Papers on Analysis:
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REFLECTIONS ON REFLECTIONS IN QUASIDISKS

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1. Reflections and Jordan domains

Ahlfors studied in [1] quasiconformal reflections in a Jordan curve C through ∞. In this note we first characterize the plane domains D that admit homeomorphic reflections f in their boundaries ∂D. We consider next what more we can say about the domain D if we know more about the restriction of the reflection f to D.

Definition 1.1. A domain $D \subset \overline{\mathbf{R}}^2$ admits a *reflection in its boundary* ∂D if there exists a homeomorphism f of $\overline{D}$ such that

$1°$ $f(D) = D^*$ where $D^* = \overline{\mathbf{R}}^2 \setminus \overline{D}$,
$2°$ $f(z) = z$ for $z \in \partial D$.

The reflection f is *quasiconformal* or *bilipschitz* if the restriction of f to D is quasiconformal or bilipschitz, respectively.

The following result characterizes the plane domains which admit a reflection.

Theorem 1.2. *A domain $D \subset \overline{\mathbf{R}}^2$ admits a reflection in ∂D if and only if it is a Jordan domain.*

Proof. By performing a preliminary self homeomorphism of $\overline{\mathbf{R}}^2$ we may assume without loss of generality that D is bounded.

If D is a Jordan domain, then there exists a self homeomorphism g of $\overline{\mathbf{R}}^2$ which maps the unit disk B onto D. If r denotes reflection in ∂B, then

$$f(z) = g \circ r \circ g^{-1}(z)$$

defines a self homeomorphism of $\overline{\mathbf{R}}^2$ which satisfies $1°$ and $2°$.

For the converse suppose that D admits a reflection f in ∂D. Then

$$D^* = f(D) \qquad \text{and} \qquad \partial D^* = f(\partial D) = \partial D. \tag{1.3}$$

Thus D is simply connected and ∂D connected. We shall prove that ∂D is locally connected at each point $z_0 \in \partial D$.

Fix $z_0 \in \partial D$ and $\epsilon > 0$, choose $\delta > 0$ so that $f(\mathbf{B}(z_0, \delta)) \subset \mathbf{B}(z_0, \epsilon)$ and suppose that

$$z_1, z_2 \in \partial D \cap \mathbf{B}(z_0, \delta).$$

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We shall show that z_1 and z_2 lie in a connected set in $\partial D \cap \mathbf{B}(z_0, \epsilon)$ and hence that ∂D is locally connected at z_0.

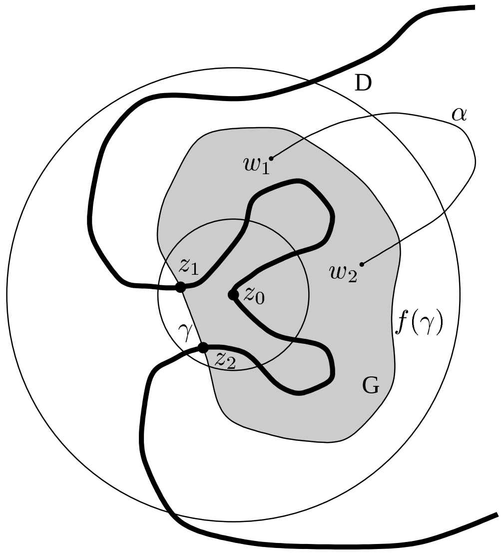

Figure 1

Let γ be the open segment (z_1, z_2) and suppose that $\gamma \cap \partial D = \emptyset$. Then

$$\gamma \cup f(\gamma) \cup \{z_1, z_2\}$$

is a Jordan curve which bounds a domain $G \subset \mathbf{B}(z_0, \epsilon)$. If w_1, w_2 are points in $\overline{G} \cap D$, then w_1 and w_2 can be joined by an arc α in D and hence by an arc

$$\beta \subset \alpha \cup \partial G \subset \overline{G} \cap D.$$

Thus $\overline{G} \cap D$ is connected. The same is true of $\overline{G} \cap D^*$ and we conclude that

$$F_1 = \overline{G} \cap \overline{D}, \quad F_2 = \overline{G} \cap \overline{D^*}$$

are closed connected sets with

$$F_1 \cap F_2 = \overline{G} \cap \partial D, \qquad F_1 \cup F_2 = \overline{G}.$$

Since $\overline{\mathbf{R}}^2 \setminus (F_1 \cup F_2)$ is a Jordan domain and hence connected, $E = F_1 \cap F_2$ is a connected set which joins z_1 and z_2 in $\partial D \cap \mathbf{B}(z_0, \epsilon)$. See, for example, Theorem V.11.5 in [13].

