Papers on Analysis:
A volume dedicated to Olli Martio on the occasion of his 60th birthday
Report. Univ. Jyväskylä 83 (2001), pp. 109–126

SOBOLEV INEQUALITIES, TRUNCATION METHOD, AND JOHN DOMAINS

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1. Introduction. The purpose of this paper is to present a modern approach to the proof of the Sobolev inequality in the subcritical case $1 \leq p < n$. The approach is very elementary. It is based on a truncation argument due to Maz'ya which leads to a surprising observation that some weak type estimates imply strong type estimates. The Maz'ya truncation argument is presented in his famous monograph [37] and the proof of Theorem 2 presented below is implicitly contained there. However the book [37] deals with very technical results which are stated in a great generality, and it is quite difficult to extract this little and beautiful observation from the results in the book. This is really a pity because the truncation argument of Maz'ya turned out to be incredibly important for the recent development of the theory of Sobolev inequalities in a general setting of metric spaces and vector fields. Since Maz'ya's argument had not been widely known it was recently rediscovered several times and Maz'ya did not gain adequate recognition.

Theorem 2 is so elementary and important that there is a need to present it in such an elementary setting in order to make it accessible to a large audience. My intention was to make the paper easily accessible to students who are just beginners in the subject of Sobolev spaces.

It is well known that the proofs of the Sobolev inequality for the cases $p = 1$ and $1 < p < n$ are different in their nature. The proof for the case $1 < p < n$ is "potential theoretic", while the proof for the case $p = 1$ is "isoperimetric". The truncation method of Maz'ya explains in a transparent way how to put the "isoperimetric" proof in the "potential theoretic" framework.

As one of the applications of the method we will give a short, elementary and self-contained proof of the Sobolev inequality for a wide class of, so called, John domains (Theorem 8). As we will explain this is a very natural and, in a sense, optimal version of the Sobolev inequality. Again, the proof does not contain new ideas, but it seems there is no single place where one would find all the details of this elementary presentation.

2000 *Mathematics Subject Classification:* 46E35. This work was supported by the KBN grant. The research was completed during the stay of the author in Department of Mathematics at University of Michigan. The author wishes to thank the University for the support and hospitality.

The development of the theory of John domains was to a large extent inspired by the research of Olli Martio. Actually it was Martio and Sarvas [34] who defined the class of John domains. Moreover Martio [33] and Reshetnyak [42] proved independently the Sobolev–Poincaré inequality for John domains when $1 < p < n$.

Now let us explain what we shall mean by a Sobolev inequality. Roughly speaking Sobolev inequality is a name assigned to a large class of integral inequalities with an integral involving a function on the left and an integral involving the gradient on the right. This description is too rough and in what follows we shall focus on two types of Sobolev inequalities that we describe more precisely now.

The following inequality was proved by Sobolev [47], [48] in the case $1 < p < n$ and extended by Gagliardo [17] and Nirenberg [40] to the case $p = 1$:

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{1/p^*} \le C(n,p)\left(\int_{\mathbb{R}^n} |\nabla u|^p\,dx\right)^{1/p} \tag{1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where $1 \le p < n$ and $p^* = np/(n-p)$. Actually inequality (1) holds for all Lipschitz[1] functions with compact support. This inequality is known as the Sobolev inequality. By this name we will also mean any inequality of the form

$$\left(\int_{\Omega} |u|^{q}\,d\mu\right)^{1/q} \le C\left(\int_{\Omega} |\nabla u|^p\,d\nu\right)^{1/p}, \tag{2}$$

where μ and ν are Borel measures in a domain $\Omega \subset \mathbb{R}^n$, with ν being absolutely continuous with respect to the Lebesgue measure, provided the inequality holds for all Lipschitz functions u compactly supported in Ω.[2]

There is yet another version of the Sobolev inequality known as the Sobolev–Poincaré inequality. By this we mean the inequality

$$\inf_{c\in\mathbb{R}}\left(\int_{\Omega} |u-c|^{p^*}\,dx\right)^{1/p^*} \le C\left(\int_{\Omega} |\nabla u|^p\,dx\right)^{1/p}, \tag{3}$$

where $1 \le p < n$ and $p^* = np/(n-p)$, which holds in any bounded and Lipschitz domain $\Omega \subset \mathbb{R}^n$ (i.e. a domain with the boundary being locally the graph of a Lipschitz function) for all locally Lipschitz functions u in Ω (and hence for all $u \in C^\infty(\Omega)$[3]). Now in contrast to inequality (1) we do not assume that u has compact support in Ω.

Again, as before, any inequality of the form

$$\inf_{c\in\mathbb{R}}\left(\int_{\Omega} |u-c|^{q}\,d\mu\right)^{1/q} \le C\left(\int_{\Omega} |\nabla u|^p\,d\nu\right)^{1/p}, \tag{4}$$

[1] We have to employ Rademacher's theorem which states that Lipschitz functions are differentiable a.e.

[2] It is necessary to assume that the measure ν is absolutely continuous with respect to the Lebesgue measure. Indeed, since u is Lipschitz, the gradient ∇u is defined a.e. with respect to the Lebesgue measure. Thus if ν were not absolutely continuous, it could happen that ∇u would not be defined on a set of a positive measure ν.

[3] Observe that each $u \in C^\infty(\Omega)$ is locally Lipschitz, but it need not be Lipschitz e.g., $x^{-1} \in C^\infty(0,1)$. On the other hand, each $u \in C_0^\infty$ is necessarily Lipschitz. For this reason we state our results either for compactly supported Lipschitz functions or for locally Lipschitz functions.

where $1 \le p \le q < \infty$, μ and ν are Borel measures in a domain $\Omega \subset \mathbb{R}^n$ and ν is absolutely continuous with respect to the Lebesgue measure, will be called Sobolev–Poincaré provided it holds for all locally Lipschitz functions u in Ω.

