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HOLOMORPHIC MOTIONS

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Abstract. The notion of holomorphic motions, introduced by Mañé, Sad and Sullivan [MSS], explains in a striking manner the many connections quasiconformal mappings have to holomorphic dynamics, Teichmüller theory and other related areas of complex analysis. In the theory of holomorphic motions a highlight is provided by Slodkowski's generalized Lambda lemma [S], which gives for any holomorphic motion of any set $E \subset \overline{\mathbf{C}}$ an extension to a motion of the whole space $\overline{\mathbf{C}}$.

Slodkowski's proof is based on the theory and techniques of several complex variables. The authors of the present article aim to explain Slodkowski's proof to specialists in quasiconformal mappings, holomorphic dynamics and related fields, and wrote this article which has been circulating as an unpublished manuscript for several years. Due to a number of demands we have now decided to publish it.

In the interim two further approahes to Slodkowski's result have been given by Douady [Do] and Chirka [C].

1. Introduction

Basically a holomorphic motion is an isotopy of a subset A of the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ analytically parametrised by a complex variable z in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. A useful feature of holomorphic motions is that the continuity assumptions can be dismissed from the definition and, in fact, analyticity alone forces strong regularity and extendability properties on the motion.

Holomorphic motions arise naturally in many situations involving complex dynamical systems. For instance the Julia sets of rational maps of $\overline{\mathbf{C}}$ often move holomorphically with holomorphic variations of the parameters (that is the coefficients of rational map). This is the situation in which holomorphic motions were first considered, by Mañé–Sad–Sullivan [MSS]. Also the limit sets of Kleinian groups often move holomorphically when one varies the associated parameters (this time the coefficients of the Möbius transformations associated to the generators are varied holomorphically). Such situations also occur when one is studying stability and genericity of these dynamical systems, for instance questions like; are structurally stable or expanding systems dense? For Kleinian groups one might read geometrically finite for structurally stable, see [Su] for a deeper discussion of these things.

The surprising fact about holomorphic motions is that they always extend to ambient holomorphic motions (that is, holomorphically parametrised isotopies of $\overline{\mathbf{C}}$) and that at each time the associated homeomorphism of the plane is quasiconformal. This is the so called extended λ–lemma. The extended λ–lemma was proven by Slodkowski [S] using techniques from several complex variables, in particular the structure of polynomial hulls of sets that fiber over the circle. The result had earlier been conjectured by Mañé–Sullivan and Sullivan–Thurston [ST]. They, along with Bers–Royden [BR], had proven partial results. In this paper we shall give a complete and self contained proof of the extended λ–lemma from the point of view of one complex variable. The proof is based on the solution of the nonlinear Riemann–Hilbert problem (NRH) by Snirelman in 1972 [Sn]. Actually, when all is said and done, this is the way Slodkowski's proof goes. He bases the proof on a result of Forstnerič [F] concerning the structure of polynomial hulls in which Forstnerič reproves NRH with different applications in mind and therefore more complication than is necessary in this setting.

We hope our approach to the generalised λ–lemma is slightly more direct and accessible. And, although the spirit of this paper is largely expository, we do obtain some new results and new proofs of older results.

We shall give precise definitions of what a holomorphic motion is and what it means for a mapping to be quasiconformal and so forth in the next section. Before this let us briefly describe why in the two cases above the important dynamical parts of the systems do move holomorphically.

Firstly, consider the case of a parametrised family of Kleinian groups $\Gamma_z, z \in \Delta$. According to a well known theorem of Jørgensen [J] members of a continuously parametrised family of discrete groups are all canonically isomorphic to Γ_0. It is also well known that the fixed point sets of loxodromic elements are dense in the limit set Λ_z of Γ_z and it is clear that these fixed points move holomorphically with the parameters involved. As two loxodromic elements in the discrete group Γ_z cannot share a single fixed point [B], loxodromic fixed points cannot collide unless both belong to the same element or else the other associated pair of fixed points also collides. In the first possibility the deformation produces parabolic elements and changes the geometric nature of the assocated orbit space $(\overline{\mathbf{C}} - \Lambda_z)/\Gamma_z$, (for instance it may become noncompact). The second possibility cannot occur since it contradicts Jørgensen's theorem: The group generated by the two colliding loxodromic elements before the collision is a discrete group with a two generator free subgroup, whereas at the time of collision, the two loxodromic elements generate a group which is virtually abelian (since it is assumed discrete!). These groups are therefore not isomorphic and so the collision cannot occur. The density of the loxodromic fixed points in the limit set therefore implies that the limit set moves holomorphically and the extended λ–lemma asserts that this motion extends to a holomorphic motion of $\overline{\mathbf{C}}$ and the isotopy is through quasiconformal mappings. From this we deduce that the Kleinian groups are canonically quasiconformally conjugate on their limit sets as long as there are no new parabolic elements produced (and this restriction is easily seen to be necessary). Thus for instance a holomorphic deformation of a Fuchsian group produces a quasiFuchsian group. The limit set moves holomorphically from a round circle to a quasicircle.

