alpha + \delta e^{2z}(1+v)}{1-v} + \frac{2Ah - h^2 v}{1-v} \end{cases}$$

Hence, by Cauchy's estimates again, w continues meromorphically across the point a.

(b) As for the subcase with $\alpha = 0$, we again define v by $w = 1/v^2$ and set $u := v'$. Substitution into (2.1) results in

$$V = \frac{4(v')^2}{(1-v^2)^2} + 2\beta v^2 + \frac{2\gamma e^z v^2 + 2\delta e^{2z} v^2}{1-v^2} + \frac{2\delta e^{2z} v^4}{(1-v^2)^2}.$$

It is now immediate to observe that

$$4(v')^2 = (1-v^2)^2 (V - v^2 Y)$$

where $Y(z_n)$ is bounded. Hence $v'(z_n) = u(z_n)$ is bounded on $\{z_n\}$ as well. Since, as in Case II, see (3.3),

$$\begin{cases} v' = u \\ u' = -\frac{2u^2 v}{1-v^2} - \frac{1}{2}v(1-v^2)^2 - \frac{1}{2}\gamma e^z v - \frac{1}{2}\delta e^{2z} v \frac{1+v^2}{1-v^2} \end{cases}$$

we may invoke the Cauchy estimates again to see that w may be continued across a, completing Case III.

Case IV. We now assume that both w and V are unbounded on the (possibly slightly modified) line segment $\Gamma := [z_0, a)$. As in [HL2], we observe that $W := V'/V$ is unbounded on Γ as well. We fix $\{z_n\}$ on Γ, converging to a, such that $W(z_n) \to \infty$. Moreover, at least one of the sequences $w(z_n)$, $1/w(z_n)$, $1/\big(w(z_n) - 1\big)$ and $w'(z_n)$ must be unbounded as $n \to \infty$, since otherwise we may apply Case I, or else we proceed to the postponed Case V.

(a) If $w(z_n) \to \infty$, possibly in a subsequence, then $V'(z_n) \to 0$ from (2.2). Therefore, $V(z_n) = V'(z_n)/W(z_n) \to 0$, and so this case reduces back to Case III.

Hence, from now on, we assume that $w(z_n)$ remains bounded as $n \to \infty$.

(b) If $w'(z_n) \to \infty$, then $W(z_n) \to 0$. This follows at once from

$$W = \frac{w[4\delta e^{2z} + (w-1)(2\gamma e^z + 4\delta e^{2z})]}{(w')^2 - 2\alpha w^2(w-1)^2 + 2\beta(w-1)^2 + 2\delta e^{2z} w + w(w-1)(2\gamma e^z + 2\delta e^{2z})}. \tag{3.5}$$

As $W(z_n) \to 0$ is a contradiction, we now proceed under the assumptions that $W(z_n) \to \infty$, $w'(z_n)$ remains bounded as $n \to \infty$, and either $w(z_n) \to 0$ or $w(z_n) \to 1$.

To proceed, we now pick any $c \in \mathbb{C} \setminus \{0,1\}$ and define

$$U := V - 2\left(\frac{w'}{w-1} - \frac{w'}{w-c}\right). \tag{3.6}$$

Then

$$U' = V' - 2\left(\frac{w''}{w-1} - \left(\frac{w'}{w-1}\right)^2 - \frac{w''}{w-c} + \left(\frac{w'}{w-c}\right)^2\right). \tag{3.7}$$

Substituting now w'' from (1.9) into (3.6) and (3.7), we observe that for some $P_j = P_j(z,w)$, $1 \le j \le 5$, U and U' take the form

$$\begin{cases} U = P_1(w')^2 + P_2 w' + P_3 \\ U' = P_4(w')^2 + P_5. \end{cases} \tag{3.8}$$

