Papers on Analysis:
A volume dedicated to Olli Martio on the occasion of his 60th birthday
Report. Univ. Jyväskylä 83 (2001), pp. 73–80

BEST CONSTANTS IN ZYGMUND'S INEQUALITY FOR CONJUGATE FUNCTIONS

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Abstract. A new proof is given of Zygmund's inequality which gives sharp constants both in the main term and in the first error term.

0. The main result

Let $F = f + i\tilde{f}$ be analytic in the unit disc U with $\tilde{f}(0) = 0$. Let $||\cdot||_1$ denote the Hardy norms in the spaces $H^1(U)$ or $h^1(U)$ (cf. [3]). A classical inequality of Zygmund [9] says there exist constants A, B and C such that

$$(0.1)\qquad ||\tilde{f}||_1 \le C \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})| \log^+ |f(re^{i\theta})|\, d\theta/2\pi + A||f||_1 + B.$$

For simplicity, we shall in the sequel write expressions as in the right hand side of this inequality as

$$C \int |f| \log^+ |f| + A||f||_1 + B.$$

The best constant C in (0.1) was found by Pichorides [8, Theorem 3.4]. He proved that for every constant $C > 2/\pi$, there exists a constant $A = A(C)$ such that (0.1) holds. Furthermore, $A(C) \to \infty$ as $C \downarrow 2/\pi$.

A general method developed by us gives an improvement of this result (cf. [5, Theorem 1], [6, Theorem 2]): there exist constants A and B such that

$$(0.2)\qquad \int |\tilde{f}| \le \frac{2}{\pi} \int |f| \log(e + |f|) + \frac{4}{\pi} \int |f| \log\log(e + |f|) + A||f||_1 + B.$$

The constant $2/\pi$ in (0.2) is sharp and examples (cf. [6, Lemma 14]) show that the constant $4/\pi$ in the error term can not be replaced by a constant less than $2/\pi$.

2000 *Mathematics Subject Classification.* Primary 42A50; Secondary 30D55, 31A05.
Key words and phrases. conjugate functions, H^1-norm estimates.

Our general method does not give the sharp constant in the error term in (0.2). Using a different argument, we can now prove

Theorem 1. *For $A \in (1,2]$, there exist absolute constants a_0, B_1 and B_2 such that*

$$\begin{aligned} ||F||_1 \le \frac{2}{\pi}\int |f|\log(a_0+|f|)+\frac{2}{\pi}\int |f|\log\log(a_0+|f|) \\ +\frac{2A}{\pi}\int |f|\log\log\log(a_0+|f|)+B_1(A-1)^{-2}||f||_1+B_2. \end{aligned} \tag{0.3}$$

The constants $2/\pi$ preceding the first two integrals in (0.3) are sharp.

1. Two Lemmas

We start with a result from [4].

Lemma A. *If $z = x+iy$ and $x>0$, we define*

$$G(z) = -\Re\{z\log z\} = y\arg z - x\log|z|.$$

We assume that $\arg z$ *vanishes on the real axis. If $x>0$, then*

$$|z| - \frac{2}{\pi}x\log|z| - x \le \frac{2}{\pi}G(z). \tag{1.1}$$

If $G_1(z) = G(|x|+iy+e^{-1})$, then G_1 is superharmonic in the complex plane $\mathbb{C}$.

We refer to [4] for details. For $a \ge e^e$, $A>0$ and $x \ge 0$, we define

$$L(x) = L(x,a,A) = \log(x+a)+\log\log(x+a)+A\log\log\log(x+a).$$

Lemma 1. *There exists an absolute constant C such that in the first quadrant*

$$|z| - \frac{2}{\pi}xL(x) \le \frac{2}{\pi}G(z)+x, \qquad L(x)\ge \log|z|, \tag{1.2}$$

$$\begin{aligned} |z| - \frac{2}{\pi}xL(x) - \frac{2x}{\pi}\log^+\left(\frac{y}{(x+a)\log(x+a)}\right)\chi_{[0,\psi(y)]}(x) \\ \le \frac{2}{\pi}G(z)+Cx, \qquad L(x)\le\log|z|, \end{aligned} \tag{1.3}$$

where

$$\psi(y) = \frac{2y}{\log y(\log\log y)^A}, \qquad y \ge b(a) = a\log a(\log\log a)^A.$$

