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On Lipschitz Continuity of Quasiconformal Mappings

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Abstract

We study the local growth of quasiconformal mappings in the plane. Estimates are given in terms of integral means of the pointwise angular dilatations. New sufficient conditions for a quasiconformal mapping f to be either Lipschitz or weakly Lipschitz continuous at a point are given.

1. Introduction

Let Ω be a domain in the complex plane $\mathbb{C}$. It is a classical result that every K-quasiconformal mapping $f : \Omega \to \mathbb{C}$ is locally Hölder continuous with exponent $1/K$, which is the best possible, see [1], namely,

$$|f(z) - f(z_0)| \le C|z - z_0|^{1/K}. \tag{1.1}$$

If one allows the exponent in (1.1) to vary with respect to the point z_0, one could get more precise information about the local behavior of f at z_0. For instance, Belinskiĭ [7, p. 37] showed that if a Beltrami coefficient μ is continuous, then the corresponding quasiconformal mapping f will be weakly Lipschitz continuous, that is, locally Hölder continuous with every exponent less than one. Iwaniec [12] extended this result to the case when μ has vanishing mean oscillation in the sense of Sarason [18]. Moreover, the weak Lipschitz continuity remains to hold for the higher-dimensional quasiconformal mappings if either the dilatation tensors or the matrix dilatations have vanishing mean oscillation [11].

In this paper we study the local behavior of plane quasiconformal mappings by employing the angular and the radial dilatations. One of our main results here is a local growth estimate for a K-quasiconformal mapping, which implies an estimate for the modulus of continuity in terms of the integral means of the angular dilatations. We will observe that the modulus of continuity of f in the form $O(|z - z_0| \log(1/|z - z_0|))$ at a given point z_0 does not necessarily follow from the continuity of μ, which implies $\mu \in \mathrm{VMO}$, for the corresponding quasiconformal mapping f (see Section 4). This observation answers negatively

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to a problem on the modulus of continuity, which was discussed during the XVIII-th Rolf Nevanlinna Colloquium held in Helsinki, 2000. Moreover, it is shown that the weak Lipschitz continuity of f at z_0 is the best possible under the assumption of the approximate continuity of μ (see Theorem 4.12).

Some of our estimates in the following are independent of the maximal dilatations K of the quasiconformal mappings. Then, in most cases, by a standard approximation procedure, we can extend the results for those homeomorphisms in the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega)$ for which the relevant quantities are bounded. For simplicity, we will not pursue this direction here.

We recall some basic definitions and notation. A sense-preserving topological embedding $f : \Omega \to \mathbb{C}$ is called quasiconformal if $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and if f satisfies the Beltrami equation

$$f_{\bar z} = \mu(z)\, f_z \quad \text{a.e. in } \Omega$$

for a complex-valued measurable function μ with $\|\mu\|_\infty < 1$, which is called the complex dilatation of f or the Beltrami coefficient of f. More specifically, f is called K-quasiconformal if $\|K_\mu\|_\infty \le K$, where $K_\mu(z) = (1+|\mu(z)|)/(1-|\mu(z)|)$.

For a given μ in Ω, we define the *angular dilatation* D_{μ,z_0} of μ at $z_0 \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$D_{\mu,z_0}(z) = \frac{\left|1 - \mu(z)\frac{\bar z - \bar z_0}{z - z_0}\right|^2}{1 - |\mu(z)|^2}$$

in the case when z_0 is finite, and by $D_{\mu,\infty}(z) = D_{\mu,0}(z)$ in the case when $z_0 = \infty$. For brevity, we will write $D_{\mu,0} = D_\mu$. The angular dilatation is a measurable function in Ω and satisfies the inequality

$$1/K_\mu(z) \le D_{\mu,z_0}(z) \le K_\mu(z) \tag{1.2}$$

a.e. in Ω for each $z_0 \in \widehat{\mathbb{C}}$. The name of D_{μ,z_0} comes from the following important relation: if f is a quasiconformal mapping in Ω with complex dilatation μ and if we write $z = z_0 + re^{i\theta}$, then

$$\left|\frac{\partial f}{\partial \theta}(z)\right|^2 = r^2 D_{\mu,z_0}(z) J_f(z) \tag{1.3}$$

holds for almost all $z \in \Omega$, where J_f stands for the Jacobian of f. The *radial dilatation* of μ at z_0 is defined as $D_{-\mu,z_0}(z)$ and its name comes from the relation

$$\left|\frac{\partial f}{\partial r}(z)\right|^2 = D_{-\mu,z_0}(z) J_f(z) \quad \text{a.e. in } \Omega. \tag{1.4}$$

The idea of employing the directional dilatations for the study of quasiconformal mappings in the plane is due to Andreian Cazacu [3] and Reich and Walczak [17], and similar ideas are used in [19], [8] and [10].

2. Growth estimates

Given a Beltrami coefficient μ in the unit disk $\mathbb{D}$ centered at the origin with $\|\mu\|_\infty \le (K-1)/(K+1)$, we define $Q_\mu(r,R)$ by

$$Q_\mu(r,R) = \frac{1}{2\pi\log(R/r)}\iint_{r<|z|<R}\frac{D_\mu(z)}{|z|^2}dxdy$$

for $0<r<R\le 1$. This quantity can be regarded as the integral average of the angular dilatation $D_\mu(z)$ over the annulus $r<|z|<R$ with respect to the logarithmic measure. In particular, since $K^{-1}\le D_\mu(z)\le K$, we see that $K^{-1}\le Q_\mu(r,R)\le K$ holds.

2.1. **Theorem.** *Let $f:\mathbb{D}\to\mathbb{D}$, $f(0)=0$, be a quasiconformal embedding with complex dilatation μ. Then*

$$|f(z)|\le C\left(\frac{|z|}{R}\right)^{1/Q_\mu(|z|,R)}, \quad |z|<R,$$

for an absolute constant $0<C\le 64$.

