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DEPARTMENT OF MATHEMATICS
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EXISTENCE AND UNIQUENESS OF $p(x)$-HARMONIC FUNCTIONS FOR BOUNDED AND UNBOUNDED $p(x)$

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0 Introduction

In this licentiate thesis we study the Dirichlet boundary value problem

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & \text{if } x \in \Omega, \\ \quad\quad u(x) = f(x), & \text{if } x \in \partial\Omega. \end{cases} \tag{0.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p : \Omega \to (1, \infty]$ a measurable function, $f : \partial\Omega \to \mathbb{R}$ the boundary data, and $-\Delta_{p(x)} u(x)$ is the $p(x)$-Laplace operator, which is written as

$$-\Delta_{p(x)} u(x) = -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2} \nabla u(x)\right)$$

for finite $p(x)$. If $p(\cdot) \equiv p$ for $1 < p < \infty$, then the $p(x)$-Laplace operator reduces to the standard p-Laplace operator

$$-\Delta_p u(x) = -\operatorname{div}\left(|\nabla u(x)|^{p-2} \nabla u(x)\right).$$

For $p(x) = \infty$ we consider the infinity Laplace operator,

$$-\Delta_\infty u(x) = -\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x), \tag{0.2}$$

which is formally derived from the p-Laplace equation $-\Delta_p u = 0$ by sending p to infinity. We study the problem (0.1) in different cases, depending on the function p, and our aim is to show that this problem has a unique solution in each of these cases. The merit of this work is in that it is the first survey of the known existence and uniqueness results for (0.1), although not complete, since the borderline cases $p = 1$ and $\inf p(x) = 1$ are outside the scope of this work.

We consider five different cases, all with different assumptions on $p(x)$, and all of them have been distributed into their own sections. The theory is presented in chronological order thus following how the research on the $p(x)$-Laplace operator has advanced.

The first case is the most well-known case where p is a constant function, $1 < p < \infty$. The p-harmonic functions (i.e., the solutions of the p-Laplace equation $-\Delta_p u = 0$) have been widely studied during the past sixty years. By studying p-harmonic functions one learns a lot about different fields of mathematics, such as calculus of variations and partial differential equations, and it is a good model case for more general nonlinear elliptic equations. The theory of p-harmonic functions is quite well developed, albeit there are some well-known open problems left, see, e.g., the survey [24]. In the first section we present the direct method of calculus of variations to find a function u that is a unique minimizer to the Dirichlet energy integral

$$I(v) := \int_\Omega |\nabla v(x)|^p \, dx$$

I would like to thank Professor Petri Juutinen for introducing me to the theory of $p(x)$-harmonic functions and for many valuable discussions and advice. I also want to thank Professor Julio D. Rossi and Docent Petteri Harjulehto for reviewing this thesis and for their valuable comments. For financial support I am indebted to the University of Jyväskylä and to the Vilho, Yrjö and Kalle Väisälä Foundation.

in the set $\{v \in W^{1,p}(\Omega) : v - f \in W_0^{1,p}(\Omega)\}$. Then we show that there is a one to one correspondence between the minimizer of the energy integral and problem (0.1), i.e., u is a minimizer of the integral if and only if u solves (0.1). The same method will be used in Sections 3 and 4.

In the second case we study infinity harmonic functions (i.e., the solutions of the infinity Laplace equation $-\Delta_\infty u = 0$). The articles by Gunnar Aronsson ([4], [5] and a handful of others) in the 1960s were the start-up point in this field. Aronsson started by studying the optimal Lipschitz extensions and found the connection between them and infinity harmonic functions. At that time viscosity solutions had not yet been discovered and the expression (0.2) could be verified only for $\mathcal{C}^2$-functions. Later, in the 1980s, the concept of viscosity solutions was presented and it gave a new view to examine the infinity harmonic functions. Using p-harmonic approximation, it was easy to prove that the problem (0.1) has a solution, but the uniqueness was harder to prove. Jensen proved that first in 1993 in his paper [21]. In 2001, Crandall, Evans and Gariepy used viscosity solutions to prove the connection that Aronsson found, see [9]. This is presented in Theorem 2.15. In 2009, Armstrong and Smart [3] found a new, easy way to prove the uniqueness. The research of infinity harmonic functions is quite intensive today. Especially, the regularity of infinity harmonic functions is still an unsolved problem in dimensions three or higher. The infinity harmonic equation also works as a model case for more general Aronsson-Euler equations. We recommend the survey [6] by Aronsson, Crandall and Juutinen to get a wider picture of infinity harmonic functions and related problems.