Next if $\gamma \cap \partial D \neq \emptyset$, then

$$\gamma = (\cup_j \gamma_j) \cup (\gamma \cap \partial D),$$

where γ_j is an open segment $(z_{1,j}, z_{2,j})$ with $z_{1,j}, z_{2,j} \in \partial D$ and $\gamma_j \cap \partial D = \emptyset$. Then as above there exists a connected set E_j which joins $z_{1,j}$ and $z_{2,j}$ in $\partial D \cap \mathbf{B}(z_0, \epsilon)$ and

$$E = (\cup_j E_j) \cup (\gamma \cap \partial D)$$

is a connected set which joins z_1 and z_2 in $\partial D \cap \mathbf{B}(z_0, \epsilon)$. Thus ∂D is locally connected at z_0.

Finally the fact that ∂D is locally connected at each point and is the common boundary of the domains D and D^* implies that ∂D is a Jordan curve. See, for example, Theorem IV.6.6 in [14]. □

2. Quasiconformal reflections

We see from Theorem 1.2 that a domain $D \subset \overline{\mathbf{R}}^2$ is a Jordan domain if and only if it admits a reflection f in its boundary. What else can we say about D if we know more about the reflection f? When, for example, can we conclude that D is a quasidisk?

One natural situation to consider is the case where f is quasiconformal.

Theorem 2.1. *A domain $D \subset \overline{\mathbf{R}}^2$ is a K-quasidisk if and only if it admits a K^2-quasiconformal reflection in ∂D.*

Proof. If D is a K-quasidisk, then there exists a K-quasiconformal self mapping g of $\overline{\mathbf{R}}^2$ which maps the upper half plane $\mathbf{H}$ onto D and

$$f(z) = g \circ r \circ g^{-1}(z), \qquad r(z) = \overline{z},$$

defines a K^2-quasiconformal reflection in ∂D.

If D admits a K^2-quasiconformal reflection f in ∂D, then D is a Jordan domain by Theorem 1.2. Hence there exists a homeomorphism h which maps $\overline{D}$ onto $\overline{\mathbf{H}}$ and is conformal in D. In this case

$$g(z) = \begin{cases} h(z) & \text{if } z \in \overline{D}, \\ r \circ h \circ f^{-1}(z) & \text{if } z \in D^* \end{cases}$$

defines a K^2-quasiconformal self mapping of $\overline{\mathbf{R}}^2$ with $g(D) = \mathbf{H}$ and D is a K^2-quasidisk.

An argument based on the general existence theorem for the Beltrami equation shows that the mapping g defined above can be replaced by K-quasiconformal mapping g^*. Thus D is actually a K-quasidisk. See [8]. □

The conformal analogue of this result characterizes the domains D which are disks or half planes: D is a disk or half plane if and only if it admits a K-quasiconformal reflection with $K = 1$.

3. Bilipschitz reflections

Ahlfors also showed in [1] and [2] that if D is a quasidisk, then

1° D admits a hyperbolic bilipschitz reflection in ∂D,

2° D admits a euclidean bilipschitz reflection in ∂D if $\infty \in \partial D$.

Examples show that the boundary of a quasidisk can be an extremely complicated Jordan curve, one which almost has positive two-dimensional measure. See [7]. It is therefore quite surprising that all such curves admit a euclidean bilipschitz reflection when D is unbounded.

We begin with the hyperbolic case.

Definition 3.1. If D and D' are Jordan domains in $\overline{\mathbf{R}}^2$, then $f : D \to D'$ is a *hyperbolic L-bilipschitz mapping* if

$$\frac{1}{L}\, h_D(z_1, z_2) \le h_{D'}(f(z_1), f(z_2)) \le L\, h_D(z_1, z_2) \tag{3.2}$$

for $z_1, z_2 \in D$.