In particular, because of (1.2), we immediately obtain

$$|f(z)|\le C\left(\frac{|z|}{R}\right)^{1/P_\mu(|z|,R)} \tag{2.2}$$

for all $|z|<R$. Here

$$P_\mu(r,R) = \frac{1}{2\pi\log(R/r)}\iint_{r<|z|<R}\frac{K_\mu(z)}{|z|^2}dxdy.$$

We remark, in particular, that $\max\{1, Q_\mu(r,R)\}\le P_\mu(r,R)\le K$. Now the aforementioned Hölder continuity estimate

$$|f(z)|\le C|z|^{1/K}, \quad z\in\mathbb{D}, \tag{2.3}$$

is a simple consequence of (2.2).

We do not discuss here the sharp constant C in the case if we allow it to depend on $K=(1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$. However, we make a few remarks here. One has the expectation that if $f:\mathbb{D}\to\mathbb{D}$ is a K-quasiconformal homeomorphism, normalized by $f(0)=0$, then inequality (2.3) holds with $C=16^{1-1/K}$. It is known that 16 is the smallest constant independent of K in this case, see, e.g., [14, p. 68]. Currently it is known that one can choose $C=64^{1-1/K}$. More details are given in [21, p. 33].

The proof of Theorem 2.1 is based on the following statement which may be of independent interest.

2.4. **Proposition.** *Let $A = \{z : r < |z| < R\}$ and $f : A \to \mathbb{C}$ be a quasiconformal embedding with complex dilatation μ. Then*

$$\frac{1}{Q_\mu(r,R)} \le \frac{\operatorname{mod} f(A)}{\operatorname{mod} A} \le Q_{-\mu}(r,R). \tag{2.5}$$

We remark that the above result is equivalent to the following inequalities:

$$\begin{aligned} -\frac{1}{2\pi}\iint_A \frac{D_{-\mu}(z)-1}{|z|^2}dxdy &\le \operatorname{mod} A - \operatorname{mod} f(A) \\ &\le \frac{\operatorname{mod} f(A)}{\operatorname{mod} A}\cdot\frac{1}{2\pi}\iint_A \frac{D_\mu(z)-1}{|z|^2}dxdy. \end{aligned}$$

Proof. For simplicity, let us assume that $r = 1$ and set $m = \operatorname{mod} A = \log R$. We first recall the following estimate due to [17]:

$$\operatorname{mod} f(A) \ge \int_1^R \frac{dr}{r\psi(r)}, \tag{2.6}$$

where

$$\psi(r) = \frac{1}{2\pi}\int_0^{2\pi} D_\mu(re^{i\theta})d\theta.$$

A complete proof of (2.6) under somewhat weaker assumptions is also contained in [10]. Here we just sketch the proof of (2.6) for the reader's convenience. First we may assume that $f(A) = \{1 < |w| < R'\}$. Let γ_r be the circle $|z| = r$. Then

$$2\pi \le \int_{\gamma_r} |d\arg f| \le \int_0^{2\pi} \left|\frac{f_\theta(re^{i\theta})}{f(re^{i\theta})}\right| d\theta$$

for almost all $r \in (1,R)$. Noting (1.3) and making use of Schwarz' inequality, we get

$$\frac{2\pi}{r\psi(r)} \le \int_0^{2\pi} J_f|f|^{-2}rd\theta.$$

If we integrate both sides of the above inequality with respect to the parameter r over $(1,R)$, we obtain the required estimate.

Next, the observation

$$\frac{1}{2\pi}\iint_A \frac{D_\mu(z)}{|z|^2}dxdy = \int_1^R \frac{\psi(r)dr}{r} = \int_0^m \psi(e^t)dt,$$

and the estimate

$$m^2 \le \int_0^m \psi(e^t)dt \cdot \int_0^m \frac{dt}{\psi(e^t)},$$

which is a consequence of Schwarz' inequality, imply

$$\int_0^m \frac{dt}{\psi(e^t)} \ge \frac{m^2}{\frac{1}{2\pi}\iint_A \frac{D_\mu(z)}{|z|^2}dxdy} = \frac{m}{Q_\mu(1,R)}. \tag{2.7}$$

Combining inequalities (2.6) and (2.7), we obtain the right-hand side in (2.5). The left-hand side of inequality (2.5) can be handled similarly by using (1.4), cf. [17]. □

The following auxiliary lemma will play an important role.

2.8. **Lemma** ([10]). *Let B be a ring domain in $\mathbb{C}$ whose complement in $\widehat{\mathbb{C}}$ consists of the bounded component E_0 and the unbounded component E_1. If* $\operatorname{mod} B > C_1$, *we have the estimate*

$$\operatorname{diam} E_0 \le C_2 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} B},$$

where C_1 and C_2 are positive absolute constants with $C_1 \le 5\log 2$ and $C_2 \le 64$.

The statement is proved by a direct application of the well-known Teichmüller modulus theorem (see [14, p. 56]).

Proof of Theorem 2.1. For a given z with $|z| < R$, we set $m = \log(R/|z|)$ and let $A = \{w : |z| < |w| < R\}$. If $\operatorname{mod} f(A) > 5\log 2$, by Lemma 2.8, we obtain

$$|f(z)| \le 64 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} f(A)} \le 64\, e^{-\operatorname{mod} f(A)}. \tag{2.9}$$

Otherwise, we see that

$$|f(z)| < 1 \le e^{5\log 2 - \operatorname{mod} f(A)} = 32\, e^{-\operatorname{mod} f(A)},$$

and hence, (2.9) holds. On the other hand, by Proposition 2.4, we have

$$\operatorname{mod} f(A) \ge \frac{m}{Q_\mu(|z|, R)}. \tag{2.10}$$

Combining (2.9) with (2.10), we arrive at the inequality

$$|f(z)| \le 64 e^{-\log(R/|z|)/Q_\mu(|z|,R)} = 64 \left(\frac{|z|}{R}\right)^{1/Q_\mu(|z|,R)},$$

and thus, we complete the proof. □

Since $D_{\pm\mu}(z) \le K_\mu(z)$ a.e. in $\mathbb{D}$, we deduce from (2.5) an estimate for the change of moduli under quasiconformal mappings in the plane with pointwise dilatation $K_\mu(z)$ established by Belinskiĭ [6] (see also [14, p. 222]).