It easily follows from the Hölder inequality that

$$(5) \qquad \frac{1}{2}\left(\int_\Omega |u-u_\Omega|^q\, d\mu\right)^{1/q} \le \inf_{c\in\mathbb{R}} \left(\int_\Omega |u-c|^q\, d\mu\right)^{1/q} \le \left(\int_\Omega |u-u_\Omega|^q\, d\mu\right)^{1/q},$$

where $1 \le q < \infty$, $\Omega \subset \mathbb{R}^n$, $\mu(\Omega) < \infty$, and u is integrable on Ω with respect to μ. Here and in what follows $u_X = \fint_X u\, d\mu = \mu^{-1}(X)\int_X u\, d\mu$ will denote the average value of u over X. Thus the Sobolev–Poincaré inequality (4) is equivalent to

$$\left(\int_\Omega |u-u_\Omega|^q\, d\mu\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p\, d\nu\right)^{1/p},$$

provided $\mu(\Omega) < \infty$.

The paper is organized as follows. In Section 2 we prove that inequalities (2) and (4) are equivalent to other inequalities which seem much weaker and much easier to verify than inequalities (2) and (4), see Theorem 2 and Theorem 4. This is where we employ the Maz'ya truncation technique. In Section 3 we prove the Sobolev–Poincaré inequality (3) in the case in which Ω is a John domain and $1 \le p < n$, see Theorem 8. To cover the case $p = 1$ we employ results from Section 2 based on the truncation technique. The case $1 < p < n$ follows then easily from the case $p = 1$. The last section, Section 4, is devoted to a discussion about a vector–valued Sobolev–type inequality due to Strauss, [50]. It is not clear how to adapt the truncation technique to this situation.

We close the paper with some problems related to the lack of the truncation argument both in the case of higher order derivatives and in the case of vector valued functions. The problems stated in the paper are not well known and it is me who takes the sole responsibility for the statements — I do not know if the problems are easy or difficult, but I would be happy to see the answers.

Notation employed in the paper is rather standard. By C we will denote a general constant which can change its value even in the same string of estimates. $|A|$ will denote the Lebesgue measure of a set $A \subset \mathbb{R}^n$. We will always use p^* to denote the Sobolev exponent $p^* = np/(n-p)$, where $1 \le p < n$. By writing $a \approx b$ we will mean that there is a constant $C \ge 1$ such that $C^{-1}b \le a \le Cb$.

2. Weak estimates versus strong estimates.

Let μ be a positive measure on a set X. The Marcinkiewicz space $L^p_w(X)$, where $0 < p < \infty$, is defined as the set of all μ-measurable functions u on X such that

$$\sup_{t>0} \mu(\{|u| > t\})t^p < \infty.$$

It is a direct consequence of Chebyschev's inequality that $L^p(X) \subset L^p_w(X)$. In general $L^p(X)$ is a proper subset of $L^p_w(X)$. For example $x^{-1} \in L^1_w(0,1) \setminus L^1(0,1)$. The Marcinkiewicz space is known also as the weak L^p space.

In this section we prove a result (Theorem 1) which gives some situations in which a function belonging to the Marcinkiewicz space L^p_w belongs to L^p as well. The proof is based on the Maz'ya truncation argument. Then we show applications of the result to Sobolev type inequalities.

First we define truncations. For a function u and $0 < t_1 < t_2 < \infty$ we set

$$\widetilde{u}_{t_1}^{t_2}(x) = \begin{cases} t_2 & \text{if } |u(x)| \geq t_2, \\ |u(x)| & \text{if } t_1 \leq |u(x)| \leq t_2, \\ t_1 & \text{if } |u(x)| \leq t_1, \end{cases}$$

and

$$u_{t_1}^{t_2}(x) = \widetilde{u}_{t_1}^{t_2}(x) - t_1.$$

The truncation $u_{t_1}^{t_2}$ has a geometric interpretation which becomes clear when one makes a picture — we cut the function $|u|$ between levels t_1 and t_2 to obtain a piece of the graph that we then lower down to touch the coordinate axis.

Theorem 1. *Let μ and ν be two positive measures defined on the same σ-algebra in a set X. Let $0 < p \leq q < \infty$, $u \in L^q_w(X, \mu)$ and $g \in L^p(X, \nu)$. Assume that $\{A_t\}_{t>0}$ is a decreasing family of measurable subsets of X (i.e., $A_{t_2} \subseteq A_{t_1}$ for $t_2 > t_1$). If for every $0 < t_1 < t_2 < \infty$*

$$\sup_{t>0} \mu(\{u_{t_1}^{t_2} > t\}) t^q \leq \|g\chi_{A_{t_1} \setminus A_{t_2}}\|^q_{L^p(X,\nu)}, \tag{6}$$

then $u \in L^q(X, \mu)$ and

$$\|u\|_{L^q(X,\mu)} \leq 4\|g\|_{L^p(X,\nu)}.$$

Proof. We have

$$\begin{aligned} \int_X |u|^q \, d\mu &\leq \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{2^{k-1} < |u| \leq 2^k\}) \\ &\leq \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{|u| \geq 2^{k-1}\}) \\ &= \sum_{k=-\infty}^{\infty} 2^{kq} \mu(\{u_{2^{k-2}}^{2^{k-1}} \geq 2^{k-2}\}) \\ &\leq \sum_{k=-\infty}^{\infty} 2^{kq} 2^{-(k-2)q} \left(\int_{A_{2^{k-2}} \setminus A_{2^{k-1}}} |g|^p \, d\nu \right)^{q/p} \\ &\leq 2^{2q} \left(\sum_{k=-\infty}^{\infty} \int_{A_{2^{k-2}} \setminus A_{2^{k-1}}} |g|^p \, d\nu \right)^{q/p} \\ &\leq 2^{2q} \|g\|^q_{L^p(X,\nu)}. \end{aligned}$$

In the second to last inequality we used the assumption $q/p \geq 1$. □

This result fairly easily applies to various types of Sobolev inequalities.