Proof. We note first that (1.2) is a direct consequence of (1.1). To discuss (1.3), we consider the domain

$$\Omega = \{z = x+iy : x>0, y>0, L(x)\le\log|z|\},$$

bounded by the curve $\Gamma = \{z = x(y) + iy\}$ in the first quadrant. Γ can be described by the equation

$$|z| = e^{L(x)} = (x+a)\log(x+a)(\log\log(x+a))^A, \qquad x > 0.$$

The following properties of Ω and Γ will be proved in Section 3.

$$\inf_{z\in\Gamma} y \geq b(a), \tag{1.4}$$

$$y \leq e^{L(x)} \leq y\sqrt{2}, \qquad z \in \Gamma, \tag{1.5}$$

$$x + a \leq \psi(y), \qquad z \in \Omega, \tag{1.6}$$

$$\frac{x+a}{y} \leq \frac{2}{\log y(\log\log y)^A}\ , \qquad z \in \Omega. \tag{1.7}$$

In the remaining part of this section, we work in Ω. We have three estimates:

$$|z| = \sqrt{x^2+y^2} \leq y + \frac{x^2}{2y}, \tag{1.8}$$

$$\log|z| \leq \log y + \frac{x^2}{2y^2}, \tag{1.9}$$

$$\arg z = \arctan\frac{y}{x} = \frac{\pi}{2} - \arctan\frac{x}{y} \geq \frac{\pi}{2} - \frac{x}{y}. \tag{1.10}$$

Let us first assume that $y > (x+a)\log(x+a)$. Using (1.8), we deduce the following upper bound for the left hand member of (1.3):

$$y + \frac{x^2}{2y} - \frac{2x}{\pi}A\log\log\log(x+a) - \frac{2}{\pi}x\log y.$$

Applying (1.9) and (1.10), we compute the following lower bound for the right hand member of (1.3):

$$y - \frac{2x}{\pi} - \frac{2x}{\pi}(\log y + \frac{x^2}{2y^2}) + Cx.$$

Comparing these two expressions, we see that several terms cancel and that (1.3) will hold if

$$0 \leq \frac{2}{\pi}A\log\log\log(x+a) + C - \frac{x}{2y} - \frac{2}{\pi} - \frac{x^2}{\pi y^2}\ , \qquad z \in \Omega.$$

Using(1.7) and (1.4), we deduce that (1.3) will hold for all $a \geq e^e$ if $C = \frac{2}{\pi} + (e+1)^{-1} + \frac{4}{\pi}(e+1)^{-2}$. We have proved Lemma 1 in the case $y > (x+a)\log(x+a)$.

In the remaining case $y \leq (x+a)\log(x+a)$, the left hand member of (1.3) is $|z| - \frac{2x}{\pi}L(x)$ while the right hand member of (1.3) is bounded from below by (cf. (1.1))

$$|z| - \frac{2x}{\pi}(1 + \log(x+a) + \log\log(x+a) + \frac{x^2}{2y^2}) + Cx.$$

A comparison of these two terms shows that Lemma 1 holds also in this case.

2. Proof of Theorem 1

We first find a_1 such that for all $a \geq a_1$ and for all $A \in (1,2]$,

$$\frac{y}{(\log\log y)^A} \leq \psi(y)\log\psi(y) \leq \frac{2y}{(\log\log y)^A}, \qquad y \geq b(a).$$

Secondly, we claim that there exists a_2 such that for all $a \geq a_2$ and for all $A \in (1,2]$,

$$\sup_{0<x<\psi(y)} x\log\left(\frac{y}{(x+a)\log(x+a)}\right) \leq \frac{2Ay\log\log\log y}{\log y(\log\log y)^A}, \qquad y \geq b(a). \tag{2.1}$$

To prove (2.1), we study for y fixed the concave function

$$K(t) = t\log\left(\frac{y}{t\log t}\right), \qquad a \leq t \leq a+\psi(y),$$

where we assume that $a \geq e^e$ and $y \geq b(a)$. If $K'(a+\psi(y)) \geq 0$, it is clear that K will be increasing on the interval $[a, a+\psi(y)]$ which implies that (2.1) holds. We note that there exists a_3 such that $\psi(y) \geq \psi(b(a)) \geq a$ if $A \in (1,2]$, $a \geq a_3$ and $y \geq b(a)$. Consequently,

$$K'(a+\psi(y)) \geq \log\left(\frac{y}{2\psi(y)\log(2\psi(y))}\right) - 2.$$

The logarithmic term will tend to ∞ as $y \to \infty$. Thus we can find $a_2 \geq a_3$ such that $K'(a+\psi(y)) \geq 0$ if $a \geq a_2$ and $y \geq b(a)$.We have proved (2.1).