2.11. **Corollary.** *Under the assumption same as in Proposition* 2.4

$$\frac{1}{P_\mu(r, R)} \le \frac{\operatorname{mod} f(A)}{\operatorname{mod} A} \le P_\mu(r, R). \tag{2.12}$$

If f is K-quasiconformal mapping, then (2.12) yields

$$\frac{\operatorname{mod} A}{K} \le \operatorname{mod} f(A) \le K \operatorname{mod} A$$

and we recognize the classical Grötzsch inequality for annuli (see, e.g., [14, p. 38]).

2.13. **Corollary.** *Under the assumptions of Proposition* 2.4

$$-\frac{1}{2\pi} \iint_A \frac{D_{-\mu}(z) - 1}{|z|^2} dxdy \le \operatorname{mod} A - \operatorname{mod} f(A) \le \frac{1}{2\pi} \iint_A \frac{D_\mu(z) - 1}{|z|^2} dxdy.$$

Proof. Set $m = \operatorname{mod} A$, $m' = \operatorname{mod} f(A)$ and $Q = Q_\mu(r, R)$. We then have to verify the inequality $m - m' \leq mQ - m$, only. The left-hand side of (2.5) implies $m'/m \geq 1/Q$. The trivial inequality $(Q-1)^2 \geq 0$ yields $1/Q \geq 2 - Q$. Hence, one can get $m'/m \geq 2 - Q$, which is nothing but the required inequality. □

In particular, we have the following simple form due to Belinskiĭ [6] (see also [14, p. 222]).

2.14. **Corollary.** *Under the assumptions of Proposition* 2.4 *the following inequality holds:*

$$|\operatorname{mod} A - \operatorname{mod} f(A)| \leq \frac{1}{2\pi} \iint_A \frac{K_\mu(z) - 1}{|z|^2} dxdy.$$

Similar growth estimates for homeomorphic solutions to the degenerate Beltrami equation were recently obtained by Chen [9].

3. Conditions for Lipschitz continuity

We will say that a function f is Lipschitz continuous at z_0 if $|f(z) - f(z_0)| \leq C|z - z_0|$ holds in a neighborhood of z_0 for a constant C. In this section, as applications of the previous results, we give a sufficient condition for a quasiconformal mapping to be Lipschitz continuous at a given point.

Suppose that $f(z)$ is a quasiconformal mapping in $\mathbb{D}$ normalized by $f(0) = 0$. It is known, e.g., [14, p. 232], that the condition

$$\iint_{|z|<R} \frac{K_\mu(z) - 1}{|z|^2} dxdy < +\infty \tag{3.1}$$

for some $R > 0$ implies that f is conformally differentiable at the origin, that is, the limit $\lim_{z\to 0} f(z)/z$ exists and differs from 0 and ∞. The existence of the limit $\lim_{z\to 0} |f(z)|/|z|$ under the hypothesis (3.1) was first proved by Teichmüller [20] and Wittich [22]. The existence of $\lim_{z\to 0} \arg(f(z)/z)$ under the assumption of (3.1) is due to Belinskiĭ (see, e.g., [7, p. 54]) and Lehto [13]. So, this result is referred to as the Teichmüller-Wittich-Belinskiĭ-Lehto regularity theorem. Under the weaker assumption than that of (3.1) we have the following statement.

3.2. **Theorem.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, is a quasiconformal embedding in $\mathbb{D}$ with complex dilatation μ satisfying the assumption*

$$\sup_{0<r<1} \iint_{r<|z|<1} \frac{D_\mu(z) - 1}{|z|^2} dxdy < +\infty. \tag{3.3}$$

Then

$$|f(z)| \leq C|z| \tag{3.4}$$

holds in $\mathbb{D}$ with some constant C depending only on the value of the above supremum, and therefore, the mapping f is Lipschitz continuous at the origin.

3.5. **Remark.** 1. If the improper integral converges to a number including $-\infty$, we can state (3.3) simply by

$$\iint_{|z|<1} \frac{D_\mu(z)-1}{|z|^2}dxdy < +\infty.$$

2. The statement of Theorem 3.2 can also be obtained by a careful observation of the proof of Theorem 3.1 in [17].

For the later convenience, we introduce the quantity

$$T_\mu(r,R) = \frac{1}{2\pi}\iint_{r<|z|<R} \frac{D_\mu(z)-1}{|z|^2}dxdy$$

for $0<r<R$. Note that the obvious relation $T_\mu(r,R) = (Q_\mu(r,R)-1)\log(R/r)$ holds. Then condition (3.3) means that $\limsup_{r\to 0} T_\mu(r,1) < +\infty$. This can also be viewed as a Dini type condition. Precisely, (3.3) holds if and only if

$$\limsup_{r\to 0} \int_r^1 \frac{\omega_\mu(t)dt}{t} < +\infty, \tag{3.6}$$

where

$$\omega_\mu(t) = \frac{1}{\pi t^2}\iint_{|z|<t}(D_\mu(z)-1)dxdy.$$

The quantity $\omega_\mu(t)$ can be considered as the integral average of $D_\mu(z)-1$ over the disk $|z|<t$ with respect to the plane Lebesgue measure. In particular, $1/K \le \omega_\mu(t)+1 \le K$ if $K_\mu(z)\le K$ a.e. in $\mathbb{D}$. As above, (3.6) can be expressed by $\int_0^1 \omega_\mu(t)t^{-1}dt < +\infty$ in the case when the improper integral is convergent. Since the relation

$$T_\mu(r,R) = \frac{\omega_\mu(R)-\omega_\mu(r)}{2} + \int_r^R \frac{\omega_\mu(t)dt}{t} \tag{3.7}$$

holds, we see that Dini type condition (3.6) is equivalent to (3.3). The formula in (3.7) can be proved in a more general setting.