Theorem 2. *Let μ and ν be two positive Borel measures on an open set $\Omega \subseteq \mathbb{R}^n$. Assume that ν is absolutely continuous with respect to the Lebesgue measure. Then for all $0 < p \leq q < \infty$, the following two conditions are equivalent:*

(1) *For every Lipschitz function u with compact support in Ω,*

$$\sup_{t>0} \mu(\{|u|>t\})t^q \le C_1 \left(\int_\Omega |\nabla u|^p \, d\nu\right)^{q/p}. \tag{7}$$

(2) *For every Lipschitz function u with compact support in Ω,*

$$\left(\int_\Omega |u|^q \, d\mu\right)^{1/q} \le C_2 \left(\int_\Omega |\nabla u|^p \, d\nu\right)^{1/p}.$$

Remarks. (1) The most interesting case is when $1 \le p \le q < \infty$. (2) The theorem is surprising since the second condition seems much stronger than the first one.

Proof of Theorem 2. Obviously 2 implies 1 by Chebyschev's inequality. The opposite implication follows from Theorem 1 with $g = |\nabla u|$, $A_t = \{|u| > t\}$, and the observation that $|\nabla u_{t_1}^{t_2}| = |\nabla u|\chi_{\{t_1<|u|\le t_2\}}$ a.e. with respect to the Lebesgue measure and hence ν a.e.[4]. □

Now we show how to use the theorem to prove the Sobolev inequality for $p = 1$ using Riesz potentials.

The well known inequality

$$|u(x)| \le C(n) \int_{\mathbb{R}^n} \frac{|\nabla u(z)|}{|x-z|^{n-1}} \, dz \quad \text{a.e.} \tag{8}$$

holds for all compactly supported Lipschitz functions u in $\mathbb{R}^n$. For a proof see[5] e.g. [19], [49], [53]. The operator

$$I_1 g(x) = \int_{\mathbb{R}^n} \frac{g(z)}{|x-z|^{n-1}} \, dz$$

which appears on the right hand side of the above inequality is called *Riesz potential.* A well known Fractional Integration Theorem, see [53], states that

$$I_1 : L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n) \tag{9}$$

is a bounded operator for $1 < p < n$. This together with inequality (8) gives the Sobolev inequality (1) for the case $1 < p < n$.

The operator I_1 is *not* bounded from L^1 to $L^{n/(n-1)}$, and so the above argument cannot be directly applied to obtain inequality (1) for $p = 1$. Fortunately we still have some estimates for the Riesz potential even when $p = 1$. Namely it is easy to prove that the operator is bounded from L^1 to $L^{n/(n-1)}_w$ in the sense that

$$\sup_{t>0} |\{|I_1 g| > t\}| \, t^{n/(n-1)} \le C \left(\int_{\mathbb{R}^n} |g(z)| \, dz\right)^{n/(n-1)}, \tag{10}$$

[4]We employed here a well known fact that if two Lipschitz functions u and v defined in a domain $\Omega \subset \mathbb{R}^n$ coincide in a measurable set $E \subset \Omega$, then $\nabla u = \nabla v$ a.e. in E. In particular if u is constant in E, then $\nabla u = 0$ a.e. in E. In what follows we will use this fact several times without notice.

[5]Below we will provide a self-contained proof of a more sophisticated version of (8) for John domains, see Theorem 10.

see Lemma 11. This leads to the following weak version of inequality (1):

$$\sup_{t>0} |\{|u|>t\}|\, t^{n/(n-1)} \le C \left(\int_{\mathbb{R}^n} |\nabla u|\, dx \right)^{n/(n-1)} .$$

Now Theorem 2 completes the proof of (1) for $p=1$.

Actually it is quite easy to deduce the case $1<p<n$ of (1) from the case $p=1$. To this end it suffices to apply inequality (1) with $p=1$ to $v=|u|^{p^*(n-1)/n}$, see the proof of Theorem 8 for a similar argument. This way we avoid the use of the Fractional Integration Theorem in the proof of (1) for the whole range $1\le p<n$.

As a direct consequence of our discussion we obtain the following corollary.

Corollary 3. *Let μ, ν and Ω be as in Theorem 2. Let $0<p\le q<\infty$. If the Riesz potential $I_1^\Omega g(x)=\int_\Omega g(z)|x-z|^{1-n}\, dz$ is bounded from $L^p(\nu)$ to $L^q_w(\mu)$, then the Sobolev inequality*

$$\left(\int_\Omega |u|^q\, d\mu \right)^{1/q} \le C \left(\int_\Omega |\nabla u|^p\, d\nu \right)^{1/p}$$

holds for all Lipschitz functions compactly supported in Ω. □

The method presented above is due to Maz'ya. Actually Maz'ya [36], (cf. [37, Section 2.3.1], [21, Theorem 1]), proved that a Sobolev inequality is equivalent to a capacitary estimate which, in turn, is a direct consequence of the weak estimate in Theorem 2. We want to point out that the method based on the truncation argument mimics the proof of the equivalence of the Sobolev inequality with the isoperimetric inequality, see [53]. Inequality (7) plays the role of the relative isoperimetric inequality and the truncation argument provides a discrete counterpart of the co-area formula. The truncation method of Maz'ya has become very useful when proving various versions of the Sobolev inequality with *sharp* exponents in the borderline case where interpolation arguments do not work. Recently the truncation argument has been employed and sometimes even rediscovered by many authors; see Adams and Hedberg [1, Theorem 7.2.1], Bakry, Coulhon, Ledoux and Saloff-Coste [2], Biroli and Mosco [4], [5], Capogna, Danielli and Garofalo [10], Cianchi [12], Coulhon [13], Franchi, Gallot and Wheeden [15], Garofalo and Nhieu [18], Hajłasz and Koskela [22, Theorem 2.1], Heinonen and Koskela [23], [24], Long and Nie [30], Maheux and Saloff-Coste [31], Malý and Pick [32], Semmes [45], and Tartar [51].

