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A NOTE ON DEGREE AND DILATATION

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In this brief essay, we point out how a method of Li and Yau [LY] can be used to show the interesting fact that it is impossible to map every closed orientable Riemannian n-manifold onto the standard n-sphere $\mathbb{S}^n$ by a quasiregular mapping with both the degree and the dilatation below some universal (dimensional) bound.

Throughout, let M^n, $n \geq 2$, denote a closed (i.e. compact and without boundary) connected orientable Riemannian n-manifold. As for quasiregular mappings, we use the notation and terminology of [R], except that here we make the simplifying convention that *all quasiregular mappings are nonconstant*.

It follows from an old construction of Alexander [A] that there always exists a quasiregular mapping $f : M^n \to \mathbb{S}^n$. Indeed, f can be chosen to be piecewise linear with respect to appropriate triangulations of M and $\mathbb{S}^n$. It moreover follows from a work of Cairns [C] that the triangulation of M can always be chosen to consist "fat" enough simplices so as to yield a quasiregular mapping $f : M \to \mathbb{S}^n$ with dilatation $K(f)$ not exceeding a dimensional constant. (Peltonen has generalized this to noncompact manifolds; see [Pe, p. 34].) Thus, we have that

$$\sup_{M^n} \mathrm{Dil}(M^n) < \infty\,, \tag{0.1}$$

where

$$\mathrm{Dil}(M^n) = \inf\{K(f) : f : M^n \to \mathbb{S}^n \text{ quasiregular}\}\,.$$

There is another fact which is less known in this context. Namely, for $n = 2, 3, 4$ we have that

$$\sup_{M^n} \mathrm{Deg}(M^n) = n, \tag{0.2}$$

where

$$\mathrm{Deg}(M^n) = \min\{\deg(f) : f : M^n \to \mathbb{S}^n \text{ quasiregular}\}\,.$$

For $n = 2$, equality (0.2) is classical and easy; every closed orientable surface can be mapped onto $\mathbb{S}^2$ by a two-to-one (piecewise linear) branched covering mapping. This can be verified by a direct construction. Alternatively, every such surface is diffeomorphic to a Riemann surface which admits a two-to-one meromorphic function onto the Riemann sphere. (For genus $g = 1$ we have the Weierstrass function, and for genus $g \geq 2$ the surfaces in question are the so called hyperelliptic curves.)

For $n = 3$, equality (0.2) was proved independently by Hilden [H] and Montesinos [M] in the 1970's. For $n = 4$, equality (0.2) is a more recent result of Piergallini [Pi].

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To be more precise here, the results of Hilden, Montesinos, and Piergallini show that $\mathrm{Deg}(M^n) \le n$ for $n = 3, 4$. Moreover, in these cases, the mappings of degree n can be chosen to be piecewise linear and not merely quasiregular. The fact that equality holds in (0.2) follows from the work of Berstein and Edmonds [BE1] who showed that $\mathrm{Deg}(T^n) \ge n$ for all $n \ge 2$, where T^n is the n-torus. Note that the definition for $\mathrm{Deg}(M^n)$ is independent of the Riemannian structure on M. Also the result in [BE1] is more general allowing for arbitrary (topological) branched coverings.

We also refer the reader to the nice paper [BE2] for useful information about branched coverings in low dimensions.

It is not known whether equality (0.2) holds for $n \ge 5$. (Compare [HS, Question 32].) In fact, the answer to the following question does not seem to be known.

Question 0.1. Is it true that

$$\sup_{M^n} \mathrm{Deg}(M^n) < \infty \tag{0.3}$$

for $n \ge 5$?

In contrast to (0.2), no numerical estimate has been given to the number (0.1).

In any case, the above discussion shows that at least in dimensions $n = 2, 3, 4$ one can map each given M^n onto $\mathbb{S}^n$ by a quasiregular mapping with either a "small" dilatation or a low degree. What we want to point out in this note is that one cannot, in general, do both simultaneously.

To this end, we shall demonstrate that the product

$$K(f)\mathrm{deg}(f)$$

for $f : M^n \to \mathbb{S}^n$ has a lower bound in terms of the volume $\mathrm{vol}(M)$ of M and the number

$$\lambda(M^n) = \inf_u \frac{(⨍_M |\nabla u|^n \, dV)^{1/n}}{(⨍_M |u|^2 \, dV)^{1/2}}, \tag{0.4}$$

where the infimum is taken over all nonconstant smooth real valued functions u on M such that

$$\int_M u \, dV = 0 \,,$$

and where the barred integral sign denotes mean value. Because

$$\Big(⨍_M |\nabla u|^2 \, dV\Big)^{1/2} \le \Big(⨍_M |\nabla u|^n \, dV\Big)^{1/n},$$

we have that

$$\lambda_1(M) \le \lambda(M)^2, \tag{0.5}$$

where $\lambda_1(M)$ is the first non-zero eigenvalue for the Laplacian on M. Recall that $\lambda_1(M)$ can be defined by the variational formula

$$\lambda_1(M^n) = \inf_u \frac{\int_M |\nabla u|^2 \, dV}{\int_M |u|^2 \, dV}, \tag{0.6}$$

with infimum over the same class of competitors u as in (0.4).