In the third section we assume that the function p is non-constant and bounded, and $1 < \inf p(x) < \sup p(x) < \infty$. The research of variable exponent Lebesgue and Sobolev spaces started in 1991, when the seminal paper by Kováčik and Rákosník [23] was published. They defined spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$ and proved the basic properties of these spaces. Although the $L^{p(\cdot)}$-functions share many properties with standard L^p-functions, they, in general, lack a so-called $p(x)$-mean continuity property. Because of this, the convolution of an $L^{p(\cdot)}$-function and a $\mathcal{C}_0^\infty$-function does not belong to $L^{p(\cdot)}$ in general. Then the standard convolution approximation, which is familiar from the L^p-spaces, cannot be generalized to variable exponent Lebesgue spaces, and hence the density of smooth functions in $L^{p(\cdot)}$ becomes an untrivial issue. However, by adding some extra conditions on $p(x)$, from which the log-Hölder continuity

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x-y|)}$$

is the most important, this problem can be avoided and the density problem solved, see [12] and [26]. The log-Hölder continuity is also a sufficient condition on $p(x)$ so that the variable exponent Sobolev-Poincaré inequality holds. This inequality is crucial when proving the existence of the solution of (0.1). After solving these difficulties, the variable $p(x)$-case becomes practically identical with the constant p-case from the standpoint of the problem (0.1). The research of variable exponent spaces is a growing field in nonlinear analysis nowadays, and there are still many open problems left and waiting to be solved. To get a comprehensive view of variable exponent Lebesgue and Sobolev spaces, we ask the reader to get acquainted with a book by Diening, Harjulehto, Hästö and Růžička [11].

In the fourth case we assume that $p(x) \equiv \infty$ in a subdomain D of Ω, and that $p|_{\Omega \setminus D}$ is $\mathcal{C}^1$-smooth and bounded. This case is based on the article [25] by Manfredi, Rossi and Urbano. Their article was the first attempt to analyze the problem (0.1) when the variable exponent is not bounded. The approach to this case is the following: first find a unique solution u_k to the problem (0.1) with $p_k(x) = \min\{k, p(x)\}$ by using the direct method of calculus of variations, then get estimates for u_k and ∇u_k that are independent of k, after that pass to the limit $k \to \infty$ to get a limit function u_∞ and find out what equation it solves.

In the final, fifth, case we consider a one-dimensional case where p is continuous and unbounded on the interval (a, b) with $\lim_{x \to b-} p(x) = \infty$, and infinity on (b, c); here $a < b < c$. The unboundedness of p on (a, b) causes different problems that we discuss. In view of these problems it is easy to see why the boundedness assumption for $p|_{\Omega \setminus D}$ was made in the fourth section. Our analysis will not be very deep, and we shall only draw some guidelines. The generalizations of our results to higher dimensions is an interesting open problem.

Next we shall say something about the cases that have been left out of this work. In the case $p = 1$ the (standard) p-Laplace operator is

$$-\Delta_1 u(x) = -\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right).$$

In the associated variational problem we should try to find a unique minimizer for the functional I, where

$$I(v) := \int_\Omega |\nabla v(x)|\, dx,$$

in the set $\{v \in W^{1,1}(\Omega) : v - f \in W_0^{1,1}(\Omega)\}$. The first problem now is that $W^{1,1}(\Omega)$ is not reflexive and I is not strictly convex, and thus the direct method of calculus of variations is no more applicable. The corresponding problem can be formulated in $BV(\Omega)$ (the class of functions $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω) as follows: for $f \in \mathcal{C}(\partial\Omega)$, find a minimizer for $\|\nabla v\|(\Omega)$ in the set $\{v \in BV(\Omega) \cap \mathcal{C}(\overline{\Omega}), u = f \text{ on } \partial\Omega\}$. Here

$$\|\nabla v\|(\Omega) = \sup\left\{\int_\Omega v \operatorname{div}\sigma\, dx : \sigma \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n), |\sigma(x)| \le 1 \text{ for } x \in \Omega\right\}.$$