3.8. **Lemma.** *For a given Lebesgue integrable function φ in the disk $|z|<R$, the function η_φ defined by*

$$\eta_\varphi(t) = \frac{1}{\pi t^2}\iint_{|z|<t}\varphi(z)dxdy, \quad 0<t\le R,$$

satisfies the relation

$$\frac{1}{2\pi}\iint_{r<|z|<R}\frac{\varphi(z)}{|z|^2}dxdy = \frac{\eta_\varphi(R)-\eta_\varphi(r)}{2} + \int_r^R \frac{\eta_\varphi(t)dt}{t}.$$

Proof. Set $\alpha(t) = \pi t^2 \eta_\varphi(t)$. This function can be written in the form $\alpha(t) = \int_0^t \psi(r)dr$, where

$$\psi(r) = r\int_0^{2\pi}\varphi(re^{i\theta})d\theta.$$

Note that $\psi \in L^1([0,R])$ by Fubini's theorem. In particular, the function $\alpha(t)$ is absolutely continuous on $[0,R]$ and $\alpha'(t) = \psi(t)$ for a.a. t. We now compute

$$\iint_{r<|z|<R} \frac{\varphi(z)}{|z|^2} dxdy = \int_r^R \frac{\psi(t)dt}{t^2} = \left[\frac{\alpha(t)}{t^2}\right]_r^R + \int_r^R \frac{2\alpha(t)}{t^3} dt$$

$$= \pi(\eta_\varphi(R) - \eta_\varphi(r)) + 2\pi \int_r^R \frac{\eta_\varphi(t)}{t} dt,$$

and thus the required identity follows. □

Theorem 3.2 can be derived from the following more general statement.

3.9. **Theorem.** *Let $f : \mathbb{D} \to \mathbb{D},\ f(0) = 0$, be a quasiconformal embedding with complex dilatation μ. Then the function f satisfies the inequality*

$$|f(z)| \le \frac{C}{R} |z| e^{T_\mu(|z|,R)}$$

for $|z| < R \le 1$, where $C \le 64$ is an absolute constant.

Proof. Fix z with $0 < |z| < R$ and let $m = \log(R/|z|)$ and $A = \{w : |z| < |w| < R\}$. By Corollary 2.14, we have $\operatorname{mod} f(A) \ge m - T_\mu(|z|,R)$. As in the proof of Theorem 2.1, we now obtain $|f(z)| \le 64\, e^{-m+T_\mu(|z|,R)} = 64\,|z|\, e^{T_\mu(|z|,R)}/R$. □

We now combine (3.7) with the above theorem to obtain the following statement.

3.10. **Corollary.** *Let $f : \mathbb{D} \to \mathbb{D},\ f(0) = 0$, be a K-quasiconformal embedding with complex dilatation μ. Then the function f satisfies the inequality*

$$|f(z)| \le C|z| \exp\left(\int_{|z|}^1 \frac{\omega_\mu(t)dt}{t}\right)$$

for $|z| < 1$, where $C = C(K)$ is a constant depending on K only.

The following result produces examples of quasiconformal mappings which satisfy (3.3) but do not necessarily satisfy (3.1).

3.11. **Proposition.** *Let ρ be an arbitrary non-negative real-valued measurable function in a neighborhood U of the origin with $\|\rho\|_\infty < 1$. Then there exists a quasiconformal embedding $f : U \to \mathbb{C},\ f(0) = 0$, which is Lipschitz continuous at the origin, with complex dilatation μ satisfying*

$$|\mu(z)| = \rho(z) \quad \text{and} \quad D_\mu(z) = 1 \quad \text{a.e. in } U$$

Proof. Given $\rho(z)$ we define a Beltrami coefficient $\mu(z)$ a.e. in U which takes one of the following two values

$$\left(\rho^2(z) \pm i\rho(z)\sqrt{1-\rho^2(z)}\right)\frac{z}{\bar z} \tag{3.12}$$

at each point $z \in U$ in a measurable way. The straightforward computation shows that $|\mu(z)| = \rho(z)$ and $D_\mu(z) = 1$ for almost all $z \in U$. By the measurable Riemann mapping theorem, there exists a quasiconformal mapping $f : U \to \mathbb{C}$, $f(0) = 0$, with complex dilatation μ. Since (3.3) trivially holds, by Theorem 3.2, the mapping f is Lipschitz continuous at the origin. □

We remark that if $\rho(z) = \rho(|z|)$ for $z \in \mathbb{D}$ and $\rho = 0$ for $z \in \mathbb{C} \setminus \mathbb{D}$, then the quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, fixing 0 and 1 with complex dilatation μ of the form (3.12) is explicitly given by the formula

$$f(z) = ze^{i\theta(|z|)}, \quad \theta(r) = \int_r^1 \frac{2\rho(t)}{\sqrt{1-\rho^2(t)}} \frac{dt}{t}$$

and represents a radial rotation. For example, if we set $\rho(z) = \varepsilon/\sqrt{4+\varepsilon^2}$, $\varepsilon \geq 0$, then we obtain the well-known spiral mapping

$$f(z) = ze^{i\varepsilon \log(1/|z|)},$$

transforming the rays $re^{i\theta}$, $0 < r < 1$, into logarithmic spirals which wind about the origin infinitely many times. Note, in particular, that $\lim_{z\to 0} \arg(f(z)/z)$ does not exist, and therefore the mapping f is not conformally differentiable at the origin, nevertheless $|f(z)| = |z|$ holds. This example was observed also by Reich and Walczak [17] in the same context.