Theorem 3.3. [2] *A domain D is a K-quasidisk if and only if it admits a hyperbolic L-bilipschitz reflection in ∂D where $K = L$ and $L \geq K^2$, respectively.*

Proof. If D admits a hyperbolic L-bilipschitz reflection in ∂D, then for each $z_0 \in D \setminus \{\infty, f^{-1}(\infty)\}$ (3.2) implies that

$$\begin{aligned}\frac{1}{L}\,\frac{\rho_D(z_0)}{\rho_{D^*}(f(z_0))} &\leq \liminf_{|z-z_0|\to 0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\\ &\leq \limsup_{|z-z_0|\to 0} \frac{|f(z)-f(z_0)|}{|z-z_0|} \leq L\,\frac{\rho_D(z_0)}{\rho_{D^*}(f(z_0))},\end{aligned}$$

where $\rho_D(z_0)$ and $\rho_{D^*}(f(z_0))$ denote the hyperbolic densities in D and D^*. Hence

$$H_f(z_0) = \limsup_{r\to 0} \frac{\sup_{|z-z_0|=r} |f(z)-f(z_0)|}{\inf_{|z-z_0|=r} |f(z)-f(z_0)|} \leq L^2.$$

Thus f is L^2-quasiconformal in D, D admits an L^2-quasiconformal reflection in ∂D and D is an L-quasidisk by Theorem 2.1.

If D is a K-quasidisk, there exists a K-quasiconformal self mapping g of $\overline{\mathbf{R}}^2$ that maps D onto the upper half plane $\mathbf{H}$. Then

$$f = g^{-1} \circ r \circ g, \qquad r(z) = \bar{z},$$

defines a natural K^2-quasiconformal reflection in ∂D which maps D onto its exterior D^*. If we apply the sharp distortion theorem for the change of hyperbolic distance under a quasiconformal mapping [11] to f we obtain

$$\tanh(h_{D^*}(f(z_1), f(z_2))) \leq \phi_{K^2}(\tanh(h_D(z_1, z_2))).$$

for $z_1, z_2 \in D$, where

$$\phi_K(t) \leq 4^{(1-1/K)}\, t^{1/K}$$

for $t > 0$. Although the above estimate is sharp, it yields no information concerning the ratio

$$\frac{h_{D^*}(f(z_1), f(z_2))}{h_D(z_1, z_2)}$$

when $h_D(z_1, z_2)$ is small or large.

The proof that a K-quasidisk admits a hyperbolic L-bilipschitz reflection depends, instead, on an important integral formula that yields for each K-quasiconformal mapping $f : \mathbf{H}\to\mathbf{H}$ a hyperbolic L-bilipschitz mapping $g : \mathbf{H}\to\mathbf{H}$ such that $g = f$ on $\partial\mathbf{H}$. See [2] and [5].

Finally we show by means of an example in Section 5 that $L \geq K^2$. □

The conformal analogue of Theorem 3.3 also characterizes the domains which are disks or half planes: D is a disk or half plane if and only if it admits a hyperbolic L-bilipschitz reflection with $L = 1$.

The situation for euclidean bilipschitz reflections is similar to the hyperbolic case considered above.

Definition 3.4. If D and D' are domains in $\mathbf{R}^2$, then $f : D\to D'$ is a *euclidean L-bilipschitz mapping* if

$$\frac{1}{L}\,|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq L\,|z_1 - z_2| \tag{3.5}$$

for $z_1, z_2 \in D$.

We then have the following analogue of Theorem 3.3

Theorem 3.6. [2] *A domain D with $\infty \in \partial D$ is a K-quasidisk if and only if it admits a euclidean L-bilipschitz reflection in ∂D where $K = L$ and*

$$L \geq \csc\left(\frac{\pi}{K^2+1}\right), \tag{3.7}$$

respectively.

Proof. If D admits a euclidean L-bilipschitz reflection f, then f is L^2-quasiconformal as above and D is an L-quasidisk.

The proof that a K-quasidisk admits a euclidean L-bilipschitz reflection depends again on the integral formula mentioned above in the proof for Theorem 3.3. Finally the above lower bound for L follows from an example in Section 5. □

4. Sector domains

We study here an example which yields lower bounds for L in Theorems 3.3 and 3.6 and hence for the ratios

$$\frac{h_{D^*}(f(z_1), f(z_2))}{h_D(z_1, z_2)} \quad \text{and} \quad \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}$$

for $z_1, z_2 \in D$, where D is a K-quasidisk and f a reflection in ∂D.