Theorem 2 concerns the Sobolev inequality for *compactly* supported Lipschitz functions. Perhaps a more interesting version of the Sobolev inequality is the Sobolev–Poincaré inequality which deals with *arbitrary* locally Lipschitz functions. The following result shows that also in this case the strong version of the Sobolev–Poincaré inequality is equivalent to its weak form.

Theorem 4. *Let μ and ν be two positive Borel measures on an open set $\Omega\subset\mathbb{R}^n$. Assume that ν is absolutely continuous with respect to the Lebesgue measure and that $\mu(\Omega)<\infty$. Then for all $0<p\le q<\infty$ the following two conditions are equivalent:*

(1) *For every locally Lipschitz function u in Ω,*

$$\inf_{c\in\mathbb{R}} \sup_{t>0} \mu(\{|u-c|>t\})t^q \le C_1 \left(\int_\Omega |\nabla u|^p\, d\nu \right)^{q/p} .$$

(2) *For every locally Lipschitz function u in Ω,*

$$\inf_{c\in\mathbb{R}}\left(\int_\Omega |u-c|^q\,d\mu\right)^{1/q} \le C_2\left(\int_\Omega |\nabla u|^p\,d\nu\right)^{1/p}.$$

Remarks. (1) As in Theorem 2, the most interesting case is $1\le p\le q<\infty$. (2) If $q\ge 1$, then we have two additional conditions equivalent to the conditions in the theorem. They are obtained from the above two conditions by replacing the left hand sides of the inequalities by $\sup_{t>0}\mu(\{|u-u_\Omega|>t\})t^q$ and $\left(\int_\Omega |u-u_\Omega|^q\,d\mu\right)^{1/q}$ respectively. This follows from (5).

Proof of Theorem 4. As before the implication from 2 to 1 is obvious, so it remains to prove the converse implication. The proof employs the same method as in the case of Theorem 2, however, it is slightly more complicated now.

First choose $b\in\mathbb{R}$, such that

$$\mu(\{u\ge b\})\ge\frac{\mu(\Omega)}{2}\qquad\text{and}\qquad \mu(\{u\le b\})\ge\frac{\mu(\Omega)}{2}. \tag{11}$$

Let $v_+=\max\{u-b,0\}$, $v_-=-\min\{u-b,0\}$. Since $|u-b|=v_++v_-$ it suffices to prove that

$$\|v_\pm\|_{L^q(\Omega,\mu)}\le C\|\nabla u\|_{L^p(\Omega,\nu)}.$$

The estimate for $\|v_+\|_{L^q}$ and $\|v_-\|_{L^q}$ goes exactly in the same way. In what follows v denotes either v_+ or v_-.

Lemma 5. *Let γ be a positive measure on X with $\gamma(X)<\infty$. If $w\ge 0$ is a measurable function such that $\gamma(\{w=0\})\ge\gamma(X)/2$, then for every $t>0$*

$$\gamma(\{w>t\})\le 2\inf_{c\in\mathbb{R}}\gamma\left(\left\{|w-c|>\frac{t}{2}\right\}\right).$$

The proof of the lemma is easy and we leave it to the reader. □

Now note that for any $\infty>t_2>t_1>0$, the function $v_{t_1}^{t_2}$ is locally Lipschitz and has the property $\mu(\{v_{t_1}^{t_2}=0\})\ge\mu(\Omega)/2$. Hence applying the lemma and inequality 1 we conclude that

$$\begin{aligned}\sup_{t>0}\mu(\{v_{t_1}^{t_2}>t\})t^q &\le C\inf_{c\in\mathbb{R}}\sup_{t>0}\mu\left(\left\{|v_{t_1}^{t_2}-c|>\frac{t}{2}\right\}\right)\left(\frac{t}{2}\right)^q\\ &\le C\|\nabla v_{t_1}^{t_2}\|^q_{L^p(\Omega,\nu)}\\ &\le C\|\nabla u\chi_{\{t_1<|u-b|\le t_2\}}\|^q_{L^p(\Omega,\nu)}.\end{aligned}$$

Now the estimate for $\|v\|_{L^q}$ follows from Theorem 1. The proof is complete. □

The following corollary is a direct consequence of the proof.

Corollary 6. *Under the assumptions of Theorem 4, if b is defined by (11), we have two additional conditions equivalent to the conditions in Theorem 4:*

(1) *For every locally Lipschitz function u in Ω,*

$$\sup_{t>0}\mu(\{|u-b|>t\})t^q\le C_1\left(\int_\Omega|\nabla u|^p\,d\nu\right)^{q/p}.$$

(2) *For every locally Lipschitz function u in Ω,*

$$\left(\int_\Omega |u-b|^q\, d\mu\right)^{1/q} \leq C_2 \left(\int_\Omega |\nabla u|^p\, d\nu\right)^{1/p}.$$

3. Sobolev spaces on John domains. Most of the textbooks on Sobolev spaces provide Lipschitz domains or the larger class of domains with the interior cone condition as examples of regions for which the Sobolev–Poincaré inequality

$$\left(\int_\Omega |u-u_\Omega|^{p^*}\, dx\right)^{1/p^*} \leq C(\Omega,p) \left(\int_\Omega |\nabla u|^p\, dx\right)^{1/p},$$

$1 \leq p < n$, holds for all $u \in C^\infty(\Omega)$. The class of domains with the interior cone condition is however a subclass of a much larger class of John domains. It turns out that the Sobolev–Poincaré inequality holds in John domains as well. The aim of this section is to provide an elementary and self-contained proof of this result.