Let f be a quasiconformal self-mapping of $\mathbb{C}$. Since f has locally integrable partial derivatives, the image Jordan curve of the circle $|z| = r$ under f, which is usually called a quasicircle, is rectifiable for almost all $r > 0$. However, we cannot say that the specific curve $f(\partial\mathbb{D})$ is rectifiable in general, see e.g., [14, p. 104] and [16, p. 249]. The following statement provides a sufficient condition for the quasicircle to be rectifiable.

3.13. **Theorem.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping with complex dilatation μ and Γ be a rectifiable curve in $\mathbb{C}$. If there are positive constants R and M such that*

$$\iint_{|z-z_0|<R} \frac{D_{\mu,z_0}(z) - 1}{|z-z_0|^2} dxdy \leq M$$

holds for each $z_0 \in \Gamma$, then f is Lipschitz continuous on Γ and therefore, the image curve $f(\Gamma)$ is rectifiable, too.

Proof. Given $z_0 \in \Gamma$, we consider the quasiconformal mapping F defined by $F(z) = f(z + z_0) - f(z_0)$. This mapping preserves the origin and has the complex dilatation $\nu(z) = \mu(z + z_0)$. In order to apply Theorem 3.2, we first calculate the angular dilatation D_ν. By definition, we see that

$$D_\nu(z) = \frac{|1 - \mu(z+z_0)\frac{\bar{z}}{z}|^2}{1 - |\mu(z+z_0)|^2} = D_{\mu,z_0}(z + z_0),$$

and therefore, we have

$$\begin{aligned}\iint_{|z|<R} \frac{D_\nu(z)-1}{|z|^2}dxdy &= \iint_{|z|<R} \frac{D_{\mu,z_0}(z+z_0)-1}{|z|^2}dxdy \\ &= \iint_{|z-z_0|<R} \frac{D_{\mu,z_0}(z)-1}{|z-z_0|^2}dxdy \le M.\end{aligned}$$

By Theorem 3.2, $|F(z)|/|z| = |f(z+z_0)-f(z_0)|/|z|$ is bounded in $|z| \le R$ uniformly with respect to $z_0 \in \Gamma$. Hence, the mapping f is Lipschitz continuous on Γ. □

More general conditions for the rectifiability of the image curve can be deduced from Theorem 3.2 by decomposing f into a suitable quasiconformal mapping and an affine mapping. We will illustrate this procedure in the next section. For related results on the rectifiability of quasicircles, see, for instance, [2], [5], [15], [4] and the references therein.

4. Weak Lipschitz continuity

We will say that a quasiconformal mapping f is weakly Lipschitz continuous at z_0 if for every $0 < \alpha < 1$ there is a constant $C > 0$ such that

$$|f(z) - f(z_0)| \le C|z-z_0|^\alpha \tag{4.1}$$

holds if $|z - z_0| < \delta$, where $\delta > 0$ is a sufficiently small number.

4.2. **Theorem.** *Let $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ satisfying*

$$\limsup_{t\to 0} \frac{1}{\pi t^2} \iint_{|z|<t} (D_\mu(z) - 1)dxdy \le 0. \tag{4.3}$$

Then f is weakly Lipschitz continuous at 0.

Proof. It suffices to show that

$$\log^+ \frac{|f(z)|}{|z|} = o\left(\log \frac{1}{|z|}\right), \quad z \to 0, \tag{4.4}$$

where $\log^+ x = \max\{\log x, 0\}$. Condition (4.3) means that $\limsup_{t\to 0} \omega_\mu(t) \le 0$, and thus,

$$\limsup_{z\to 0} \frac{1}{\log(1/|z|)} \int_{|z|}^1 \frac{\omega_\mu(t)dt}{t} \le 0.$$

Now, from Corollary 3.10, the conclusion follows. □

Note that this result is qualitatively best possible as we will see in Theorem 4.12 below. Corollary 3.10 can be considered to give more precise information about the behavior of $f(z)/z$ at the origin in terms of the local quantity $\omega_\mu(t)$ defined by the complex dilatation.

As might be expected in view of the above proof, if the limit in (4.3) is negative, we may obtain even a stronger consequence.

4.5. **Theorem.** *Let $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, be a quasiconformal embedding with complex dilatation μ. Suppose that*

$$\limsup_{t\to 0} \frac{1}{\pi t^2} \iint_{|z|<t} D_\mu(z)\,dxdy \le \frac{1}{\alpha} \tag{4.6}$$

holds for a constant $\alpha > 0$. Then f is weakly Hölder continuous with exponent α, that is, for each $0 < \alpha' < \alpha$, there is a constant C such that

$$|f(z)| \le C|z|^{\alpha'}, \quad z \in \mathbb{D}.$$

Proof. By assumption, there is a non-negative function $\varepsilon(t)$ with $\lim_{t\to 0} \varepsilon(t) = 0$ such that $\omega_\mu(t) \le \alpha^{-1} - 1 + \varepsilon(t)$ holds for $t > 0$. By (3.7), we have

$$(Q_\mu(r,1) - 1)\log(1/r) = \int_r^1 \frac{\omega_\mu(t)dt}{t} + O(1) \le (\alpha^{-1} - 1)\log(1/r) + o(\log(1/r))$$

as $r \to 0$. This implies that

$$Q_\mu(r,1) \le \frac{1}{\alpha} + o(1)$$

as $r \to 0$. We can now apply Theorem 2.1 to get the conclusion. □

The following statement produces examples of quasiconformal mappings which satisfy (4.3) but do not necessarily satisfy the stronger assumption

$$\lim_{t\to 0} \frac{1}{t^2} \iint_{|z|<t} (K_\mu(z) - 1)\,dxdy = 0.$$

4.7. **Proposition.** *Let $\rho(z)$ be an arbitrary non-negative real-valued measurable function in $\mathbb{D}$ such that $\rho(z) \le (K-1)/(K+1)$ a.e. in $\mathbb{D}$. Then the quasiconformal embedding $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, with complex dilatation*

$$\mu(z) = \rho(z)\frac{z}{\bar{z}}$$

satisfies (4.6) with $\alpha = K$, in particular, f is weakly Hölder continuous with exponent K at the origin.