In fact, one may easily find by straightforward computation that

$$\begin{cases} P_1 = \frac{1}{w(w-1)^2}, \\ P_2 = -2\left(\frac{1}{w-1} - \frac{1}{w-c}\right), \\ P_3 = -2\alpha w + \frac{2\beta}{w} + \frac{2\gamma e^z + 2\delta e^{2z}}{w-1} + \frac{2\delta e^{2z}}{(w-1)^2}, \\ P_4 = \left(\frac{1}{2w} + \frac{1}{w-1}\right) P_2 + 2\left(\frac{1}{(w-1)^2} - \frac{1}{(w-c)^2}\right), \\ P_5 = \frac{2\gamma e^z + 4\delta^{2z}}{w-1} + \frac{4\delta e^{2z}}{(w-1)^2} + \left\{(w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \gamma e^z w + \frac{\delta e^{2z} w(w+1)}{w-1}\right\} P_2. \end{cases} \tag{3.9}$$

We now wish to find $Q_j = Q_j(z,w)$, $1 \le j \le 4$, such that

$$(U' - Q_1 - Q_2 U)^2 = Q_3 + Q_4 U. \tag{3.10}$$

It appears, after some calculation that (3.10) holds if (and only if)

$$\begin{cases} Q_1 = \frac{P_2^2 P_4}{2P_1^2} - \frac{P_3 P_4}{P_1} + P_5, \\ Q_2 = \frac{P_4}{P_1}, \\ Q_3 = \frac{P_2^2 - 4P_1 P_3}{4P_1^4} P_2^2 P_4^2, \\ Q_4 = \frac{P_2^2 P_4^2}{P_1^3}. \end{cases} \tag{3.11}$$

Before proceeding, it is important to note that, for any z, the functions Q_j are polynomials in $(w-c)^{-1}$, and so are bounded on any path on which $w-c$ is bounded away from zero. This turns out to be the case for any $c \in \mathbb{C} \setminus \{0,1\}$. In the present Case IV, any such c may be used; we fix $c=-1$. However, in the final Case V, we need to select c such that $|c|$ is sufficiently large.

For $c=-1$, the functions Q_j are found to be, by a substitution of (3.9) in (3.11),

$$\left\{\begin{aligned}
&Q_1 = \tfrac{16w(w-1)^2}{(w+1)^4} - \tfrac{2\gamma e^z(3w-1)+8(\alpha w+\beta)(w-1)+4w\delta e^{2z}}{(w+1)^2},\\
&Q_2 = 2\tfrac{(w-1)^2}{(w+1)^2},\\
&Q_3 = 128(w+1)^{-8}\{32-128(w+1)+4(w+1)^2[4\alpha-4\beta\\
&\quad -2\gamma e^z-\delta e^{2z}+52]+4(w+1)^3[-44+8(\beta-2\alpha)+5\gamma e^z+3\delta e^{2z}]\\
&\quad +(w+1)^4[82+104\alpha-24\beta-18\gamma e^z-13\delta e^{2z}]\\
&\quad +(w+1)^5[-20-88\alpha+8\beta+7\gamma e^z+6\delta e^{2z}]\\
&\quad +(w+1)^6[2+41\alpha-\beta-\gamma e^z-\delta e^{2z}]-10\alpha(w+1)^7+\alpha(w+1)^8\},\\
&Q_4 = \tfrac{64w(w-1)^4}{(w+1)^6}.
\end{aligned}\right. \tag{3.12}$$

For a general $c \in \mathbb{C}\setminus\{0,1\}$, we refer to the Appendix, where we offer, as Mathematica printouts, the formulas for Q_j, $1 \le j \le 4$, in terms of $t := w-c$.

As in [HL2] for the case of the modified third Painlevé equation, we may need, in Case IV, to replace the path Γ by a rectifiable arc $\tilde{\Gamma}$ from z_0 to a such that

$$\liminf_{\tilde{\Gamma} \ni z \to a} |w(z)-c| > 0.$$

The construction of such an arc $\tilde{\Gamma}$ will be offered in the final section below.