In the case of the Julia sets $J(R)$ of rational maps $R : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ we illustrate their holomorphic deformations by an example, by considering the quadratic polynomials. Any quadratic polynomial is conjugate by a Möbius transformation to a mapping of the form $R_c(z) = z^2 + c$ and so this family exhibits all the dynamical phenomena possible for the iteration of a quadratic polynomial. For such a mapping R(z) we recall that the Fatou set F(R) is where the dynamics are stable, that is the maximal domain where the family $\{R^n; n \in \mathbf{Z}_+\}$ is normal. Then $J(R) = \mathbf{C} - F(R)$. The Mandelbrot set consists of those parameters $c \in \mathbf{C}$ such that $J(z^2 + c)$ is connected. Inside the Mandelbrot set we have the hyperbolic regions consisting of those c's for which R_c has an attracting periodic cycle, that is for some $n \in \mathbf{Z}$ the mapping $R_c^n(z)$ has a fixed point z_0 such that $|(R_c^n)'(z_0)| < 1$. It is conjectured that the Mandelbrot set is the closure of the hyperbolic regions. These regions are simply connected (and so are themselves holomorphically parametrised by the disk!). Suppose we fix a hyperbolic region U, vary $c \in U$ and study the dynamical system associated to iteration. Firstly recall that every attracting cycle attracts a critical point of our mapping. As the mapping is quadratic there is exacly one finite attracting cycle. Recall too that the repelling periodic points, those points z_0 such that for some $n \in \mathbf{Z}_+$, $R_c^n(z_0) = z_0$ and $|(R_c^n)'(z_0)| > 1$, are dense in the Julia set. As we move c in U these points remain repelling: They cannot become attracting as there is already one such cycle nor can they become indifferent, $|(R_c^n)'(z_0)| = 1$, as such cycles also "attract" critical points as well. [Actually, for every $c \in U$ the closure of the critical orbit does not meet the Julia set, so by a theorem of Fatou the mapping R_c is expanding on the Julia set, that is $|R_c'(z)| \geq \lambda > 1$ for all $z \in J(R_c)$, This too forces repelling periodic points to remain repelling].

It is clear the repelling points move holomorphically with the parameter c. We need to show that they don't collide. To see this, note that a repelling periodic point z_{c_0} is a solution of the equation $\Psi(c_0, z) = R_{c_0}^n(z) - z = 0$. As $\partial_z \Psi(c_0, z_{c_0}) \neq 0$, by the implicit function theorem there is a neighbourhood of (c_0, z_{c_0}) where for each c we have a unique z_c with $R_c^n(z_c) = z_c$. An alternative geometric argument is to observe that if repelling points of period m and n collide $(m < n)$ then by continuity two points on the period n cycle collide. At such a point w, $R_c^n(z) - w$ has a double root and derivative zero. Thus for nearby time the derivative was less than one in modulus, a contradiction. This then shows that the repelling points move holomorphically and therefore so does the Julia set. This is the reason why Julia sets seem to be quasiconformally similar (because they are!). See the Figures 1 and 2 below for some illustrations of this.

Finally we note that the extended λ–lemma is important for many other reasons as well. As an instance it implies that a holomorphic perturbation of the complex structure of a Riemann surface is necessarily induced by a quasiconformal mapping, showing that quasiconformal mappings are indespensible tools in the study of Teichmüller theory. It shows too that quasiconformal mappings are precisely those mappings which are obtained by a holomorphic perturbation of the identity mapping.

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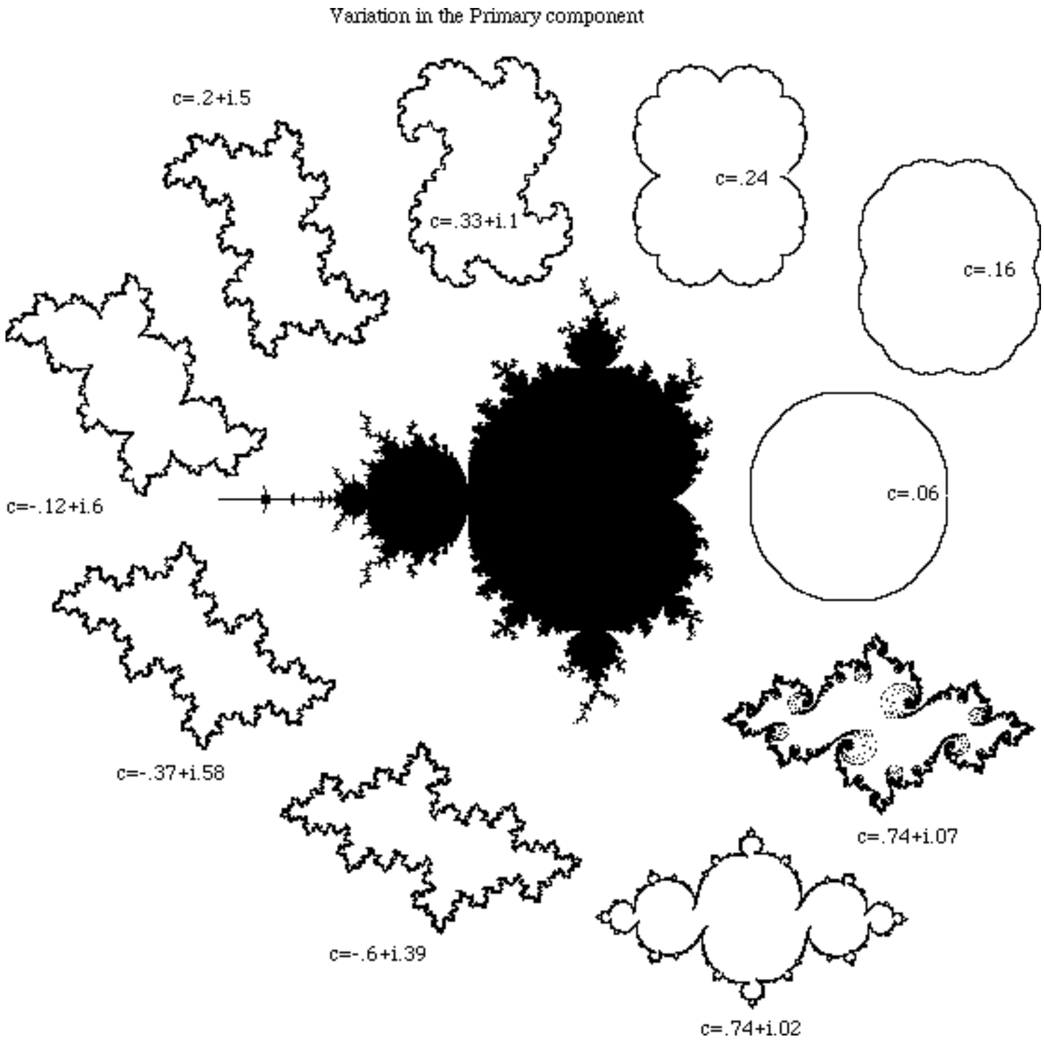
Variation in the Primary component
c=.2+i.5
c=.33+i.1
c=.24
c=.16
c=-.12+i.6
c=.06
c=-.37+i.58
c=.74+i.07
c=-.6+i.39
c=.74+i.02