Proof. Given $\rho(z)$, we see that $D_\mu(z) = 1/K_\mu(z) \le 1/K$ a.e. in $\mathbb{D}$ and thus the conclusion follows from Theorem 4.5. □

In particular, if we set $\rho(z) = (K-1)/(K+1)$, where $K \ge 1$ is a constant, then we see that

$$f(z) = z|z|^{K-1}.$$

Then μ fails to be continuous at 0 and it does not have vanishing mean oscillation, however, f is Lipschitz continuous at the origin.

By using the composition procedure with affine mappings, we now deduce more general statement from the previous theorem.

4.8. **Theorem.** *Let $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, be a K-quasiconformal mapping. Suppose that for some $a \in \mathbb{D}$ the complex dilatation μ of f satisfies*

$$\limsup_{t\to 0} \frac{1}{\pi t^2} \iint_{|w|<t} (D_{\mu_a}(w) - 1)dudv \le 0, \tag{4.9}$$

where

$$\mu_a = \frac{\mu - a}{1 - \bar{a}\mu} \circ L^{-1}$$

and L is the affine mapping defined by $L(z) = z + a\bar{z}$. Then f is weakly Lipschitz continuous at the origin.

Proof. Let us consider the quasiconformal mapping $F(w) = f \circ L^{-1}(w)$ whose complex dilatation agrees with μ_a. By Theorem 4.2, assumption (4.3) implies F to be weakly Lipschitz continuous at the origin, which is equivalent to the condition that

$$\log^+ \frac{|F(w)|}{|w|} = o\left(\log \frac{1}{|w|}\right)$$

as $w \to 0$. Hence

$$\log^+ \frac{|f(z)|}{|L(z)|} = o\left(\log \frac{1}{|L(z)|}\right)$$

as $z \to 0$, which is equivalent to (4.4). Thus, f is weakly Lipschitz continuous at the origin. □

Note that the above number a necessarily satisfies $|a| \le (K-1)/(K+1)$.

4.10. **Corollary.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a quasiconformal embedding whose complex dilatation μ is approximately continuous at the origin. Then f is weakly Lipschitz continuous at the origin.*

Proof. The approximate continuity of μ at the origin means that there exists a point $a \in \mathbb{D}$ such that

$$\frac{1}{\pi t^2} \iint_{|z|<t} |\mu(z) - a| dxdy \to 0$$

as $t \to 0$. The chain of elementary inequalities

$$\begin{aligned} \frac{1}{\pi t^2} \iint_{|w|<t} (D_{\mu_a}(w) - 1)dudv &\le \frac{1}{\pi t^2} \iint_{|w|<t} (K_F(w) - 1)dudv \\ &\le \frac{C}{\pi t^2} \iint_{|z|<t/(1-|a|)} |\mu(z) - a| dxdy \end{aligned}$$

implies (4.9), where C is a constant depending on K only. Thus, the required weak Lipschitz continuity of f at the origin follows from Theorem 4.8. □

4.11. **Remark.** If the complex dilatation is approximately continuous uniformly on a compact set E, then the mapping f is weakly Lipschitz continuous uniformly on E, that is, for each $0 < \alpha < 1$ there exists a constant $C = C(E) > 0$ such that $|f(z_1) - f(z_2)| \le C|z_1 - z_2|^\alpha$ holds whenever $z_1, z_2 \in E$.

The example of the quasiconformal radial stretching

$$f(z) = z\left(\log\frac{1}{|z|}\right)^2$$

defined on a neighborhood of the origin, whose complex dilatation

$$\mu(z) = \frac{1}{1-\log(1/|z|)}\frac{z}{\bar z}$$

is *continuous* at the origin, tells us that the weak Lipschitz continuity cannot be replaced by the stronger estimate $|f(z)| \le C|z|\log(1/|z|)$ in Corollary 4.10. Moreover, we cannot say anymore than (4.1) in the following sense.

4.12. **Theorem.** *Let $h : (0,\delta] \to \mathbb{R}$ be a continuous function such that*

(i) $h(t) = o(\log(1/t))$ *as* $t \to 0$, *and*

(ii) $h(t) \to \infty$ *as* $t \to 0$.

Then there exists a quasiconformal mapping $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$ such that

$$\limsup_{z\to 0}\frac{\log^+(|f(z)|/|z|)}{h(|z|)} > 0,$$

while the complex dilatation μ is continuous in $\mathbb{D}$, vanishes at the origin, and therefore, satisfies (4.3).