We now proceed to consider the various possible modes of behaviour of $w(z)$ on the arc $\tilde{\Gamma}$. It is immediate to observe that when, at least for a sequence $\{z_n\}$ on $\tilde{\Gamma}$, converging to a, Cases I, II, III, IV(a) or IV(b) appear, the above reasoning applies with no changes. Therefore, we may proceed to the remaining Cases IV(c) and IV(d), while Case V will be treated separately.

(c) We now assume that $W(z_n) \to \infty$, $w(z_n) \to 0$ and $w'(z_n)$ remains bounded as $n \to \infty$, where $\{z_n\}$ is a sequence on $\tilde{\Gamma}$ with $z_n \to a$. From the expression (3.5) of W, we observe at once that $w'(z_n)^2 + 2\beta \to 0$.

Recalling that $c=-1$, the expression of U in (3.8) and (3.9) may be applied to obtain

$$\begin{aligned}
-\frac{(w')^2+2\beta}{w} &= -4\beta - 2\gamma e^z - U + \frac{4w'}{w+1}\\
&\quad + w\left(2\beta - 2\alpha(w-1)^2 + 2\gamma e^z + 2\delta e^{2z} - wU + 2U - \frac{4w'}{w+1}\right).
\end{aligned} \tag{3.13}$$

We may now choose A such that $A^2=-2\beta$ and, at least on a subsequence converging to a, $w'(z_n) \to A$ as $n \to \infty$. Observe that the possibility $A=0$ is not excluded. We now write

$$w' = A + uw. \tag{3.14}$$

Looking at the expressions (3.12) of Q_j, $1 \le j \le 4$, the function U is then bounded on $\tilde{\Gamma}$ as can be seen from

$$|U'| \le K_1|U| + K_2$$

on $\tilde{\Gamma}$ for some positive constants K_1, K_2 exactly as in the case of the modified third Painlevé equation, see [HL2]. Factoring $(w')^2 + 2\beta = (w' - A)(w' + A)$, and substituting (3.14) in (3.13), we obtain

$$-u(w' + A) = (-1 + 2w - w^2)U + \text{ a bounded term.}$$

Therefore, as U is bounded, and $w(z_n) \to 0$, $w'(z_n) \to A$, we infer that $u(z_n)$ remains bounded as $n \to \infty$ if $\beta \neq 0$. It is now straightforward to see that

$$u' = -\tfrac{1}{2}u^2 + \gamma e^z + \delta e^{2z}\frac{w+1}{w-1} + \alpha(w-1)^2 + \frac{2Au + u^2 w}{w-1} + \beta - \frac{2\beta}{w-1}. \tag{3.15}$$

Combining (3.14) and (3.15), we may apply the Cauchy estimates again to see that w continues beyong the point a.

If $\beta = 0$, we define v by $w = v^2$ and then u by $w' = uv$. Now the left hand side of (3.13) equals $-u^2$ so that $u(z_n)$ is bounded. A calculation shows that

$$\begin{cases} v' = u/2 \\ u' = \frac{u^2 v - \delta e^{2z} v(1+v^2)}{v^2-1} + \alpha v(v^2-1)^2 + \gamma e^z v. \end{cases}$$

Hence Cauchy's estimates show again that w can be analytically continued beyond the point a.

(d) We now suppose that $w(z_n) \to 1$, in addition to $W(z_n) \to \infty$ and the fact that $w'(z_n)$ remains bounded as $z_n \to a$ on $\tilde{\Gamma}$. From (3.5), we see that $(w')^2 + 2\delta e^{2z} \to 0$ on z_n as $n \to \infty$.