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is called a *domain with the interior cone condition* if there exists a finite cone

$$C = \{x \in \mathbb{R}^n : x_1^2 + \ldots + x_{n-1}^2 \leq a x_n^2,\ 0 \leq x_n \leq b\},$$

such that any point of Ω is a vertex of a cone that is congruent to C and is entirely contained in Ω.

We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is a *John domain* if there is a constant $C_J \geq 1$ and a distinguished point $x_0 \in \Omega$ (called a central point) so that each point $x \in \Omega$ can be joined to x_0 by a curve (called John curve) $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq C_J^{-1} |x - \gamma(t)|$$

for every $t \in [0,1]$.

The above definition and the usual one are slightly different, but they are equivalent.

The class of John domains was considered for the first time by Fritz John in [29]. Later the class was named after John by Martio and Sarvas, [34], who realized how important the class was. Since that time the theory developed a great deal and now it is difficult to find anyone working in the area of geometric analysis who has not heard about John domains. The class of John domains plays an important role in the theory of quasiconformal mappings and in holomorphic dynamics. For references see e.g. [11], [35], [39], [52].

Lemma 7. *Every domain with the interior cone condition is John.* □

The lemma is obvious, simply extend the cores of the cones to suitable curves that end at a fixed point x_0 that is selected to be as far from the boundary as possible. Roughly speaking the difference between an interior cone condition domain and a John domain is in replacing the "rigid cone" by a "twisted cone". The difference is huge however. For example one can easily construct John domains with fractal boundary of Hausdorff dimension strictly greater than $n-1$ (von Koch snowflake is an example), while boundary of a domain with the interior cone condition consists of a finite number of graphs of Lipschitz functions[6] and hence is of finite $(n-1)$-dimensional Hausdorff measure.

The following theorem was proved independently by Martio [33] and Reshetnyak [42] for the case $1 < p < n$ and was extended to the case $p = 1$ by Bojarski [6].

Theorem 8 (Bojarski–Martio–Reshetnyak). *If $\Omega \subset \mathbb{R}^n$ is a John domain, and $1 \leq p < n$, then*

$$\left(\int_\Omega |u - u_\Omega|^{p^*} \right)^{1/p^*} \leq C(C_J, n, p) \left(\int_\Omega |\nabla u|^p \right)^{1/p} \tag{12}$$

for all locally Lipschitz functions u in Ω.

Martio and Reshetnyak proved Theorem 8 for the case $1 < p < n$ as a direct consequence of Theorem 10 below and the Fractional Integration Theorem. To cover the case $p = 1$ Bojarski employed a powerful Boman chain technique. Another proof was given in [21]. In the proof that we are going to present below we will show that also the case $p = 1$ of Theorem 8 is a consequence of Theorem 10.

Actually the class of domains for which the Sobolev–Poincaré inequality holds is not much larger than the class of John domains. Indeed, Buckley and Koskela [8] proved the following striking result (see [8] and [9] for further generalizations, including those for higher dimensional domains).

Theorem 9 (Buckley–Koskela). *Let Ω be a bounded simply connected plane domain. Fix $1 \leq p < 2$. Then Ω satisfies the Sobolev–Poincaré inequality (12) if and only if Ω is a John domain.*

The plan for the proof of Theorem 8 is the following. First we prove Theorem 10 which is a generalization of inequality (8) and Lemma 11 which can be viewed as a borderline case of the Fractional Integration Theorem.

Theorem 10 (Martio–Reshetnyak). *Let $\Omega \subset \mathbb{R}^n$ be a John domain. Then for every locally Lipschitz function u in Ω and all $x \in \Omega$*

$$|u(x) - u_\Omega| \leq C(C_J, n) \int_\Omega \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz.$$

Lemma 11. *Let $\Omega \subset \mathbb{R}^n$ be open and $g \in L^1(\Omega)$. Then*

$$\sup_{t>0} |\{x \in \Omega :\ I_1^\Omega g(x) > t\}| t^{n/(n-1)} \leq C(n) \left(\int_\Omega |g|\, dx \right)^{n/(n-1)},$$

where $I_1^\Omega g(x) = \int_\Omega g(z)|x - z|^{1-n}\, dz$.

Lemma 11 and Theorem 10 imply a weak type estimate

$$\sup_{t>0} |\{x \in \Omega :\ |u(x) - u_\Omega| > t\}| t^{n/(n-1)} \leq C(C_J, n) \left(\int_\Omega |\nabla u|\, dx \right)^{n/(n-1)},$$

[6]The boundary need not be locally a graph of a Lipschitz function — think for example of a disc with one radius removed.

which together with Theorem 4 completes the proof of Theorem 8 for the case $p = 1$. Thus we are left with the proofs of Lemma 11, Theorem 10 and with an argument that shows how to reduce the case $1 < p < n$ to the case $p = 1$.

Lemma 11 is a limiting case of the Fractional Integration Theorem, however the proof given below is much simpler than the proof of the Fractional Integration Theorem. The method goes back to Santalo. I learned it from Jan Malý.

Proof of Lemma 11. We start with a useful general observation. If $E \subset \mathbb{R}^n$ is a measurable set of the finite Lebesgue measure, then

$$\int_E \frac{dz}{|x-z|^{n-1}} \leq C(n)|E|^{1/n}. \tag{13}$$

Indeed, let $B = B(x,r)$ be a ball such that $|B| = |E|$. Then

$$\int_E \frac{dz}{|x-z|^{n-1}} \leq \int_B \frac{dz}{|x-z|^{n-1}} = Cr = C'|E|^{1/n}.$$

Now we can return to the proof of the lemma. Replacing g by g/t we may assume that $t = 1$. Let $E = \{I_1^\Omega g > 1\}$. Then

$$|E| \leq \int_E I_1^\Omega g = \int_\Omega \int_E \frac{dx}{|x-z|^{n-1}}\, g(z)\, dz \leq C|E|^{1/n} \int_\Omega |g|.$$

The proof is complete. $\square$

Proof of Theorem 10. In the proof we will need the following familiar version of the Sobolev–Poincaré inequality. Actually we will need the case $p = 1$ only, but we provide a proof for all $1 \leq p < \infty$.