If Ω is a Lipschitz domain, then it is quite easy to construct the minimizer for this problem in the space $BV(\Omega)$, but the question that the minimizer is continuous and satisfies the boundary condition, is more subtle. The behaviour of $\partial\Omega$ plays an essential role in this question. By assuming that $\partial\Omega$ has non-negative mean curvature in a weak sense and that $\partial\Omega$ is not locally area-minimizing, the continuity and the boundary condition of the minimizer can be verified. On the other hand, if neither condition on $\partial\Omega$ is true, then it is possible to construct a boundary data f such that the corresponding problem has no solution. The uniqueness of solutions can also be obtained under the aforementioned assumptions on $\partial\Omega$. Since the nature of this problem is quite different from the cases we treat, we do not consider it any further in this work. See [27] for a detailed study of this case and [22], where the minimizers of the above problem are approximated by p-harmonic functions as $p \to 1$. See also [18], in which the authors consider the minimization problem related to (0.1) in the case where the variable exponent $p : \Omega \to [1, \infty)$ attains the value 1.

The regularity of solutions of (0.1) is beyond the scope of this thesis. If the reader is interested in regularity, we recommend the following articles/surveys: [24] for p-harmonic functions, [1], [2] and [7] for variable $p(x)$-harmonic functions, and [13] and [14] for infinity harmonic functions.

0.1 Notation and prerequisities

We shall use the following notation throughout this thesis. For a set A in the euclidean space $\mathbb{R}^n$, ∂A is its boundary and $\overline{A}$ is its closure. The notation $A \subset\subset B$ means that A is an open subset of B whose closure $\overline{A}$ is a compact subset of B. The euclidean distance between two sets, A and B, is denoted by $\operatorname{dist}(A, B)$, and $\operatorname{dist}(x, A)$ is the distance from x to A. The diameter of A is $\operatorname{diam}(A)$.

Throughout this work we shall assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain (=open and connected) with $n \geq 2$, except in Section 5 where Ω is an open interval in $\mathbb{R}$. The Lebesgue measure of a measurable set A is denoted by $|A|$. The open ball with center at $x \in \mathbb{R}^n$ and radius $r > 0$ is $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. The boundary of the ball, $\{y \in \mathbb{R}^n : |x - y| = r\}$, is denoted by $\partial B(x, r)$ or, equivalently, by $S(x, r)$.

The class of continuous functions in A is denoted by $\mathcal{C}(A)$. For an open set A, the class $\mathcal{C}^k(A)$ consists of all k times continuously differentiable functions $u : A \to \mathbb{R}$. When we say that $u : A \to \mathbb{R}$ is a classical solution (or smooth solution) to some equation, we mean that u is at least twice continuously differentiable, and when we talk about the density of smooth functions, we mean by the word *smooth* that the functions are infinitely many times continuously differentiable, that is, they are members of the class $\mathcal{C}^\infty(A)$. By $\mathcal{C}_0^k(A)$ we denote all the functions $u \in \mathcal{C}^k(A)$, for which the support of u, $\operatorname{spt} u$, is a compact subset of A. A function $u \in \mathcal{C}^k(A)$ belongs to class $\mathcal{C}^{k,\alpha}(A)$, if all k-th order derivatives of u are Hölder continuous with exponent $0 < \alpha < 1$. For a Lipschitz continuous function $f : A \to \mathbb{R}$ we denote the Lipschitz constant of f by

$$\operatorname{Lip}(f, A) = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y \right\}.$$