We first suppose that $\delta \neq 0$, and choose $A \neq 0$ with $A^2 = -2\delta$ such that $w'(z_n) - Ae^{z_n} \to 0$ as $n \to \infty$, at least on a subsequence. From the expression (3.8), (3.9) of U, we conclude that

$$\begin{aligned} -\frac{(w')^2 + 2\delta e^{2z}}{w-1} &= 2\beta(w-1) - 2w'w + 2\frac{w'w(w-1)}{w+1} \\ &\quad - w(w-1)(U + 2\alpha w) + 2\delta e^{2z} + w(2\gamma e^z + 2\delta e^{2z}). \end{aligned} \tag{3.16}$$

We now define u by

$$w' = Ae^z + (w-1)\left(1 - \frac{\gamma}{A} + Ae^z\right) + u(w-1)^2. \tag{3.17}$$

Invoking (1.9), we obtain

(3.18)

$$\begin{aligned} u' = {} & \frac{(\gamma - A)e^z + \delta e^{2z}}{w} + \alpha w + \frac{\beta}{w} + \frac{u^2(w-1)^2}{2w} + \frac{(1 - \frac{\gamma}{A} + Ae^z)^2}{2w} \\ & - u^2(w-1) - 3u\left(1 - \frac{\gamma}{A} + Ae^z\right) + \frac{u}{w}\left(2Ae^z + (3w-1)\left(1 - \frac{\gamma}{A} + Ae^z\right)\right). \end{aligned}$$

Now, the Cauchy estimates reasoning applies to (3.17) and (3.18), provided we first show that $u(z_n)$ remains bounded as $n \to \infty$.

Comparing (3.16) and (3.17), we observe that

$$\begin{aligned}
u(w' + Ae^z) &= \frac{(w')^2 + 2\delta e^{2z}}{(w-1)^2} - \frac{w' + Ae^z}{w-1}\left(1 - \frac{\gamma}{A} + Ae^z\right)\\
&= -2\beta + \frac{2w'w}{w-1} - \frac{2w'w}{w+1} + w(U + 2\alpha w) - \frac{2\delta e^{2z}}{w-1}\\
&\quad - \frac{w}{w-1}(2\gamma e^z + 2\delta e^{2z}) - \frac{w' + Ae^z}{w-1}\left(1 - \frac{\gamma}{A} + Ae^z\right)\\
&= -2\beta + 2w' - \frac{2w'w}{w+1} + w(U + 2\alpha w) - 2\gamma e^z - 2\delta e^{2z}\\
&\quad + \frac{1}{w-1}\left(w' - \gamma e^z - 2\delta e^{2z} - Ae^z + \frac{\gamma}{A}w' - Ae^z w'\right)\\
&= X_1 + \frac{X_2}{w-1},
\end{aligned}$$

say. Obviously, $X_1(z_n)$ remains bounded as $n \to \infty$. We may write X_2 in the form

$$\begin{aligned}
X_2 &= (w' - Ae^z) + Ae^z - \gamma e^z - 2\delta e^{2z} - Ae^z\\
&\quad + \frac{\gamma}{A}(w' - Ae^z) + \gamma e^z - Ae^z\big((w' - Ae^z) + Ae^z\big)\\
&= (w' - Ae^z)\left(1 + \frac{\gamma}{A} - Ae^z\right).
\end{aligned}$$

By (3.16), $(w' - Ae^z)/(w-1)$ remains bounded on $\{z_n\}$ as $n \to \infty$. Since

$$u(w' + Ae^z) = X_1 + \frac{w' - Ae^z}{w-1}\left(1 + \frac{\gamma}{A} - Ae^z\right),$$

we observe that $u(z_n)$ remains bounded, completing the case $\delta \neq 0$.