Lemma 12. *Let $B \subset \mathbb{R}^n$ be a ball of radius r and $1 \leq p < \infty$. Then*

$$\left(\fint_B |u - u_B|^p \, dx \right)^{1/p} \leq C(n,p) r \left(\fint_B |\nabla u|^p \, dx \right)^{1/p}$$

for all locally Lipschitz functions u in B.

Proof. We can assume that B is centered at the origin. For $x, y \in B$ we have

$$\begin{aligned} |u(y) - u(x)| &= \left| \int_0^1 \frac{d}{dt} u(x + t(y-x))\, dt \right| = \left| \int_0^1 \langle \nabla u(x + t(y-x)), y - x \rangle \, dt \right| \\ &\leq 2r \int_0^1 |\nabla u(x + t(y-x))| \, dt. \end{aligned}$$

Hence integrating with respect to y and then applying Hölder's inequality yield

$$\begin{aligned} |u(x) - u_B| &\leq 2r \int_0^1 \fint_B |\nabla u(x + t(y-x))| \, dy \, dt \\ &\leq 2r \left(\int_0^1 \fint_B |\nabla u(x + t(y-x))|^p \, dy \, dt \right)^{1/p}. \end{aligned}$$

Now

$$\int_B |u-u_B|^p \leq C\frac{r^p}{|B|}\int_0^1\int_B\int_B |\nabla u(x+t(y-x))|^p\, dy\, dx\, dt.$$

Changing variables $(x,y)\in B\times B$ to $(\xi,\eta)\in B\times 2B$ by the formula

$$\xi = x+t(y-x), \qquad \eta = y-x,$$

we easily see that the Jacobian of the transformation is one and hence

$$\int_B |u-u_B|^p \leq C\frac{r^p}{|B|}\int_0^1\int_{2B}\int_B |\nabla u(\xi)|^p\, d\xi\, d\eta\, dt = C'r^p\int_B |\nabla u(\xi)|^p\, d\xi.$$

□

Now we can complete the proof of Theorem 10. Let $x_0\in\Omega$ be a central point. Let $B_0 = B(x_0, \operatorname{dist}(x_0,\partial\Omega)/4)$. We will prove that there is a constant $M = M(C_J,n) > 0$ such that to every $x\in\Omega$ there is a sequence of balls (chain) $B_i = B(x_i,r_i)\subset\Omega$, $i = 0,1,2,\ldots$ with the properties

(1) $|B_i\cup B_{i+1}|\leq M|B_i\cap B_{i+1}|$, $i=0,1,2,\ldots$
(2) $\operatorname{dist}(x,B_i)\leq Mr_i$, $r_i\to 0$, $x_i\to x$ as $i\to\infty$.
(3) No point of Ω belongs to more than M balls B_i.

To prove it, assume first that x is far enough from x_0, say $x\in\Omega\setminus 2B_0$. Let γ be a John curve that joins x with x_0. We construct a chain of balls as follows.

All balls in the chain are centered on γ. The ball B_0 is already defined. Assume that balls $B_0,\ldots,B_i$ are defined. Starting from the center x_i of B_i we trace along γ towards x until we leave B_i for the last time. Denote by x_{i+1} the point on γ when it happens and define $B_{i+1} = B(x_{i+1}, |x-x_{i+1}|/4C_J)$.

It easily follows that $B_i\subset\Omega$. Property 1 and the inequality $\operatorname{dist}(x,B_i)\leq Cr_i$ in 2 follow from the fact that consecutive balls have comparable radii and that the radii are comparable to the distances of centers of the balls to x.

To prove 3 suppose that $y\in B_{i_1}\cap\ldots\cap B_{i_k}$. Observe that the radii of the balls B_{i_j}, $j=1,2,\ldots,k$ are comparable to $|x-y|$. It follows from the construction that if $m_1<m_2$, then the center of B_{m_2} does not belong to B_{m_1}. This results in the fact that distances between centers of the balls B_{i_j} are comparable to $|x-y|$. The number of the points in $\mathbb{R}^n$ with pairwise comparable distances is bounded i.e. if $z_1,\ldots,z_N\in\mathbb{R}^n$ satisfy $c^{-1}r<\operatorname{dist}(z_i,z_j)<cr$ for $i\neq j$, then $N\leq C(c,n)$. Hence k is bounded by a constant depending on n and C_J, so 3 follows.

Now 3 easily implies that $r_i\to 0$ and hence $x_i\to x$ as $i\to\infty$, which completes the proof of 2.

The case $x\in 2B_0$ is easy and we leave it to the reader.