Let A be a measurable set and $1 \leq p \leq \infty$. We define a Lebesgue space $L^p(A)$ by

$$L^p(A) := \{u : A \to \mathbb{R} \text{ is measurable } : \|u\|_{L^p(A)} < +\infty\},$$

where the L^p-norm of u is

$$\|u\|_{L^p(A)} := \left(\int_A |u(x)|^p \, dx \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$ and

$$\|u\|_{L^\infty(A)} := \operatorname*{ess\,sup}_{x \in A} |u(x)|$$

for $p = \infty$. Then $L^p(A)$ is Banach space. The space $L^p_{\text{loc}}(A)$ consists of all measurable functions u for which $u \in L^p(K)$ for every compact $K \subset A$. If

$p \in (1,\infty)$, then the dual space of $L^p(A)$ is $L^q(A)$, where $1/p + 1/q = 1$, and the space $L^p(A)$ is reflexive.

For an open set B, the Sobolev space $W^{1,p}(B)$ consists of functions $u \in L^p(B)$, whose weak gradient ∇u belongs to $L^p(B)$. The space $W^{1,p}(B)$ is Banach space with the norm

$$\|u\|_{W^{1,p}(B)} := \|u\|_{L^p(B)} + \|\nabla u\|_{L^p(B)}.$$

The local space $W^{1,p}_{\text{loc}}(B)$ is defined in the same way as the local L^p-space, and the Sobolev space with zero boundary values, $W^{1,p}_0(B)$, is defined as the closure of $\mathcal{C}_0^\infty(B)$ with respect to the Sobolev norm $\|\cdot\|_{W^{1,p}(B)}$. For $p \in (1,\infty)$, the space $W^{1,p}(B)$ is reflexive and the dual space is $W^{1,q}(B)$; here $1/p + 1/q = 1$.

We present the inequalities that will be used frequently. The Hölder inequality

$$\int_A |f(x)g(x)|\ dx \leq \|f\|_{L^p(A)}\|g\|_{L^q(A)}$$

holds for $f \in L^p(A)$ and $g \in L^q(A)$, where $1/p+1/q = 1$. The Sobolev inequality

$$\|\nabla u\|_{L^p(B)} \leq C(n,p,B)\|\nabla u\|_{L^p(B)}$$

is true if B is an open set with finite measure, $u \in W^{1,p}_0(B)$ and $1 < p < \infty$. If $n < p < \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$, then Morrey's inequality

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{1-\frac{n}{p}}\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

holds for every $x, y \in \mathbb{R}$. For the proofs of the inequalities above and for more information concerning Lebesgue and Sobolev spaces, we ask reader to see [16] and [29].

We assume that the reader of this work knows the basics from functional analysis and from measure and integration theory. Also the knowledge on calculus of variations and on partial differential equations helps the reader to follow the text. For example, the book by Giusti [16] is of great help and gives more than sufficient prerequisities.

1 Constant p, $1 < p < \infty$

In this section, we consider the case where the function p is constant, $1 < p < \infty$. Then the Dirichlet boundary value problem is written as

$$\begin{cases} -\Delta_p u(x) = 0, & \text{if } x \in \Omega, \\ \quad\;\; u(x) = f(x), & \text{if } x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here $f : \partial\Omega \to \mathbb{R}$ is the boundary data and

$$\begin{aligned} -\Delta_p u(x) &= -\operatorname{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right) \\ &= -|\nabla u(x)|^{p-4}\left[|\nabla u(x)|^2\Delta u(x) + (p-2)\sum_{i,j=1}^{n} u_{x_i}(x)u_{x_j}(x)u_{x_ix_j}(x)\right] \end{aligned}$$

is the p-Laplace operator. Our aim is to show that the problem (1.1) has a unique solution.

Since the class of classical solutions (i.e., $\mathcal{C}^2$-functions for which $-\Delta_p u(x) = 0$ and $u(x) = f(x)$ could be verified pointwise) is too small to treat the problem (1.1) properly, we need to use the concept of weak solutions. Then the boundary values are determined by the function $f \in W^{1,p}(\Omega)$. The fact that an arbitrary function u is a weak solution to the problem (1.1) means two things:

1) u is a weak solution of the equation $-\Delta_p u = 0$ in Ω.