We shall show below the following: *if $f : M^n \to \mathbb{S}^n$ is a quasiregular mapping, then*

$$\lambda(M)^n \mathrm{vol}(M) \leq A(n)K(f)\deg(f), \tag{0.7}$$

where

$$A(n) = (n+1)^{n/2}\mathrm{vol}(\mathbb{S}^n). \tag{0.8}$$

In particular, in terms of the first eigenvalue λ_1, we get from (0.5) and (0.7) that

$$\lambda_1(M)^{n/2}\mathrm{vol}(M) \leq A(n)K(f)\deg(f) \tag{0.9}$$

if $f : M^n \to \mathbb{S}^n$ is a quasiregular mapping, where $A(n)$ is given in (0.8).

Now in each dimension $n \geq 2$ there are examples of manifolds M^n such that the left hand side of (0.9) exceeds any prescribed bound. This is a rather deep fact for $n = 2$; see [BBD] for a discussion. For $n = 2$, large values for the left hand side of (0.9) require large genus, for we have the estimate

$$\lambda_1(M_g)\mathrm{vol}(M_g) \leq 8\pi(g+1)$$

if M_g is a surface of genus g. This is a result of Yang and Yau [YY]. (See also [LY, (2.25)].)

For $n \geq 3$, the curious fact is that *every* closed smooth orientable n-manifold can be equipped with Riemannian metrics with constant volume but arbitrarily large λ_1; see [CD].

In view of the above discussion, we thus conclude as our main claim that

$$\sup_{M^n} \inf\{K(f)\deg(f) : f : M^n \to \mathbb{S}^n \text{ quasiregular}\} = \infty \tag{0.10}$$

for each $n \geq 2$.

The following question seems worth investigation.

Question 0.2. For which collections $\mathcal{M} = \{M^n\}$ of closed connected orientable Riemannian n-manifolds we have that

$$\sup_{M^n \subset \mathcal{M}} \inf\{K(f)\deg(f) : f : M^n \to \mathbb{S}^n \text{ quasiregular}\} < \infty, \tag{0.11}$$

and what geometric data would provide such finite bounds?

We now turn to the proof of (0.7). As mentioned earlier, the idea is that of Li and Yau [LY, Section 2].

Thus, let $f : M^n \to \mathbb{S}^n$ be a quasiregular mapping. By using the embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we write $f = (f_1, \ldots, f_{n+1})$. By a continuity argument as in [LY, p. 274], we find a conformal self-mapping $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ such that

$$F = \varphi \circ f = (F_1, \ldots, F_{n+1}) : M \to \mathbb{S}^n$$

satisfies

$$\int_M F_i \, dV = 0$$

for each $i = 1, \ldots, n+1$. Because of this, and because each F_i belongs to the Sobolev space $W^{1,n}(M)$, we have by approximation that

$$\mathrm{vol}(M)^{(2-n)/n}\lambda(M)^2\Big(\int_M F_i^2\,dV\Big) \le \Big(\int_M |\nabla F_i|^n\,dV\Big)^{2/n}.$$

On the other hand, because F is quasiregular with $K(F) = K(f)$ and $\deg(F)$=$\deg(f)$, we have that

$$\begin{aligned}\int_M |\nabla F_i|^n\,dV &\le \int_M |\nabla F|^n\,dV \\ &\le K(f)\int_M \det DF\,dV = K(f)\deg(f)\mathrm{vol}(\mathbb{S}^n)\,.\end{aligned}$$

To finish, we combine the fact that

$$\mathrm{vol}(M)^{2/n}\lambda(M)^2 = \mathrm{vol}(M)^{(2-n)/n}\lambda(M)^2\int_M\sum_{i=1}^{n+1}F_i^2\,dV$$

with the preceding two inequalities, to obtain

$$\mathrm{vol}(M)^{2/n}\lambda(M)^2 \le (n+1)\mathrm{vol}(\mathbb{S}^n)^{2/n}K(f)^{2/n}\deg(f)^{2/n}.$$

This gives (0.7) and the proof is thereby complete.

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