It follows from (1.2), (1.3) and (2.1) that for all $z \in \mathbb{C}$,

$$|z| \leq \frac{2}{\pi}(|x|+e^{-1})L(|x|+e^{-1}) + \frac{4A|y|\log\log\log|y|\chi_{[b(a),\infty)}(|y|)}{\pi\log|y|(\log\log|y|)^A} + \frac{2}{\pi}G_1(z) + C(|x|+e)\ .$$

Replacing z by $F(z)$, integrating over circles and using the superharmonicity of G_1, we see that there are constants C' and C'' such that

$$||F||_1 \leq \frac{2}{\pi}G_1(F(0)) + \frac{2}{\pi}\int(|f|+e^{-1})L(|f|+e^{-1}) + \frac{4A}{\pi}\int_{|\tilde f|>b(a)}|\tilde f|\log\log\log|\tilde f|\left(\log|\tilde f|(\log\log|\tilde f|)^A\right)^{-1} + C'||f||_1 + C''. \tag{2.2}$$

We note that $G_1(F(0)) \leq e^{-1}$. A weak type estimate due to Kolmogorov [7] says that

$$|\{e^{i\theta} : |\tilde f(e^{i\theta})| > t\}| \leq C_0||f||_1/t. \tag{2.3}$$

The sharp constant C_0 is known (cf. [2] and [1]). Applying (2.3), we can estimate the integral in (2.2) containing $\tilde f$ by

$$2C_0||f||_1\int_{b(a)}^{\infty}\log\log\log t(t\log t(\log\log t)^A)^{-1}dt$$
$$\leq 2C_0||f||_1(A-1)^{-2}C\left((\log\log b(a))^{1-A}\log\log\log b(a)\right).$$

Combining these results, we obtain (0.3) with $a_0 = \max\{e^e, a_1, a_2\} + e^{-1}$ and we have proved Theorem 1.

3. Proofs of estimates (1.4)–(1.7)

On Γ, we have $|z| = e^{L(x)}$. Let us for y fixed study the function $h(x) = \sqrt{x^2+y^2} - e^{L(x)}$, $x > 0$ which vanishes on Γ. Since $a \geq e^e$, it is easy to see that h is decreasing. A line $\{y = y_0\}$ intersects Γ in the first quadrant only if $h(0) = y_0 - e^{L(0)} > 0$. Thus (1.4) holds (we note that $b(a) = e^{L(0)}$). It is now easy to see that $h(y) < 0$ which implies that $x(y) \leq y$ and that (1.5) holds. To prove (1.6), we note that if $x + a = \psi(y)$,

$$\begin{aligned}\frac{e^{L(x)}}{y} &= \frac{\psi(y)}{y}\log\psi(y)(\log\log\psi(y))^A\\ &= 2\frac{\log\psi(y)}{\log y}\left(\frac{\log\log\psi(y)}{\log\log y}\right)^A \to 2, \qquad y\to\infty.\end{aligned}$$

A computation shows that if $a \geq e^e$, then

$$\frac{e^{L(x)}}{y} > \sqrt{2}, \qquad \psi(y) \geq a.$$

Since (1.5) holds for $z \in \Gamma$, it is clear that (1.6) must be true. Finally, (1.7) is a direct consequence of (1.6).

4. An alternative argument

The "best" choice of the "correction term" in (1.3) depends in a crucial way on the choice of $L(x)$. If we use L_1 defined by

$$L_1(x) = L_1(x,a,A) = \log(x+a) + A\log\log(x+a),$$

and replace the correction term in (1.3) by

$$\frac{2x}{\pi}\log^+\left(\frac{y}{x+a}\right)\chi_{[0,\psi_1(y)]}(x),$$

where $\psi_1(y) = 2y(\log y)^{-A}$, $y \geq a(\log a)^A$, our method shows that for $A \in (1,2]$, there exist absolute constants a_0, B_1 and B_2 such that

$$||F||_1 \leq \frac{2}{\pi}\int|f|\log(a_0+|f|) + \frac{2A}{\pi}\int|f|\log\log(a_0+|f|) + \frac{B_1}{(A-1)^2}||f||_1 + B_2.$$

Alternatively, we can argue in the following way. Let $L : [0,\infty) \to (0,\infty)$ be a strictly increasing and concave function which is such that

$$L(x) \geq \log(x\sqrt{2}), \quad x \geq e^{L(0)}, \tag{4.1}$$

$$L'(x)e^{L(x)} \geq 1, \quad x \geq e^{L(0)}, \tag{4.2}$$

$$xL(x) \text{ is convex}, \quad x > 0. \tag{4.3}$$

A direct consequence of (4.3) is that

$$x^2L'(x) \text{ is increasing}, \quad x > 0. \tag{4.4}$$

Theorem 2. *Assume that (4.1)-(4.3) hold and that furthermore*

$$\int^{\infty} t^2 L'(t)^2 e^{-L(t)} dt < \infty. \tag{4.5}$$

Then there exist constants C' and C'' only depending on L such that

$$||F||_1 \le \frac{2}{\pi}\int |f|L(|f|) + C'||f||_1 + C''. \tag{4.6}$$

We need the following lemma.