FIGURE 1

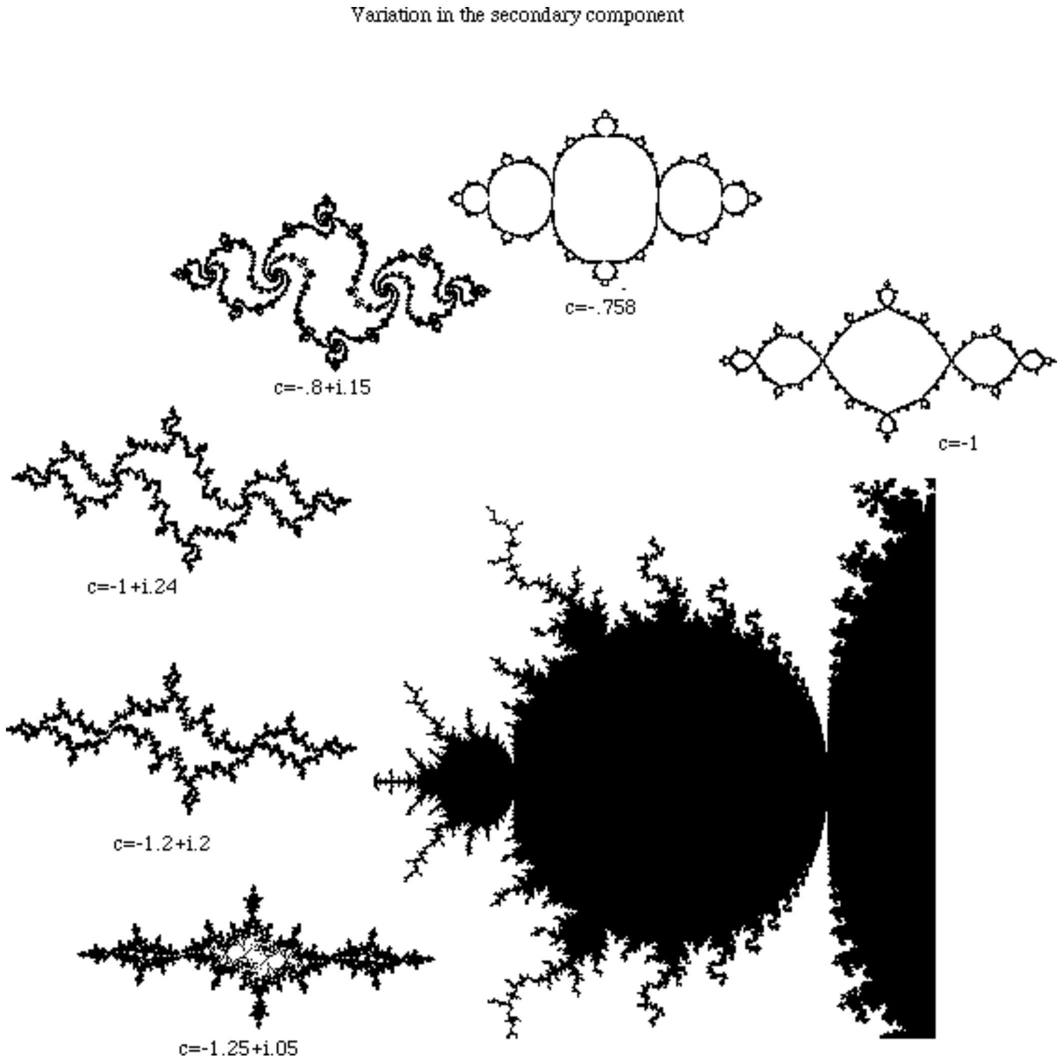
Variation in the secondary component
c=-.758
c=-.8+i.15
c=-1
c=-1+i.24
c=-1.2+i.2
c=-1.25+i.05

FIGURE 2

2. Quasiconformal Mappings

As a basic class of mappings we shall be talking about here are quasiconformal mappings we take a few moments to give their definition and recall a few basic facts from [L] and [V].

Let $\Omega \subset \mathbf{C}$ be a planar domain and $f : \Omega \to \mathbf{C}$ be an orientation preserving homeomorphism. Define

$$H_f(z) = \limsup_{r\to 0} \frac{\max_{|h|=r} |f(z+h) - f(z)|}{\min_{|h|=r} |f(z+h) - f(z)|}. \tag{1}$$

Then f is said to be quasiconformal if there is $H < \infty$ such that $H_f(z) < H, z \in \Omega$. The essential supremum of $H_f(z)$ is called the dilatation of f and the letter K is usually reserved for this quantity. When $K = 1$ we obtain a planar conformal mapping. Quasiconformal mappings have locally L^2–integrable derivatives $\partial_z f$ and $\partial_{\overline{z}} f$, the change of variable formula works and f preserves sets of zero Lebesgue measure. Furthermore, given a quasiconformal mapping f we set

$$\mu_f = \frac{\partial_{\overline{z}} f}{\partial_z f} \tag{2}$$

and note that μ_f is an element of the open unit ball of $L^\infty(\Omega)$ (as f is orientation preserving, the Jacobian is positive a.e. so that $|\partial_{\overline{z}} f| < |\partial_z f|$). The function μ_f is called the complex dilatation of f. Conversely, given a μ in the unit ball of $L^\infty(\Omega)$, there is a (unique up to normalisation) quasiconformal mapping f with μ as its complex dilatation. The relationship between the complex dilatation and the number K is

$$K = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}. \tag{3}$$

When a homeomorphism $f : A \to \overline{\mathbf{C}}$ is defined on a set without interior, the above definition makes no sense. In this case the linear dilatation condition (1) is usually replaced by the requirement that the mapping distorts cross ratios by a bounded amount: Writing

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{4}$$

we say a homeomorphism f defined on the set A is quasiconformal if

$$|(f(z_1), f(z_2), f(z_3), f(z_4))| \le \varphi(|z_1, z_2, z_3, z_4|) \tag{5}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is continuous increasing and onto.