Example 4.1. For $0 < \alpha < 2\pi$ let $\mathbf{S}(\alpha)$ denote the angular sector

$$\mathbf{S}(\alpha) = \left\{ z = r\, e^{i\,\theta} : 0 < r < \infty,\ |\theta| < \frac{\alpha}{2} \right\}.$$

Then $\mathbf{S}(\alpha)$ is a K-quasidisk where

$$K = \max\left(\sqrt{\frac{2\pi - \alpha}{\alpha}}, \sqrt{\frac{\alpha}{2\pi - \alpha}} \right). \tag{4.2}$$

The bound in (4.2) is sharp.

To prove this let

$$f(r\, e^{i\,\theta}) = r^p\, e^{i\,\phi(\theta)}$$

for $0 < r < \infty$ and $|\theta| \leq \pi$, where

$$p = \frac{\pi}{\sqrt{(2\pi - \alpha)\,\alpha}}$$

and

$$\phi(\theta) = \begin{cases} \dfrac{\pi\,\theta}{\alpha} & \text{if} \quad 0 \leq \theta \leq \dfrac{\alpha}{2}, \\ \pi - \dfrac{\pi\,(\pi - \theta)}{2\pi - \alpha} & \text{if} \quad \dfrac{\alpha}{2} \leq \theta \leq \pi, \\ -\phi(-\theta) & \text{if} \quad -\pi \leq \theta \leq 0. \end{cases}$$

An elementary calculation shows that f is K-quasiconformal, where K is as in (4.2), and that f maps $\mathbf{S}(\alpha)$ onto the right half plane.

To show that bound in (4.2) is best possible, suppose that f is a K-quasiconformal mapping of $\overline{\mathbf{R}}^2$ which maps $\mathbf{S}(\alpha)$ onto the right half plane. Then

$$g = f^{-1} \circ r \circ f, \qquad r(z) = -\bar{z}$$

defines a K^2-quasiconformal mapping of $\overline{\mathbf{R}}^2$ which maps $\mathbf{S}(\alpha)$ onto its exterior $\mathbf{S}(\alpha)^*$.

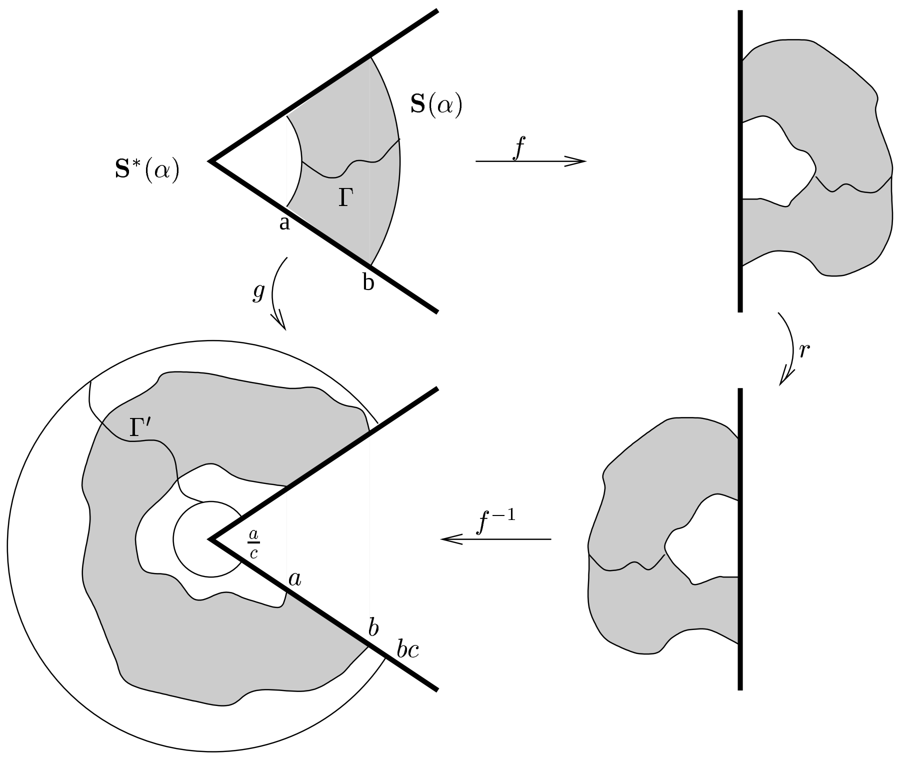

Figure 2

Next fix $0 < a < b < \infty$ and let Γ denote the family of arcs which join the circles $\{z : |z| = a\}$ and $\{z : |z| = b\}$ in $\{z : a \le |z| \le b, |\arg(z)| < \alpha/2\}$. Then it is not difficult to check that