The complete Case IV(d), suppose now that $\delta = 0$. By Section 2, we may assume that $\gamma \neq 0$. Since $\delta = 0$, we now have $w'(z_n) \to 0$ as $n \to \infty$. Writing $w - 1 = v^2$, we see that $w' = 2vv'$ and $v(z_n) \to 0$. Substituting these to the definition (3.6) of U, we get

(3.19)
$$v^2\left(U + \frac{4v'}{v} - \frac{4vv'}{v^2+2} + 2\alpha(v^2+1) - \frac{2\beta}{v^2+1}\right) = \frac{4}{v^2+1}\left((v')^2 + \frac{\gamma}{2}e^z + \frac{\gamma}{2}e^z v^2\right).$$

Hence $(v')^2 + (\gamma/2)e^z \to 0$ on z_n as $n \to \infty$. We may now choose $A \neq 0$ with $A^2 = -\gamma/2$ so that, on a subsequence of z_n at least, $v' - Ae^{z/2} \to 0$. Define u by

$$v' = Ae^{z/2} + \tfrac{1}{2}v + uv^2. \tag{3.20}$$

Invoking (1.9), we see that

(3.21)
$$\begin{aligned}
u' = &-\frac{v}{1+v^2}\left(-\tfrac{1}{2}\gamma e^z + \tfrac{1}{4}v^2 + u^2v^4 + Ave^{z/2} + 2Auv^2e^{z/2} + uv^3\right) - u\left(\tfrac{3}{2} + 2uv\right)\\
&+ \frac{v}{2}\left(\alpha(v^2+1) + \frac{\beta}{v^2+1}\right) + \tfrac{1}{4}v + u^2v^3 + Ae^{z/2} + 2Ae^{z/2}uv + uv^2 + u^2v + u.
\end{aligned}$$

The Cauchy estimates reasoning now applies to (3.20) and (3.21), and so w continues across the point a, provided again that we know that $u(z_n)$ remains bounded as $n \to \infty$. To this end, we return back to (3.19). This may be rewritten in the form

$$\begin{aligned} 4\frac{(v')^2 + \frac{1}{2}\gamma e^z}{v^2+1} &= v^2 \left\{ U + \frac{4v'}{v} - \frac{4vv'}{v^2+2} + 2\alpha(v^2+1) - \frac{2\beta}{v^2+1} - \frac{2\gamma e^z}{v^2+1} \right\} \\ &= 4v'v + v^2 \left\{ U - \frac{4vv'}{v^2+2} + 2\alpha(v^2+1) - \frac{2\beta}{v^2+1} - \frac{2\gamma e^z}{v^2+1} \right\} \\ &= 4v'v + v^2 Y_1. \end{aligned}$$

Hence

$$(v')^2 + \tfrac{1}{2}\gamma e^z = v'v + v^2 Y_2, \tag{3.22}$$

where Y_1 and $Y_2 = \frac{1}{4}Y_1(1+v^2) + v'v$ are bounded on $\{z_n\}$, since $v(z_n) \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned} \frac{v' - Ae^{z/2}}{v} &= \frac{v'}{v' + Ae^{z/2}} + \frac{vY_2}{v' + Ae^{z/2}} \\ &= \tfrac{1}{2} + \frac{\frac{1}{2}(v' - Ae^{z/2}) + vY_2}{v' + Ae^{z/2}} \\ &= \tfrac{1}{2} + \frac{v}{v' + Ae^{z/2}} \left\{ Y_2 + \tfrac{1}{2}\frac{v' - Ae^{z/2}}{v} \right\} \\ &= \tfrac{1}{2} + \frac{v}{v' + Ae^{z/2}} \left\{ Y_2 + \tfrac{1}{2}\frac{v' + vY_2}{v' + Ae^{z/2}} \right\}. \end{aligned} \tag{3.23}$$

The last equality here follows immediately from (3.22) by factorizing the left-hand side $(v')^2 + \frac{1}{2}\gamma e^z = (v' - Ae^{z/2})(v' + Ae^{z/2})$. On the other hand, by (3.20),

$$\frac{v' - Ae^{z/2}}{v} = \tfrac{1}{2} + uv. \tag{3.24}$$

Equating (3.24) and the last expression in (3.23) immediately implies that $u(z_n)$ is bounded as $n \to \infty$. This completes the whole Case IV.