In fact if A is a planar domain then (5) implies that the linear dilatation $H_f(x)$ is uniformly bounded and the converse holds when $A = \mathbf{C}$. However in general domains $K = K(f) < \infty$ implies (5) only locally. Therefore the notion of quasiconformality we shall use in the sequel is slightly stronger than the usual assumption $\sup H_f(x) < \infty$. The results we obtain are thus a little stronger.

3. The Extended λ–Lemma

Definition Let A be a subset of $\overline{\mathbf{C}}$. A holomorphic motion of A is a map $f : \Delta \times A \to \overline{\mathbf{C}}$ such that
(i) for any fixed $a \in A$, the map $\lambda \to f(\lambda, a)$ is holomorphic in Δ
(ii) for any fixed $\lambda \in \Delta$, the map $a \to f(\lambda, a) = f_\lambda(a)$ is an injection and
(iii) the mapping f_0 is the identity on A.

Note especially that there is no assumption regarding the continuity of f as a function of a or the pair (λ,a). That such continuity occurs is a consequence of the following remarkable λ–*lemma* of Mañé–Sad–Sullivan [MSS].

Theorem 3.1. *If $f : \Delta \times A \to \overline{\mathbf{C}}$ is a holomorphic motion, then f has an extension to $F : \Delta \times \overline{A} \to \overline{\mathbf{C}}$ such that*
(i) F is a holomorphic motion of $\overline{A}$
(ii) each $F_\lambda(\cdot) : \overline{A} \to \overline{\mathbf{C}}$ is quasiconformal
(iii) F is jointly continuous in (λ, a).

Proof:- Let ρ denote the hyperbolic metric of the triply punctured sphere $\mathbf{C} - \{0,1\}$. Since this metric is complete, given $z, w \in \mathbf{C} - \{0,1\}$ a bounded hyperbolic distance apart we see that $|z| \to 0$ implies that $|w| \to 0$. Thus there is a continuous function $\eta : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ such that for each fixed $M < \infty$, $\eta(M, \epsilon) \to 0$ as $\epsilon \to 0$ and

$$|w| \leq \eta(M, |z|) \tag{6}$$

whenever $z, w \in \mathbf{C} - \{0,1\}$ and $\rho(z, w) < M$. If x, y, z and w are distinct points of A, the holomorphic motion f gives rise to the cross ratio function

$$g(\lambda) = (f_\lambda(x), f_\lambda(y), f_\lambda(z), f_\lambda(w)) \tag{7}$$

which is holomorphic in Δ with values in $\mathbf{C} - \{0,1\}$. By the generalised Schwarz lemma of Ahlfors, see eg. [N], the mapping g is a contraction of hyperbolic metrics. That is

$$\rho(g(\lambda), g(0)) \leq \rho_\Delta(\lambda, 0) = \log \frac{1 + |\lambda|}{1 - |\lambda|} \tag{8}$$

where ρ_Δ is the Poincaré (hyperbolic) metric of the disk Δ. Since $g(0) = (x, y, z, w)$ we find

$$|(f_\lambda(x), f_\lambda(y), f_\lambda(z), f_\lambda(w))| \leq \eta(M, |(x, y, z, w)|) \tag{9}$$

with $M = \log \frac{1+|\lambda|}{1-|\lambda|}$. Therefore each f_λ is uniformly continuous in A and so extends continuously to $F_\lambda : \overline{\mathbf{A}} \to \overline{\mathbf{C}}$. Permuting the x, y, z and w entries in the equation (9) shows this extension is injective and so each $F_\lambda(\cdot)$ is a homeomorphism onto its image. For each $a \in \overline{A} - A$ the function $F(\cdot, a)$ is holomorphic since it is the local uniform limit of holomorphic functions; the joint continuity in (λ, a) follows since for every $r < 1$ the family $\{F_\lambda(\cdot) : \lambda \in r\Delta\}$ is equicontinuous. Moreover, by definition equation (9) establishes the quasiconformality of F_λ in $\overline{A}$ and hence the proof is complete. $\square$

Using fairly sophisticated results from Teichmüller space theory Bers and Royden extended Theorem 3.1 in two directions [BR]. Firstly they showed that actually each

$f_\lambda(\cdot), \lambda \in \Delta$ is the restriction of a quasiconformal self map of $\overline{\mathbf{C}}$ and secondly they provided a sharp dilatation estimate: There is an extension to $\overline{\mathbf{C}}$ of $f_\lambda(\cdot)$ whose dilatation does not exceed

$$K = \frac{1+|\lambda|}{1-|\lambda|}. \tag{10}$$

Notice that the existence of a global quasiconformal extension follows from Theorem 3.1 if one could show that $f(\lambda, a)$ was the restiction to A of a holomorphic motion of $\overline{\mathbf{C}}$. Similarly the dilatation estimate can be obtained by showing that the Beltrami coefficients of the quasiconformal mappings also vary holomorphically. The question of whether or not each holomorphic motion of A is actually the restriction of a holomorphic motion of $\mathbf{C}$ was posed by Sullivan and Thurston [ST]. They showed that there is a universal constant $a > 0$ such that if f is a holomorphic motion of a set A, then $f|\{|\lambda| < a\} \times A$ is the restiction of a holomorphic motion $F : \{|\lambda| < a\} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$. Bers and Royden then further showed that $a \geq \frac{1}{3}$.