Lemma 2. *Assume that (4.1) and (4.2) hold. Then there exists a constant $C = C(L)$ such that in the first quadrant*

$$|z| - \frac{2x}{\pi}L(x) - \frac{2x}{\pi}\log^+(ye^{-L(x)}) \le \frac{2}{\pi}G(z) + Cx. \tag{4.7}$$

Proof of Lemma 2. If $L(x) \ge \log|z|$, (4.7) is a direct consequence of (1.1). As in the proof of Lemma 1, it suffices to work in the domain Ω bounded by the curve $\Gamma = \{|z| = e^{L(x)}, x > 0, y > 0\}$. For y fixed, we consider $h(x) = |z| - e^{L(x)}$. According to (4.2), h decreases and will be negative for x large. In particular, it is clear from (4.1) that we have $h(y) \le 0$. It follows that if $z \in \Gamma$, we must have $y \ge e^{L(0)}$ which means that

$$\inf_{z\in\Gamma} y = e^{L(0)}, \tag{4.8}$$

and that

$$x \le y, \qquad z \in \Gamma. \tag{4.9}$$

In Ω, estimates (1.8)-(1.10) and (4.9) hold. Arguing exactly as in the proof of Lemma 1, we deduce Lemma 2. We have to separate the cases $\log y \le L(x)$ and $\log y > L(x)$.

Proof of Theorem 2. We begin by defining $\eta(y) = L^{-1}(\log y)$, $y \ge e^{L(0)}$. Let $Q(x) = x(\log y - L(x))$. We claim that

$$\sup_{x>0} Q(x) = Q(x_M(y)) \le \eta(y)^2 L'(\eta(y)). \tag{4.10}$$

To prove (4.10), we first note that it follows from (4.3) that $Q'(x)$ is decreasing. Since $Q'(\eta(y)) = -\eta(y)L'(\eta(y)) < 0$, we conclude that $x_M(y) \le \eta(y)$ and that (cf. (4.4))

$$Q(x_M(y)) = x_M(y)^2 L'(x_M(y)) \le \eta(y)^2 L'(\eta(y)). \tag{4.11}$$

From Lemma 2 and (4.10), we deduce that

$$|z| \leq \frac{2}{\pi}(|x|+e^{-1})L(|x|+e^{-1}) + \frac{2}{\pi}\eta(y)^2 L'(\eta(y)) + \frac{2}{\pi}G_1(z) + C(|x|+e^{-1}).$$

Arguing as in the proof of Theorem 1, we see that we have to estimate the integral

$$\int_{|\tilde{f}|>e^{L(0)}} \eta(|\tilde{f}|)^2 L'(\eta(|\tilde{f}|). \tag{4.12}$$

From (4.4) and (4.5), we deduce that

$$x^2 L'(x)e^{-L(x)} \to 0, \qquad x \to \infty. \tag{4.13}$$

Since we know that (4.13) is true, we can use (2.3) to estimate (4.12): an upper bound (modulo constants) is

$$2\int_{e^{L(0)}}^{\infty} \eta(y)^2 L'(\eta(y))dy/y^2.$$

Now, the change of variable $\eta(y) = t$ gives us the integral in (4.5) which we have assumed to be convergent. This completes the proof of Theorem 2.

As an application, we give a generalization of Theorem 1. Let $\log^{[k]} x = \log^{[k-1]}(\log x)$, $k = 2, 3, \ldots$, $\log^{[1]}(x) = \log x$ and

$$L(x) = \sum_{k=1}^{p} \log^{[k]}(x+b) + A\log^{[p+1]}(x+b).$$

Corollary. *For each $A > 1$ and p, we can find b such that L will satisfy the assumptions of Theorem 2 and (4.6) will hold.*

In this example, xL' is essentially constant for x large and (4.5) says that $\int^{\infty} exp(-L(t))dt$ is convergent.

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