$$\operatorname{mod}(\Gamma) = \frac{\alpha}{\log(b/a)}.$$

Similarly,

$$\operatorname{mod}(\Gamma') = \frac{2\pi - \alpha}{\log(b/a) + 2\log(c)},$$

where Γ' is the family of arcs which join $\{z : |z| = a/c\}$ and $\{z : |z| = bc\}$ in $\{z : a/c \le |z| \le bc, \alpha/2 < |\arg(z)| \le \pi\}$, where $c \ge 1$. If $c = \lambda(K)$, where

$$\lambda(K) = \left(\frac{1}{4}\, e^{\pi K/2} - e^{-\pi K/2}\right)^2 + \delta(K), \qquad 0 < \delta(K) < e^{-\pi K}, \tag{4.3}$$

then for each arc $\gamma' \in \Gamma'$ there exists an arc $\gamma \in \Gamma$ such that $g(\gamma) \subset \gamma' \in \Gamma'$. See [3] or [12]. Thus $\operatorname{adm}(g(\Gamma)) \subset \operatorname{adm}(\Gamma')$ whence

$$\operatorname{mod}(g(\Gamma)) \ge \operatorname{mod}(\Gamma')$$

and

$$K^2 \geq \frac{\operatorname{mod}(g(\Gamma))}{\operatorname{mod}(\Gamma)} \geq \frac{2\pi - \alpha}{\alpha} \frac{\log(b/a)}{\log(b/a) + 2\log(c)}.$$

We conclude that

$$K^2 \geq \frac{2\pi - \alpha}{\alpha}$$

by letting $b/a \to \infty$.

Finally reversing the roles of $\mathbf{S}(\alpha)$ and $\mathbf{S}(\alpha)^*$ in the above argument yields

$$K^2 \geq \frac{\alpha}{2\pi - \alpha}$$

and hence (4.2) □

5. Bilipschitz reflections in sector domains

We conclude by deriving here lower bounds for L when

1° f is a hyperbolic L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$,
2° f is a euclidean L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$.

These will then yield the lower bounds for L in Theorems 3.3 and 3.6.

We begin with the hyperbolic case.

Lemma 5.1. *If f is a hyperbolic L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$, then*

$$L \geq \max\left(\frac{2\pi - \alpha}{\alpha}, \frac{\alpha}{2\pi - \alpha}\right) = L(\alpha). \tag{5.2}$$

This bound is sharp.

Proof. Suppose without loss of generality that $\mathbf{S} = \mathbf{S}(\alpha)$ where $0 < \alpha \leq \pi$ and that $f : \mathbf{S} \to \mathbf{S}^*$ where

$$\frac{1}{L} h_{\mathbf{S}}(z_1, z_2) \leq h_{\mathbf{S}^*}(f(z_1), f(z_2)) \leq L h_{\mathbf{S}}(z_1, z_2) \tag{5.3}$$

for $z_1, z_2 \in \mathbf{S}$. To establish (5.2) choose for $j = 1, 2$ $z_j \in \mathbf{S}$ and $w_j = f(z_j)$ so that $w_j = -|w_j|$ and $|w_1| < |w_2|$. The hyperbolic densities at $z \in \mathbf{S}$ and $w \in \mathbf{S}^*$ are given by

$$\rho_{\mathbf{S}}(z) = \frac{\pi}{\alpha|z|} \sec\left(\frac{\pi \arg(z)}{\alpha}\right),$$
$$\rho_{\mathbf{S}^*}(w) = \frac{\pi}{(2\pi - \alpha)|w|} \sec\left(\frac{\pi \arg(w)}{2\pi - \alpha}\right).$$

Hence

$$h_{\mathbf{S}^*}(w_1, w_2) = \frac{\pi}{2\pi - \alpha} \int_{|w_1|}^{|w_2|} \frac{|dw|}{|w|} = \frac{\pi}{2\pi - \alpha} \log\left(\frac{|w_2|}{|w_1|}\right)$$

while

$$h_{\mathbf{S}}(z_1, z_2) \geq \frac{\pi}{\alpha} \log\left(\frac{|z_2|}{|z_1|}\right).$$

Next since f is L^2-quasiconformal with $f(0) = 0$ and $f(\infty) = \infty$,

$$\frac{1}{c} \leq \frac{|w_j|}{|z_j|} \leq c$$

for $j = 1, 2$, where $c = \lambda(L^2)$ and $\lambda(K)$ is as in (4.3). Hence

$$h_{\mathbf{S}}(z_1, z_2) \geq \frac{\pi}{\alpha} \log\left(\frac{|w_2|}{c^2|w_1|}\right) \geq L(\alpha)\, h_{\mathbf{S}^*}(w_1, w_2) - \frac{\pi}{\alpha} \log c^2$$

and we obtain (5.2) by letting $h_{\mathbf{S}^*}(w_1, w_2) \to \infty$. A similar argument yields (5.2) if $\pi \leq \alpha < 2\pi$.