Using innovative ideas from several complex variables Slodkowski proved that the constant a above can indeed be taken to be equal to 1.

Theorem 3.2. *Every holomorphic motion of a set $A \subset \mathbf{C}$ is the restriction of a holomorphic motion of $\mathbf{C}$.*

As a consequence one directly obtains the complete version of the extended λ–lemma.

Theorem 3.3. *If $f : \Delta \times A \to \overline{\mathbf{C}}$ is a holomorphic motion of $A \subset \mathbf{C}$, then f has an extension to $F : \Delta \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that*

(i) F is a holomorphic motion of $\overline{\mathbf{C}}$

(ii) each $F_\lambda(\cdot) : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a quasiconformal self homeomorphism of dilatation not exceeding $\frac{1+|\lambda|}{1-|\lambda|}$

(iii) F is jointly continuous in (λ, a)

As we shall see in the sequel, the proof of this theorem is a little indirect. In the set up we have chosen the main steps are as follows:

(1) Standard compactness results and the λ–lemma , Theorem 3.1, proved above reduce the problem to the case that A is a finite point set. (In this case the result is called the holomorphic axiom of choice).

(2) For $r < 1$ extend the motion $f : \mathbf{S}^1(r) \times A \to \overline{\mathbf{C}}$ to a diffeotopy $\Psi : \mathbf{S}^1(r) \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ of the sphere (as only smoothness is required and as A is only a finite point set the construction of Ψ is relatively straightforward).

(3) Use the solution to the nonlinear Riemann–Hilbert problem to show that a smoothly varying family of smooth Jordan curves (parameterised by the circle) can be realised as the boundary values of an essentially unique holomorphic motion.

(4) Foliate the plane by such Jordan curves (essentially arbitrarily) and show that under the induced holomorphic motion they do not collide. This gives a holomorphic motion of the plane which agrees with the original motion of A by uniqueness.

The technically difficult part of this proof is of course (3). The fact that any smooth family of Jordan curves can be reparameterised to move holomorphically is in itself already quite remarkable.

4. Nonlinear Riemann–Hilbert Problem

As we have noted above an essential part of the proof of the extended λ–lemma is the solution to the nonlinear Riemann–Hilbert Problem, Theorem 4.1 below. As far as we can tell, this was first solved by Snirelman in 1972 [Sn]. Forstnerič reproved this result in his paper [F] with a different approach and with better control on the regularity. However, both proofs are based on the continuity method in one form or another.

We give here a restatement of the result proved by Snirelman-Forstnerič best suited to our ends and sketch the proof more along the lines of [F].

Theorem 4.1. *Let $\{C_\lambda : \lambda \in \mathbf{S}^1\}$ be a smoothly varying family of smooth Jordan curves in the complex plane, each separating 0 from ∞. Then for every $\zeta \in C_1$ there is a unique holomorphic function $g_\zeta : \Delta \to \mathbf{C}$, continuous and nonzero in $\overline{\Delta}$ and with vanishing winding number about 0, such that $g_\zeta(1) = \zeta$ and for each $\lambda \in S^1$*

$$g_\zeta(\lambda) \in C_\lambda$$

Moreover for each $\lambda \in \mathbf{S}^1$ the set $\{g_\zeta(\lambda) : \zeta \in C_1\} = C_\lambda$ and for each $z \in \overline{\Delta}$, $g_\zeta(z) \neq g_\eta(z)$ when $\zeta \neq \eta$.

Sketch of Proof:- By a smoothly varying family $\{C_\lambda\}$ of Jordan curves we mean that there is a smooth map $\Phi : \mathbf{S}^1 \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that if we set $\Phi_\lambda(z) = \Phi(\lambda, z)$, then

$$\Phi_\lambda^{-1}(\mathbf{S}^1) = C_\lambda$$

and each Φ_λ fixes 0 and ∞. We therefore seek a function g continuous in $\overline{\Delta}$ and holomorphic in Δ such that

$$|\Phi|(\lambda, g(\lambda)) = 1, \qquad \lambda \in \mathbf{S}^1. \tag{11}$$

As everything is smooth and as 0 is contained in the domains separated from ∞ by C_λ, it is clear we may assume that Φ_λ is the identity in a neighbourhood of 0. Then there is t_0 such that for all $t < t_0$ there exists a family of solutions g_t, namely $g_t(z) \equiv \zeta \in \mathbf{S}^1(t)$, with

$$|\Phi|(\lambda, g_t(\lambda)) = t, \quad \lambda \in \mathbf{S}^1. \tag{12}$$

We want to show that the set of t's for which the above equation has solutions g_t, continuous in t and with the desired properties, is both open and closed. To see that this set is open we use the implicit function theorem in a Banach space on which the Hilbert transform is bounded. For our purposes the most convenient choice is

$$E = C^{1,\alpha}(\mathbf{S}^1)$$

the Banach space of functions on the circle whose first derivatives are Hölder continuous with exponent α. Now assuming that we have solutions for $0 < t \leq t_0$ define $F : \mathbf{R} \times E \to E$ by

$$F(t,u)(\lambda) = |\Phi|(\lambda, g_{t_0}(\lambda) + (u + \imath Hu)(\lambda)X(\lambda)) - t \tag{13}$$

where Hu is the Hilbert transform of u (and so $u + \imath Hu$ admits a holomorphic extension to Δ which is continuous in $\overline{\Delta}$) and where $X \in E$ is a holomorphic function to be defined later. Then,

$$\begin{aligned}D_uF(t_0,0)u &= \partial_z|\Phi|(\lambda, g_{t_0}(\lambda))(u+\imath Hu)(\lambda)X(\lambda)\\ &\quad + \ \partial_{\bar z}|\Phi|(\lambda, g_{t_0}(\lambda))\overline{(u+\imath Hu)(\lambda)X(\lambda)}\\ &= 2\Re\left(\partial_z|\Phi|(\lambda, g_{t_0}(\lambda))(u+\imath Hu)(\lambda)X(\lambda)\right)\end{aligned}$$