Before proving the theorem we will introduce standard radial stretchings. Let $M : (0,\delta) \to [1,+\infty)$ be a bounded measurable function. For convenience, we extend $M(t)$ to be identically 1 for $t \ge \delta$. We consider the two homeomorphisms f_M and g_M of the complex plane onto itself defined by

$$f_M(z) = \frac{z}{|z|}\exp\left(\int_1^{|z|}\frac{dt}{tM(t)}\right) \text{ and } g_M(z) = \frac{z}{|z|}\exp\left(\int_1^{|z|}\frac{M(t)dt}{t}\right). \tag{4.13}$$

Then the complex dilatations of f_M, g_M are given by μ_M, $-\mu_M$, respectively, where

$$\mu_M(z) = -\frac{M(|z|)-1}{M(|z|)+1}\cdot\frac{z}{\bar z}.$$

Therefore, we have $D_{\mu_M}(z) = M(|z|)$, $D_{-\mu_M}(z) = 1/M(|z|)$ and $K_{\mu_M}(z) = K_{-\mu_M}(z) = M(|z|)$ for almost all z.

4.14. **Lemma.** *Let h be as in Theorem* 4.12. *Then there exists a function $g : (0,\delta] \to \mathbb{R}$ with properties* (a), (b), (c) *and either* (d) *or* (d') *below:*

(a) *g is differentiable, strictly decreasing and satisfies* $\lim_{t\to 0} g(t) = +\infty$.

(b) *g' is absolutely continuous on the interval* $[\delta_1,\delta]$ *for each* $0 < \delta_1 < \delta$.

(c) $g''(t) = o(1/t^2)$ *as* $t \to 0$.

(d) $g(t_n) = h(t_n)$ *for some sequence* t_n *in* $(0,\delta]$ *with* $t_n \to 0$.

(d') $g(t) \le h(t) + O(1)$ *as* $t \to 0$.

Note that condition (c) implies that $g'(t) = o(1/t)$ and $g(t) = o(\log 1/t)$ as $t \to 0$.

Proof of Lemma 4.14. Set $H(x) = h(\delta e^{-x}) - h(\delta) + 1$ for $x \geq 0$. Since $h(t) = o(\log 1/t)$, we note that $H(x) = o(x)$ as $x \to +\infty$. For a given $y > 0$, we define $H^*(y) = \max\{x \geq 0 : H(x) \leq y\}$. One can then see that $H^*(y) \to +\infty$ while $y/H^*(y) \to 0$ as $y \to +\infty$. Set $y_0 = \min\{H(x) : x \geq 0\}$ and $x_0 = 0$. Let $y_n,\ n = 1, 2, \dots$, be a monotone sequence of positive numbers such that $y_1 > y_0$ and $y_n \to \infty$ as $n \to \infty$. Also let $x_n = H^*(y_n)$. It is easy to see that $H(x_n) = y_n$ for $n \geq 1$, $H(x) \geq y_n$ for $x \geq x_n$, $x_0 < x_1 < x_2 < \dots$ and $x_n \to +\infty$ as $n \to \infty$. Choosing a suitable sequence, we can make the following two conditions valid:

(1) $x_{n+1} - x_n \to +\infty$ if $n \to \infty$,

(2) $p_n := (y_{n+1} - y_n)/(x_{n+1} - x_n)$ decreases to 0 as $n \to \infty$.

First, we give a function g satisfying (a), (b), (c) and (d). Define the preliminary function G_0 by

$$G_0(x) = y_n + p_n(x - x_n) + \frac{p_{n+1} - p_n}{2(x_{n+1} - x_n)}(x - x_n)^2, \quad x \in [x_n, x_{n+1}), \tag{4.15}$$

for $n = 0, 1, \dots$. Then, $G_0(x_n) = y_n,\ G_0'(x_n) = p_n$ and $G_0'(x_{n+1} - 0) = p_{n+1}$. This function has the gap

$$D_n := G_0(x_{n+1}) - \lim_{x \to x_{n+1}-} G_0(x) = \frac{p_n - p_{n+1}}{2}(x_{n+1} - x_n)$$

at the point x_{n+1}. So, we will fill it up by adding a suitable term to G_0. To this end, we consider an auxiliary function $\varphi : [0, 1] \to [0, 1]$ with the following properties:

(i) $\varphi \in C^2[0, 1]$,

(ii) $\varphi(0) = 0,\ \varphi'(0) = 0, \varphi(1) = 1$, and $\varphi'(1) = 0$, and

(iii) $0 \leq \varphi'(x) \leq C_1$ and $|\varphi''(x)| \leq C_2$ for $x \in [0, 1]$, where C_1 and C_2 are constants.

For example, we can take $\varphi(x) = \sin^2(\pi x/2)$.

We now define the continuous function $G : [0, +\infty) \to \mathbb{R}$ by

$$G(x) = G_0(x) + D_n \varphi\left(\frac{x - x_n}{x_{n+1} - x_n}\right) \quad x \in [x_n, x_{n+1}),$$

for $n = 0, 1, \dots$. Furthermore, G has the locally absolutely continuous derivative

$$G'(x) = p_n \frac{x_{n+1} - x}{x_{n+1} - x_n} + p_{n+1} \frac{x - x_n}{x_{n+1} - x_n} + \frac{p_n - p_{n+1}}{2} \varphi'\left(\frac{x - x_n}{x_{n+1} - x_n}\right)$$

on $[x_n, x_{n+1}]$. Note here that $G'(x) \geq 0$ and $G'(x) \to 0$ as $x \to +\infty$ by (2). From the expression

$$G''(x) = -\frac{p_{n+1} - p_n}{x_{n+1} - x_n}\left[1 - \frac{1}{2}\varphi''\left(\frac{x - x_n}{x_{n+1} - x_n}\right)\right], \quad x \in (x_n, x_{n+1}),$$

we obtain $G''(x) \to 0$ as $x \to +\infty$. We now set $g(t) = G(\log(\delta/t))$ for $0 < t \le \delta$. Then, $g'(t) = -G'(\log(\delta/t))/t < 0$ and $g''(t) = (G''(\log(\delta/t)) - G'(\log(\delta/t)))/t^2$, and thus, properties (a), (b) and (c) follow. Let $t_n = \delta e^{-x_n}$. Then $g(t_n) = G(x_n) = H(x_n) = h(t_n)$, and (d) has been shown.