Finally if we set

$$f(re^{i\theta}) = -re^{-iL(\alpha)\theta}$$

for $z = re^{i\theta} \in \mathbf{S}$, then

$$\limsup_{z \to z_0} \frac{h_{\mathbf{S}}(z, z_0)}{h_{\mathbf{S}^*}(f(z), f(z_0))} = L(\alpha)$$

while

$$\limsup_{z \to z_0} \frac{h_{\mathbf{S}^*}(f(z), f(z_0))}{h_{\mathbf{S}}(z, z_0)} = 1.$$

It then follows that f is a hyperbolic $L(\alpha)$-bilipschitz map in $\mathbf{S}$ and hence the bound in (5.2) is sharp. □

We turn next to the euclidean case.

Lemma 5.4. *If f is a euclidean L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$, then*

(5.5) $$L \geq \csc(\alpha/2).$$

Proof. Suppose for $\alpha \neq \pi$ that f is a euclidean L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$ and let γ be the segment joining the points $z_1 = e^{-i\alpha/2}$ and $z_2 = e^{i\alpha/2}$. Then $f(z_j) = z_j$ for $j = 1, 2$ and hence

$$L \geq \frac{\text{length}(f(\gamma))}{\text{length}(\gamma)} \geq \csc(\alpha/2). \qquad \square$$

Remark 5.6. If we make use of the fact that

$$\text{dist}(f(\gamma), 0) \geq \frac{|\cos(\alpha/2)|}{L}$$

in the above argument, we obtain

$$\left(1 + \frac{1}{L}\right)^2 \cos^2(\alpha/2) + \sin^2(\alpha/2) \leq L^2 \sin^2(\alpha/2)$$

whence

$$\cot^2(\alpha/2) \leq L^2 \frac{L-1}{L+1} \leq L^2 - L$$

and

(5.7) $$L \geq 1/2 + \sqrt{\cot^2(\alpha/2) + 1/4} > \csc(\alpha/2)$$

for $\alpha \neq \pi$. Thus (5.5) is not sharp in general. On the other hand if $0 < \alpha \leq \pi$, then

(5.8) $$f(x + iy) = -x + 2\cot(\alpha/2)\,|y| + i\,y$$

is a euclidean L-bilipschitz reflection in $\partial\mathbf{S}(\alpha)$ with

$$L = \cot(\alpha/4) = \cot(\alpha/2) + \csc(\alpha/2).$$

Thus the bound in (5.5) is within a factor of 2 of being best possible. Moreover since

$$\cot(\alpha/4) = \frac{\cos(\alpha/4) + \sin(\alpha/4)}{\sin(\alpha/4)} - 1 \le \frac{\sqrt{2}}{\sin(\alpha/4)} - 1 < \frac{2\pi - \alpha}{\alpha}$$

for $0 < \alpha < \pi$, the extremal for hyperbolic bilipschitz reflection in $\partial\mathbf{S}(\alpha)$ is not extremal for euclidean bilipschitz reflection in $\partial\mathbf{S}(\alpha)$.

We conjecture that the piecewise linear map in (5.8) is extremal in the euclidean case and hence that Lemma 5.4 holds with

$$\max(\cot(\alpha/4), \tan(\alpha/4)) \qquad \text{in place of} \qquad \csc(\alpha/2).$$

Lemmas 5.1 and 5.4 together with (4.2) yield the following lower bounds for the bilipschitz constants L in Theorems 3.3 and 3.6.

Corollary 5.9. *Suppose that D is a K-quasidisk. If f is a hyperbolic L-bilipschitz reflection in ∂D, then*

$$L \ge K^2.$$

If f is a euclidean L-bilipschitz reflection in ∂D, then

$$L \ge \csc\left(\frac{\pi}{K^2+1}\right).$$

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