Now if we can choose X so that $\partial_z|\Phi|(\lambda, g_{t_0}(\lambda))X(\lambda))$ is real and nonvanishing, then we obtain

$$D_uF(t_0,0)u = \mathcal{A}(\lambda)u(\lambda) \tag{14}$$

from which it is clear that $D_uF(t_0,0)$ is invertible. But for t small $\partial_z|\Phi|(\lambda, g_{t_0}(\lambda))$ is a nonzero constant and so by continuity (in t) it has vanishing winding number about 0 even for large t. Therefore we can write

$$\partial_z|\Phi|(\lambda, g_{t_0}(\lambda)) = e^{\alpha(\lambda)+\imath\beta(\lambda)} \tag{15}$$

and simply choose

$$X(\lambda) = e^{-\imath\beta(\lambda)+(H\beta)(\lambda)} = e^{-\imath(\beta+\imath H\beta)(\lambda)}. \tag{16}$$

Now briefly, by the implicit function theorem there is $\varepsilon > 0$ for which the equation $|\Phi|(\lambda, g_t(\lambda)) = t$ admits solutions $g_t \in E$ varying continuously in t for $0 < t < t_0 + \varepsilon$. Similarly, as g_t depends smoothly on the initial data g_{t_0} and $g_t(\lambda) \in \Phi_\lambda^{-1}(\mathbf{S}^1(t))$, the values $g_t(\lambda)$ cover this Jordan curve as long as every initial solution admits the continuation g_t.

Next we consider the last part of the statement, the pointwise uniqueness of the solutions to NRH; this part of the theorem is, of course, crucial in constructing the holomorphic motions. By compactness it is clear that the set of t's such that $g_{t,\eta}(z) \neq g_{t,\zeta}(z)$ for all $z \in \overline{\Delta}$, is open. To see that this condition is closed in t we first apply Hurwitz's theorem: the limit of nonzero holomorphic functions is either nonzero or identically zero. Thus we only need to consider the cases $z = \lambda \in \mathbf{S}^1$.

In proving the boundary uniqueness we use the *linear* Riemann-Hilbert problem. Namely, we have for each $\lambda \in \mathbf{S}^1$

$$\Re(a(\lambda)(g_{t,\eta} - g_{t,\zeta})(\lambda)) = 0,$$

where

$$a(\lambda) = \int_0^1 \partial_z|\Phi|(\lambda, g_{t,\eta}(\lambda) + s(g_{t,\eta} - g_{t,\zeta})(\lambda))ds.$$

Since $\partial_z|\Phi|(\lambda, z)$ is smooth and nonzero when $z \neq 0$, by the mean value theorem $a(\lambda)$ is nonzero when $g_{t,\eta}$ and $g_{t,\zeta}$ are sufficiently close in C^α. Continuity in t proves that then $a(\lambda)$ has zero winding number about 0 and thus $a(\lambda) = e^{\alpha_1(\lambda)+i\beta_1(\lambda)}$ where α_1 and β_1 are Hölder continuous. If $g_{t,\eta} - g_{t,\zeta}$ is not identically zero, by Hurwitz's theorem it was nonzero in the open unit disk. Therefore we obtain

$$\Im(\log(g_{t,\eta} - g_{t,\zeta}(\lambda)) = \frac{\pi}{2} - \beta_1(\lambda).$$

The harmonic conjugate of β_1 is continuous, hence bounded and it follows that $g_{t,\eta}(\lambda) \neq g_{t,\zeta}(\lambda)$ for all $\lambda \in \mathbf{S}^1$, at least when $g_{t,\eta}$ and $g_{t,\zeta}$ are close enough. Thus each value $g_{t,\eta}(\lambda)$ determines $g_{t,\eta}$ locally. But since $g_{t,\eta}(\lambda)$ must stay on the curve

$\Phi_\lambda^{-1}(\mathbf{S}^1(t))$, continuity in t proves the global claim, that for different solutions of (12), $g_{t,\eta}(\lambda) \neq g_{t,\zeta}(\lambda)$ at every $\lambda \in \mathbf{S}^1$.

It now remains to show that the set of t's for which there is a solution g_t, is closed. One should not immediately jump to the conclusion that this follows from a standard compactness argument: We must wind up with a limit function for which the above implicit function and uniqueness arguments work, in other words, the limit function must be of the appropriate smoothness class $C^{1,\alpha}$. The major technical part of the proof is therefore to find *a'priori* estimates for the solutions of equation (12). That is assuming $g_t \in C^{1,\alpha}$ we must prove that

$$\|g_t\|_{C^{1,\alpha}} \leq C_0, \tag{17}$$

where C_0 is an absolute constant independent of t.

Letting $\lambda = e^{\imath\theta}$ and differentiating (12) with respect to θ gives

$$\partial_\theta|\Phi|(\lambda, g_t(\lambda)) - 2\Im\left(\lambda g_t'(\lambda)\partial_z|\Phi|(\lambda, g_t(\lambda)\right) = 0. \tag{18}$$

And as above $\partial_z|\Phi|(\lambda, g_t(\lambda)) = e^{\alpha(\lambda)+\imath\beta(\lambda)}$ so that

$$2\Im\left(\lambda g_t'(\lambda)e^{\imath(\beta+\imath H\beta)(\lambda)}\right) = e^{-(\alpha+H\beta)(\lambda)}\partial_\theta|\Phi|(\lambda, g_t(\lambda)). \tag{19}$$