Since $u_{B_i} = \fint_{B_i} u \to u(x)$ as $i \to \infty$, we obtain

$$\begin{aligned}
|u(x) - u_{B_0}| &\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
&\leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_i \cap B_{i+1}}| + |u_{B_{i+1}} - u_{B_i \cap B_{i+1}}| \\
&\leq \sum_{i=0}^{\infty} \frac{|B_i|}{|B_i \cap B_{i+1}|} \fint_{B_i} |u - u_{B_i}| + \frac{|B_{i+1}|}{|B_i \cap B_{i+1}|} \fint_{B_{i+1}} |u - u_{B_{i+1}}| \\
&\quad \text{(property 1)} \\
&\leq C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}| \\
&\quad \text{(Lemma 12)} \\
&\leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(z)|}{r_i^{n-1}}\, dz.
\end{aligned}$$

Observe that 2 implies $|x - z| \leq C r_i$ for $z \in B_i$ and hence $1/r_i^{n-1} \leq C/|x - z|^{n-1}$. Thus

$$|u(x) - u_{B_0}| \leq C \sum_{i=0}^{\infty} \int_{B_i} \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz \leq C \int_{\Omega} \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz. \tag{14}$$

The last inequality follows from 3. Now it is easy to complete the proof. Since

$$|u(x) - u_{\Omega}| \leq |u(x) - u_{B_0}| + |u_{B_0} - u_{\Omega}|, \tag{15}$$

it remains to estimate $|u_{B_0} - u_{\Omega}|$. We have

(16)

$$|u_{B_0} - u_{\Omega}| \leq \fint_{\Omega} |u - u_{B_0}| \leq C \int_{\Omega} \fint_{\Omega} \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dx\, dz \leq C |\Omega|^{-(n-1)/n} \int_{\Omega} |\nabla u(z)|\, dz.$$

In the last inequality we employed inequality (13). By the John condition we have

$$C|\Omega|^{1/n} \geq \operatorname{dist}(x_0, \partial\Omega) \geq C_J^{-1} |x - x_0|.$$

Taking the supremum over $x \in \Omega$ yields

$$\operatorname{diam} \Omega \leq C(n, C_J) |\Omega|^{1/n},$$

and hence

$$|\Omega|^{-(n-1)/n} \leq \frac{C}{|x - z|^{n-1}},$$

for all $z \in \Omega$. This and (16) give

$$|u_{B_0} - u_{\Omega}| \leq C \int_{\Omega} \frac{|\nabla u(z)|}{|x - z|^{n-1}}\, dz. \tag{17}$$

Now the theorem follows from estimates (15), (14) and (17). The proof is complete. □

Proof of Theorem 8. We have already proved the theorem for the case $p = 1$ and we are left with the case $1 < p < n$. Let $b \in \mathbb{R}$ be as in (11) with μ being the Lebesgue measure now. Denote the corresponding subsets of Ω by Ω_+ and Ω_- respectively. According to Corollary 6 we know that

$$\left(\int_\Omega |u-b|^{n/(n-1)}\,dx\right)^{(n-1)/n} \le C\int_\Omega |\nabla u|\,dx \tag{18}$$

and we want to prove that

$$\left(\int_\Omega |u-b|^{p^*}\,dx\right)^{1/p^*} \le C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \tag{19}$$

for all $1 < p < n$. Let $\alpha = p(n-1)/(n-p)$ and set

$$v = \begin{cases} |u-b|^\alpha & \text{on } \Omega_+ \\ -|u-b|^\alpha & \text{on } \Omega_-. \end{cases}$$

Since $\alpha > 1$ we conclude that v is locally Lipschitz. The exponent α was chosen in such a way that

$$|v|^{n/(n-1)} = |u-b|^{p^*}.$$

Since

$$|\{v \ge 0\}| \ge |\Omega|/2 \quad \text{and} \quad |\{v \le 0\}| \ge |\Omega|/2,$$

applying Corollary 6 to v yields

$$\begin{aligned} \left(\int_\Omega |u-b|^{p^*}\right)^{(n-1)/n} &= \left(\int_\Omega |v|^{n/(n-1)}\right)^{(n-1)/n} \\ &\le C\int_\Omega |\nabla v| \\ &= C\int_\Omega \frac{p(n-1)}{n-p}|u-b|^{n(p-1)/(n-p)}|\nabla u| \\ &\le C'\left(\int_\Omega |u-b|^{p^*}\right)^{(p-1)/p}\left(\int_\Omega |\nabla u|^p\right)^{1/p}, \end{aligned}$$

where we employed Hölder's inequality in the last step. The above estimates readily yield (19). The proof is complete. $\square$

The truncation argument easily applies to Sobolev inequalities with first order derivatives. For higher order derivatives we still have estimates by Riesz potentials[7] and hence in various situations we can obtain weak type estimates. However the truncation argument does not apply now, mainly because, in general, the function $u_{t_1}^{t_2}$ is not k-times differentiable when $k \ge 2$, even for $u \in C_0^\infty$. Thus I do not know if there are counterparts of Theorem 2, Corollary 3 and Theorem 4 for higher order derivatives.

Question 1. *Let μ, ν and $\Omega \subset \mathbb{R}^n$ be as in Theorem 2 and let $k \ge 2$ be an integer. Is it true that for any $0 < p \le q < \infty$ the following two conditions are equivalent?*

[7]Namely $|u(x)| \le C\int_{\mathbb{R}^n} |\nabla^k u(z)||x-z|^{k-n}\,dz$ for $u \in C_0^\infty$, see [37], [53].

(1) *For every* $u \in C_0^\infty(\Omega)$

$$\mu(\{|u| > t\})t^q \leq C \left(\int_\Omega |\nabla^k u|^p \, d\nu \right)^{q/p}.$$

(2) *For every* $u \in C_0^\infty(\Omega)$

$$\left(\int_\Omega |u|^q \, d\mu \right)^{1/q} \leq C \left(\int_\Omega |\nabla^k u|^p \, d\nu \right)^{1/p}.$$

If not, then how can we characterize those μ, ν, k, p *and* q *for which the conditions 1 and 2 are equivalent?*

4. The Strauss inequality. To a vector function $u = (u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}^n$, we associate the deformation tensor ε defined as the symmetric part of the gradient of u, i.e., $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, or in terms of components,

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The well known Korn inequality states that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C \|\varepsilon(u)\|_{L^p(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n), \tag{20}$$

provided $1 < p < \infty$, see [3], [16], [38], [43, Theorem 12.20], [46] and references therein.