Case V. We now assume that w and $1/w$ are bounded on Γ. To avoid Case I, there must be a sequence $\{z_n\}$ on Γ, tending to a, such that $w(z_n) \to 1$ as $n \to \infty$. Since w is bounded on Γ, we may take $c \in \mathbb{C} \setminus \{0,1\}$ with $|c|$ large enough to satisfy $|w - c| > 1$ on Γ. We use this c to define U as in (3.6) and we proceed as in Case IV(c) to show that U is bounded on Γ. Note that the expressions to be applied for Q_j, $1 \le j \le 4$, here, have been given in the Appendix. Observe also that there is no need to modify Γ, due to $|w - c| > 1$.

If there exists a sequence $\{\zeta_n\}$ on Γ, converging to a such that $W(\zeta_n) \to \infty$, then $w'(\zeta_n)$ is bounded as $n \to \infty$. This can be seen as in Case IV(b). To avoid Case I, we must have $w(\zeta_n) \to 1$, and we may now argue as in the Case IV(d).

The only case remaining now is when $W = V'/V$ is bounded on Γ, while $w(z_n) \to 1$ as $n \to \infty$. Since $V(z) = V(z_0)\exp(\int_{z_0}^{z} W(t)\,dt)$, we conclude that V is bounded on Γ, and from $V' = WV$, the function V' is bounded on Γ as well. If now $\delta \ne 0$, then $V'(z_n) \to \infty$ by (2.2), a contradiction. Therefore, $\delta = 0$ and (2.2) reduces to

$$V' = \frac{2\gamma e^z}{w-1},$$

implying that $\gamma = 0$, and this reduces back to Section 2.

4. Path modification in Case IV(c) and Case IV(d)

Recall first our reasoning following Case IV(b), where we fixed $c = -1$. Suppose $0 < \varepsilon \le 1/2$. Let now the points z'_k, z''_k divide Γ into arcs (z'_k, z''_k) on which $|w - c| = |w+1| < \varepsilon$ and arcs (z''_k, z'_{k+1}) on which $|w+1| > \varepsilon$. Pick any k and write $w_0 = w(z'_k)$ so that $|w_0 + 1| = \varepsilon$. Let f be the local inverse function of w defined in a neighbourhood of w_0. Then

$$-f''(w) = \left(\frac{1}{2w} + \frac{1}{w-1}\right) f'(w) + \Phi(w,f)\bigl(f'(w)\bigr)^3, \tag{4.1}$$

where

$$\Phi(w,f) = (w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \gamma w e^f + \frac{\delta e^{2f} w(w+1)}{w-1}.$$

There is $M > 1$ such that

$$\left|\frac{1}{2w} + \frac{1}{w-1}\right| \le M$$

whenever $|w| \ge \frac14$ and $|w-1| \ge \frac14$, and such that at the same time,

$$|\Phi(w,f)| \le M$$

whenever $|w| \ge \frac14$, $|w-1| \ge \frac14$, and $|f| \le |z_0| + R + 1$. Pick now $\psi \in [0, 2\pi)$, and define

$$s(r) = |f'(w_0 + re^{i\psi})|.$$

Then, as long as r is so small that $s(r)$ is defined, we have

$$|s'(r)| \le |f''(w_0 + re^{i\psi})| \le M\bigl(s(r)^3 + s(r)\bigr),$$

and so

$$-M(s^3+s) \le s'(r) \le M(s^3+s).$$

Observe that the initial value problem

$$s'(r) = \pm M(s^3+s), \qquad s(0) = |f'(w_0)| = s_0$$

has the solution

$$s_\pm(r) = s_0 e^{\pm Mr}\bigl(1 + s_0^2(1 - e^{\pm 2Mr})\bigr)^{-1/2},$$

and so

$$s_-(r) \le s(r) \le s_+(r)$$

for all r small enough to be considered, see [W], p. 69.