Now it will be enough to prove that for each $p \in (1,\infty)$

$$\|e^{\pm H\beta(\lambda)}\|_{L^p(S^1)} \leq C(p) < \infty \tag{20}$$

with C(p) independent of t. Indeed as the Hilbert transform is bounded on L^p and Φ is smooth equations (19) and (20) give

$$\|g_t'(\lambda)e^{\imath(\beta+\imath H\beta)(\lambda)}\|_{L^{2p}(S^1)} \leq C_1C(2p)$$

and so

$$\|g_t'(\lambda)\|_{L^p(S^1)} \leq C_1C(2p)\|e^{H\beta(\lambda)}\|_{L^{2p}(S^1)} \leq C_1C(2p)^2. \tag{21}$$

According to a theorem of Hardy and Littlewood [D p. 84] we then obtain a uniform bound for $\|g_t\|_\alpha$, where $\alpha = 1 - \frac{1}{p}$, which from the previous estimates yields the desired bounds on $\|g_t'\|_\alpha$.

Therefore to complete the proof of Theorem 4.1 we must obtain the estimate of equation (20) for

$$\beta = \Im\left(\log(\partial_z|\Phi|(\lambda, g_t(\lambda)))\right). \tag{22}$$

It is of course clear that the L^p–norm of (20) is at least bounded. To obtain the uniform estimate that we need, note that a continuity argument implies that $\partial_z|\Phi|(\lambda, z)$ is null homotopic as a map from $M_t = \{(\lambda, z) : |\Phi|(\lambda, z) = t\}$ to $\mathbf{C}\backslash\{0\}$. Therefore one can write

$$\Im\left(\log(z\partial_z|\Phi|(\lambda, z))\right) = \mathcal{B}(\lambda, z)$$

where $\mathcal{B}$ is continuous on M_t. Applying Mergeleyan's Theorem we have

$$\mathcal{B} = \Re(P_t) + R_t$$

on M_t, where P_t is a polynomial in λ, $\overline{\lambda}$, and z and the remainder term can be made small $|R(\lambda, z)| < \epsilon$. By smoothness P_t varies continuously in t and as $\beta \equiv 0$ for t small we can choose P_t with a constant degree $m = \deg(P_t)$ and such that the coefficients have upper bounds also independent of $t \in (0,1]$. For simplicity set

$$u(\lambda) = P(\lambda, g_t(\lambda)) \quad \text{and} \quad v(\lambda) = R(\lambda, g_t(\lambda))$$

As g_t is holomorphic in Δ and continuous in $\overline{\Delta}$ by an explicit term by term calculation we see that

$$\|Hu(\lambda)\|_\infty \leq C_0 \|g_t\|_\infty^m \tag{23}$$

where C_0 depends only on P and so can be bounded by a fixed constant. To estimate the other term note that if we choose ϵ so small that $p\epsilon \leq \frac{\pi}{2}$, then $|v(\lambda)| < \epsilon$ implies that

$$\int_{\mathbf{S}^1} e^{pHv(\lambda)} |d\lambda| \leq \frac{1}{\cos(\epsilon\pi)} \tag{24}$$

c.f. [D p. 57]. Finally, as

$$\beta(\lambda) = -\arg g_t(\lambda) + \mathcal{B}(\lambda, g_t(\lambda))$$

it follows that $H\beta = \log|g_t| + Hv + Hu$. And as $|g_t(z)|$ is bounded away from 0 and ∞ when t is not close to 0 we get (20). This now proves Theorem 4.1. □

It is the following consequence, in itself quite surprising, that we shall need.

Theorem 4.2. *Let $\phi : S^1 \times \mathbf{C} \to \mathbf{C}$, $\phi(0) = 0$ be a diffeotopy. Then the curves $\phi(\lambda, \mathbf{S}^1(t))$, $0 < t < \infty$, can be uniquely reparameterised and extended to be a holomorphic motion Φ. That is, there is a holomorphic motion $\Phi : \overline{\Delta} \times \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ and a foliation of $\mathbf{C} - \{0\}$ by Jordan curves Σ_t, $0 < t < \infty$, each separating 0 from ∞, such that for $\lambda \in \mathbf{S}^1$ and $0 < t < \infty$*

$$\Phi(\lambda, \Sigma_t) = \phi(\lambda, \mathbf{S}^1(t)). \tag{25}$$

Proof:- Firstly $C_\lambda^t = \phi(\lambda, S_t)$ is a smoothly varying family of smooth Jordan curves each separating 0 from ∞. If $\zeta \in C_1^t$, apply Theorem 4.1 to the family $\{C_\lambda^t : \lambda \in \mathbf{S}^1\}$ and let $g_\zeta^t(z)$ be the holomorphic fuction in Δ solving the corresponding nonlinear Riemann–Hilbert problem with $g_\zeta^t(1) = \zeta$. Then define

$$\Psi(z, w) = g_\zeta^t(z), \quad \text{if } \ w = g_\zeta^t(0). \tag{26}$$

We claim that Ψ is a well defined holomorphic motion of the whole complex plane.

To see this choose distinct points $\zeta \in C_1^t$ and $\eta \in C_1^s$. If $t = s$, we see $g_\zeta^t(z) \neq g_\eta^s(z)$ as $\zeta \to g_\zeta^t(z)$ is injective by Theorem 4.1. If $t < s$, g_η^s is nonvanishing and $g_\zeta^t(z) \in C_z^t$ for $|z| = 1$. Hence the maximum principle implies that for t sufficiently small

$$|g_\zeta^t(z)| < |g_\eta^s(z)|$$

for all $z \in \overline{\Delta}$. Then of course $g_\zeta^t(z) \neq g_\eta^s(z)$ in $\overline{\Delta}$ and as $g_\zeta^t(z)$ is continuous in ζ (by the implicit function theorem argument of Theorem 4.1), we can again use Hurwitz's Theorem, that the limit of nonvanishing analytic functions is either nonvanishing analytic or identically zero. It follows that the set of $t < s$ for which $g_\zeta^t(z) \neq g_\eta^s(z)$ is both open and closed and so of course

$$g_\zeta^t(z) \neq g_\eta^s(z) \quad \text{whenever } \ \zeta \neq \eta$$

Next note that $\Psi(z, w) = g_\zeta^t(z)$ is holomorphic in z, injective in w and $\Psi(0, w) = w$; thus it is a holomorphic motion of the set

$$A = \{w \in \mathbf{C} : w = g_\zeta^t(0), 0 < t < \infty, \zeta \in C_1^t\}. \tag{27}$$