This and the Sobolev inequality (1) imply that for all $1 < p < n$

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\varepsilon(u)\|_{L^p(\mathbb{R}^n)}, \qquad u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n). \tag{21}$$

Ornstein, [41], proved that inequality (20) fails for $p = 1$. However Strauss [50] proved a surprising fact that inequality (21) holds even for $p = 1$.

Theorem 13 (Strauss)**.** *Inequality*

$$\left(\int_{\mathbb{R}^n} |u|^{n/(n-1)} \, dx \right)^{(n-1)/n} \leq C \int_{\mathbb{R}^n} |\varepsilon(u)| \, dx$$

holds for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

Strauss' proof was a clever modification of Nirenberg's proof, [40], of (1) for $p = 1$.

We will not prove this theorem here but instead we will show some arguments which could perhaps lead to the proof. However, there are some missing steps. These missing steps lead us to some open problems that we will state later.

Let $u = (u_1, u_2, \ldots, u_n) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. For $k = 1, 2, \ldots, n$, we have the well known integral formula, see [37, Theorem 1.1.10/2],

$$u_k = \frac{2}{n\omega_n} \sum_{1 \leq i \leq j \leq n} \frac{\partial^2 u_k}{\partial x_i \partial x_j} * K_{ij}, \tag{22}$$

where $K_{ij}(x) = x_i x_j / |x|^n$ and ω_n denotes the volume of the unit ball. Note that

$$\frac{\partial \varepsilon_{jk}}{\partial x_i} - \frac{\partial \varepsilon_{ij}}{\partial x_k} + \frac{\partial \varepsilon_{ki}}{\partial x_j} = \frac{\partial^2 u_k}{\partial x_i \partial x_j}.$$

Placing this identity in (22) and integrating by parts we obtain

$$u_k = \frac{2}{n\omega_n} \sum_{1\le i\le j\le n} \left(\varepsilon_{jk} * \frac{\partial K_{ij}}{\partial x_i} - \varepsilon_{ij} * \frac{\partial K_{ij}}{\partial x_k} + \varepsilon_{ki} * \frac{\partial K_{ij}}{\partial x_j} \right). \tag{23}$$

Thus we obtained an explicit integral formula to represent u in terms of $\{\varepsilon_{ij}\}$.

Note that $|\partial K_{ij}/\partial x_l| \le C|x|^{1-n}$, so the Fractional Integration Theorem implies inequality (21) for $1 < p < n$. In the case $p = 1$ Lemma 11 gives only a weak type estimate

$$\sup_{t>0} |\{|u| > t\}| t^{n/(n-1)} \le C \left(\int_{\mathbb{R}^n} |\varepsilon(u)| \right)^{n/(n-1)} . \tag{24}$$

We would like to apply the truncation method of Maz'ya to conclude the proof of Theorem 13 from (24). Unfortunately there is no clear way how to do it. This is due to the fact that we do not deal with a single function but with a vector valued mapping.

Question 2. *How can one modify the argument based on the truncation method in order to deduce Theorem 13 from the estimate (24)?*

Question 3. *What is the best constant in Theorem 13?*

Question 4. *Is there any "isoperimetric" interpretation of Theorem 13?*

Estimate (24) is a special case of a much more general class of inequalities in which we replace $\varepsilon(u)$ by other first order differential operators.

Let $P_j = (P_{j1}, \ldots, P_{jM})$, $j = 1, \ldots, N$ be linear homogeneous partial differential operators of order 1, with constant coefficients, acting on vector functions

$$u = (u_1, \ldots, u_M) \quad \text{and} \quad P_j u = \sum_{k=1}^{M} P_{jk} u_k.$$

Homogeneity of order 1 means $P_{jk} = \sum_{i=1}^n a_i^{jk} \partial/\partial x_i$. By $p_{jk}(\xi)$ we will denote the characteristic polynomial of P_{jk}.

The following result is due to Smith [46, Theorem 4.1].

Theorem 14 (Smith)**.** *If for every $\xi \in \mathbb{C}^n \setminus \{0\}$, the matrix $\{p_{jk}(\xi)\}$ has rank M, then there exist $K_{ij} \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $K_{ij}(x) = |x|^{1-n} K_{ij}(x/|x|)$ when $x \ne 0$, such that for $u = (u_1, \ldots, u_M) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^M)$ we have*

$$u_i = \sum_{j=1}^{N} K_{ij} * P_j u.$$

Note that formula (23) is a particular case of the Smith theorem. Indeed

$$\varepsilon_{ij} = \sum_{k=1}^{n} P_{(ij),k} u_k,$$

where

$$P_{(ij),k} = \frac{1}{2}\left(\delta_{ki}\frac{\partial}{\partial x_j} + \delta_{kj}\frac{\partial}{\partial x_i}\right)$$

and $\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ otherwise. Thus $M = n$, $N = n^2$ and ε_{ij} plays the role of P_j. It is easy to check that the rank of a suitable matrix is n.

Now Smith's theorem and the Fractional Integration Theorem lead to

Corollary 15. *Under the assumptions of Theorem 14 for every* $1 < p < n$ *the inequality*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\sum_{j=1}^{N}\|P_j u\|_{L^p(\mathbb{R}^n)} \tag{25}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^M)$.

For the case $p = 1$ Lemma 11 yields only a weak type estimate

$$\sup_{t>0}|\{|u| > t\}|t^{n/(n-1)} \leq C\left(\sum_{j=1}^{N}\int_\Omega |P_j u|\, dx\right)^{n/(n-1)}.$$

Now it is very natural to ask if there is a counterpart of Strauss' theorem in the current setting i.e. if the above weak type estimate can be replaced by the strong one.

Question 5. *Does inequality (25) hold with* $p = 1$*?*

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