Now, w, $1/w$ and $1/(w-1)$ are bounded on the sequence z'_k. To avoid getting into Case I, we may assume that $|w'(z'_k)| \to \infty$ as $k \to \infty$. Hence we may restrict ourselves to considering such large k only, say $k \ge k_0$, that $|w'(z'_k)| > 2$, and so $s_0 < 1/2$. Now we have

$$e^{-Mr}\bigl(1+s_0^2(1-e^{-2Mr})\bigr)^{-1/2} = \frac{s_-(r)}{s_0} \le \frac{s(r)}{s_0} \le \frac{s_+(r)}{s_0} = e^{Mr}\bigl(1+s_0^2(1-e^{2Mr})\bigr)^{-1/2}.$$

Choose now $\eta \in (0, \frac{1}{8}]$ so that for all $r \in [0, \eta]$, we have

$$\tfrac{1}{2} < \frac{s_-(r)}{s_0} \leq \frac{s(r)}{s(0)} \leq \frac{s_+(r)}{s_0} < 2.$$

Let t be the largest number with $0 < t \leq \eta$ such that f can be continued analytically to the disk $B(w_0, t)$ and such that

$$|f(w)| \leq |z_0| + R + 1$$

for all $w \in B(w_0, t)$. Clearly,

$$\tfrac{1}{2} < \frac{s(r)}{s(0)} = \frac{|f'(w)|}{|f'(w_0)|} < 2$$

for all $w \in B(w_0, t)$. Hence $|f'(w)| < 2s_0$ and so, by (4.1),

$$|f''(w)| \leq 2s_0 M(1 + 4s_0^2) < 4s_0 M$$

for all $w \in B(w_0, t)$.

Set now $T = f'$ and note that the system

$$\left\{ \begin{aligned} f' &= T \\ T' &= \left(\tfrac{1}{2w} + \tfrac{1}{w-1}\right) T + \Phi(w, f) T^3, \end{aligned} \right. \tag{4.2}$$

corresponding to (4.1), is satisfied in $B(w_0, t)$. Now f, T, and the right hand side functions in (4.2) are uniformly bounded in $B(w_0, t)$. Therefore, by Cauchy's estimates, f and T can be continued analytically to a disk $B(w_0, t')$ with $t' > t$. If $t < \eta$, choose t' so that $t < t' < \eta$ and so that we still have $|f'(w)| < 3s_0$ for all $w \in B(w_0, t')$. Integrating along a line segment, we get

$$|f(w)| \leq |f(w_0)| + \int_{w_0}^{w} |f'(\zeta)||d\zeta| < |z_0| + R + 3s_0\eta < |z_0| + R + \tfrac{3}{2}\eta < |z_0| + R + 1.$$

This contradicts the maximality of t, and so $t = \eta$. Hence, if $|w_0 + 1| = \varepsilon \leq 1/2$ and $\eta = \eta(M)$ is small enough, we can continue f analytically to $B(w_0, \eta)$ so that $1/2 < |f'(w)|/|f'(w_0)| < 2$ holds there.

Assume now M is given, as z_0, R, α, β, γ, δ are known, hence η can be chosen. Next, choose $\varepsilon \in (0, 1/2]$ so that $\overline{B(-1, 3\varepsilon)} \subset B(w_0, \eta)$ for all w_0 with $|w_0 + 1| = \varepsilon$. As in [HL2], we may now modify Γ to a rectifiable arc $\tilde{\Gamma}$ from z_0 to a, so that $|w - c| = |w + 1| \geq \varepsilon$ on $\tilde{\Gamma}$ (or at least on a terminal part of $\tilde{\Gamma}$). As earlier in [HL1] and [HL2], since $\tilde{\Gamma}$ may leave the disk $B(z_0, R)$ infinitely often, we may need to construct a Riemann surface S as in [HL2] to make sure that w remains single-valued.

This completes the proof of Theorem 1.

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