But the mapping $\zeta \to g^t_\zeta(0)$ extends to a homeomorphism of $\overline{\mathbf{C}}$ onto $A \cup \{0, \infty\}$. Thus $A = \mathbf{C} - \{0\}$ and Ψ defines a holomorphic motion of $\overline{\mathbf{C}}$.

Finally, let

$$\Sigma_t = \{w \in \mathbf{C} : w = g^t_\zeta(0), \zeta \in C^t_1\}$$

Then for $\lambda \in \mathbf{S}^1$

$$\Psi(\lambda, \Sigma_t) = \phi(\lambda, \mathbf{S}^1(t))$$

so that $\{\Sigma_t : 0 < t < \infty\}$ is a foliation of $\mathbf{C} - \{0\}$ consisting of Jordan curves each separating 0 from ∞. Thus Ψ is a reparameterisation of ϕ.

5. The Completion of the Proof

The proof of Theorem 3.2 is now more or less complete. One first establishes the following compactness result for holomorphic motions. It is quite straightforward using the λ–lemma, Theorem 3.1.

Lemma 5.1. *Let $\{A_n\}$ be an increasing sequence of subsets of $\overline{\mathbf{C}}$ and Φ_n a sequence of holomorphic motions of $\mathbf{C}$ with*

$$\Phi_{n+1}|A_n = \Phi_n|A_n. \tag{28}$$

Then there is a limit holomorphic motion $\Phi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that $\Phi|A_n = \Phi_n|A_n$.

Next one proves the holomorphic axiom of choice. It simply says that a holomorphic motion of a finite point set can be extended to include any other arbitrary point.

Theorem 5.1. *Let $f_i : \Delta \to \overline{\mathbf{C}}$, $i = 1, 2, \ldots, n$ be holomorphic functions such that for each $z \in \Delta$ we have $f_i(z) \neq f_j(z), i \neq j$. Then for each $z_{n+1} \in \mathbf{C} - \{f_i(0) : i = 1, 2, \ldots, n\}$ there is a holomorphic function $f_{n+1} : \Delta \to \mathbf{C}$ such that $f_{n+1}(0) = z_{n+1}$ and for all $i = 1, 2, \ldots, n$ and $z \in \Delta$ we have $f_{n+1}(z) \neq f_i(z)$*

Proof:- A straightforward normalisation will imply that we can assume $f_1(z) \equiv 0$. Choose $r < 1$ and consider the diffeotopy of the finite point set $A = \{f_i(r) : i = 1, 2, \ldots, n\}$ defined by

$$\phi(\lambda, f_i(r)) = f_i(r\lambda) \tag{29}$$

which is, as $r < 1$, parametrised smoothly by the circle $|\lambda| = 1$. It is easy to see how to extend this diffeotopy of a finite point set to an ambient diffeotopy of $\mathbf{C}$. For instance one may integrate a suitable vector field (an extension of the vector field for which ϕ already gives the integral curves) as in [S] p 350.

Now let $\Psi : \Delta \times \mathbf{C} \to \mathbf{C}$ be the holomorphic motion and Σ_t, $0 < t < \infty$ the foliation with

$$\Psi(\lambda, \Sigma_t) = \phi(\lambda, \mathbf{S}^1(t)) \quad \lambda \in \mathbf{S}^1,\ t > 0,$$

that was constructed in Theorem 4.2. As $f_i(rz)$ is already a solution to the nonlinear Riemann–Hilbert problem

$$g(\lambda) \in \phi(\lambda, \mathbf{S}^1(t)) = \Psi(\lambda, \Sigma_t), \quad \lambda \in \mathbf{S}^1, \tag{30}$$

where $t = |f_i(r)|$, we have by uniqueness that $\Psi(z, w) = f_i(rz)$ for $w = f_i(0)$ and for all $i = 1, 2, \ldots, n$. In particular of course, $\Psi(z, z_{n+1}) \neq f_i(rz)$ whenever $z \in \Delta$. The result now follows by compactness as we let $r \to 1$.$\square$

The proof of Theorem 3.3 is now clear. Given a holomorphic motion Φ of a set A we choose a countable dense subset $\{a_1, a_2, \dots, a_n, \dots\}$ of A, set $A_n = \{a_1, a_2, \dots, a_n\}$ and define $\Phi_n = \Phi|A_n$. Extend each Φ_n to a holomorphic motion of the plane and use the lemma above to wind up with a limiting holomorphic motion. It's easy to check (using Theorem 3.1) that the limit is an extension of the initial motion. Once we have this we see that for each $\lambda \in \Delta$ the map $\Phi(\lambda, \cdot)$ is quasiconformal by the ordinary λ–lemma. But now the derivatives of $\Phi(\lambda, \cdot)$ are also easily seen to move holomorphically and therefore so does the complex dilatation μ_λ. But $\mu_0 \equiv 0$ and $\|\mu_\lambda\|_\infty < 1$ for all $\lambda \in \Delta$. Therefore the Schwarz lemma gives

$$\|\mu_\lambda\|_\infty \leq |\lambda|$$

from which we deduce that the dilatation at time λ is

$$K_\lambda = \frac{1 + \|\mu_\lambda\|_\infty}{1 - \|\mu_\lambda\|_\infty} \leq \frac{1 + |\lambda|}{1 - |\lambda|}.$$

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