Next, we construct a function g satisfying (a), (b), (c) and (d'). First, we change the definition of p_n by setting $p_n = (y_n - y_{n-1})/(x_{n+1} - x_n)$, and then, in the definition of G_0 above, we replace y_n by y_{n-1}, where we set $y_{-1} = y_0$, and define g in the exactly same way as above. Properties (a), (b) and (c) now follow similarly. Also, in this case, we see that $G_0(x) \le y_n \le H(x)$ for $x \in [x_n, x_{n+1})$. If D_n is bounded, we obtain the estimate $G(x) = G_0(x) + O(1) \le H(x) + O(1)$, and hence, we confirm property (d'). Otherwise, taking a positive constant C, we set $E_n = \max\{D_n - C, 0\}$ for each n. Note that $E_n > 0$ for infinitely many n, in this case. Then, we modify the above definition of G by

$$G(x) = G_0(x) + \min\{D_n, C\}\,\varphi\left(\frac{x - x_n}{x_{n+1} - x_n}\right) - \sum_{k=0}^{n-1} E_k, \quad x \in [x_n, x_{n+1}),$$

for $n = 0, 1, \dots$ and set $g(t) = G(\log(\delta/t))$. Then, by definition, $G(x) \le G_0(x) + O(1)$ as $x \to +\infty$. Since $G(x_{n+1}) - G(x_n) \ge C$ if $E_n > 0$, we can see that $G(x) \to +\infty$ as $x \to +\infty$. Now properties (a), (b), (c) and (d') are easy to see for this function g. □

Proof of Theorem 4.12. Let g be a function satisfying properties (a), (b), (c) and (d) in Lemma 4.14. Since $\varepsilon(t) = -tg'(t)$ is non-negative and converges to zero as $t \to 0$, we can choose a small $0 < \delta_1 \le \delta$ so that $M(t) = 1/(1 - \varepsilon(t))$ is bounded in $0 < t < \delta_1$. Now we extend M by setting $M(t) = M(\delta_1)$ for $t \ge \delta_1$. Then the radial stretching f_M has continuous complex dilatation μ_M. We also note that f_M has the form $f_M(z) = Cz\exp(g(|z|))$ for $|z| < \delta_1$, where C is a constant. Let t_n be the sequence in property (d). Then

$$\limsup_{z \to 0} \frac{\log^+(|f(z)|/|z|)}{h(|z|)} \ge \limsup_{n \to \infty} \frac{g(t_n) + \log C}{h(t_n)} = 1 > 0.$$

Thus, the assertions of the theorem have been proved. □

Now we show that (3.1) is the qualitatively best possible condition for a function f to be Lipschitz continuous at the origin. The same is true for (3.3). Recall that (3.3) is equivalent to Dini type condition (3.6).

4.16. **Theorem.** *Let $h : (0, \delta] \to \mathbb{R}$ be a continuous function satisfying $h(t) \to +\infty$ if $t \to 0$. Then there exist a quasiconformal radial stretching $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, with complex dilatation μ and a constant C such that $D_\mu = K_\mu$ and that*

$$\int_r^1 \frac{\omega_\mu(t)dt}{t} \le h(r) + C$$

for $r > 0$ while f fails to be Lipschitz continuous at the origin.

Proof. The proof is somewhat similar to that of the previous theorem. First, we can assume that $h(t) = o(\log(1/t))$ as $t \to 0$. Let g be a function satisfying (a), (b), (c) and (d') in Lemma 4.14. If we set $\varepsilon(t) = -tg'(t)$, by (b) we then have $\varepsilon(t) = o(1)$ and $t\varepsilon'(t) = -tg'(t) - t^2g''(t) = o(1)$ as $t \to 0$. In particular, we can choose a sufficiently small $\delta_1 > 0$ so that $\varepsilon(t) < 1/2$ and $|t\varepsilon'(t)| < 1$ for $t \in (0, \delta_1)$. We now define $M(t)$ by $M(t) = 1 + \varepsilon(t) + t\varepsilon'(t)/2$ for $t \in (0, \delta_1)$ and by $M(t) = 1$ for $t \geq \delta_1$. Then M is a bounded measurable function with $1 \leq M \leq 2$. Let us consider the function f_M given by (4.13), which has complex dilatation $\mu = \mu_M$. We compute

$$\omega_\mu(t) = \frac{2}{t^2}\int_0^t (M(s)-1)s\,ds = \frac{1}{t^2}\int_0^t (s^2\varepsilon(s))'ds = \varepsilon(t)$$

for $0 < t < \delta_1$, and by (d') we obtain

$$\begin{aligned}\int_r^1 \frac{\omega_\mu(t)dt}{t} &= \int_r^{\delta_1} \frac{\varepsilon(t)dt}{t} + O(1) = -\int_r^{\delta_1} g'(t)dt + O(1)\\ &= g(r) + O(1) \leq h(r) + O(1)\end{aligned}$$

as $r \to 0$. On the other hand, since $1/M(t) \leq 1 - (M(t)-1)/2$, we see that

$$\begin{aligned}\int_r^1 \frac{dt}{tM(t)} &\leq -\log r - \frac{1}{2}\int_r^{\delta_1} \frac{\varepsilon(t)dt}{t} - \frac{1}{4}\int_r^{\delta_1} \varepsilon'(t)dt\\ &= -\log r + \frac{g(\delta_1)-g(r)}{2} - \frac{\varepsilon(\delta_1)-\varepsilon(r)}{4}\end{aligned}$$

for sufficiently small $r > 0$, and therefore, $\log|f_M(z)/z| \geq g(|z|)/2 + O(1) \to \infty$ as $z \to 0$. □

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