2) u equals f on $\partial\Omega$ in the Sobolev sense, that is, $u - f \in W_0^{1,p}(\Omega)$.

The concept of weak solutions of the equation $-\Delta_p u = 0$ is deduced from the classical solutions of that equation. Indeed, suppose that $u \in \mathcal{C}^2(\Omega)$ and $-\Delta_p u(x) = 0$ pointwise in Ω. Then

$$\int_\Omega -\Delta_p u(x)\,\varphi(x)\,dx = 0$$

for every $\mathcal{C}_0^\infty(\Omega)$. Using integration by parts, we obtain

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,dx = 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. This expression is the natural interpretation of $-\Delta_p u = 0$ in the weak sense.

Definition 1.1. We say that a function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a *weak solution* (respectively, *subsolution, supersolution*) of the equation $-\Delta_p u = 0$ in Ω, if

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla\varphi(x)\,dx = 0 \text{ (respectively, } \le 0,\ \ge 0)$$

for every test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (respectively, for every non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$).

A continuous weak solution of $-\Delta_p u = 0$ is called a *p-harmonic function.*

We want to remark that every weak solution of $-\Delta_p u = 0$ can be redefined in a set of zero Lebesgue measure such that the new function is continuous. Even more can be said about the regularity of weak solutions. In fact, they belong to class $\mathcal{C}^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha > 0$, see [10].

We consider $|0|^{p-2}0$ as 0 also when $1 < p < 2$. It would be *a priori* enough to assume that $\nabla u \in L^{p-1}_{\text{loc}}(\Omega)$ to ensure that the integral is finite. However, not much can be done with this relaxation (for example, Cacciopoli estimates), so we stick with the space $W^{1,p}_{\text{loc}}(\Omega)$.

In Definition 1.1 we used $\mathcal{C}^\infty_0(\Omega)$-functions to the test if the integral is zero. Sometimes it is useful to use a wider class of test functions. This is possible if we assume that the weak solution belongs to the global space $W^{1,p}(\Omega)$; then $\mathcal{C}^\infty_0(\Omega)$ can be replaced by $W^{1,p}_0(\Omega)$. Indeed, suppose that $u \in W^{1,p}(\Omega)$ is a weak solution of $-\Delta_p u = 0$ and $v \in W^{1,p}_0(\Omega)$. Since the function v can be approximated by a sequence of $\mathcal{C}^\infty_0(\Omega)$-functions (φ_j) in the norm $\|\cdot\|_{W^{1,p}(\Omega)}$, we have that

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx \\
&\qquad = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla\varphi_j) \, dx + \underbrace{\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi_j \, dx}_{=0} \\
&\qquad (\star) \leq \left(\int_\Omega |\nabla u|^p \, dx \right)^{p-1} \left(\int_\Omega |\nabla v - \nabla\varphi_j|^p \, dx \right)^{\frac{1}{p}},
\end{aligned}$$

where $(\star)$ follows from Hölder's inequality. Since the first term on the right hand side is bounded and the second term tends to zero as $j \to \infty$, we finally get

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = 0.$$

Note that the step $(\star)$ is no longer true for $u \in W^{1,p}_{\text{loc}}(\Omega)$.

Example 1.2. a) Constant functions and linear functions are p-harmonic.

b) The function $u_F : \Omega \to \mathbb{R}$, where

$$u_F(x) := \begin{cases} |x|^{\frac{p-n}{p-1}}, & \text{when } p \neq n, \\ \log|x|, & \text{when } p = n, \end{cases}$$

is p-harmonic in Ω, if $0 \notin \Omega$. This is true since $u_F \in \mathcal{C}^2(\Omega)$ and $-\Delta_p u_F(x) = 0$ pointwise in Ω. If $0 \in \Omega$, then $u_F \notin W^{1,p}_{\text{loc}}(\Omega)$ since $u_F \notin W^{1,p}(D)$ for any open D that contains zero. Thus u_F cannot be a weak solution since it does not belong to right space.

Next we formulate the main theorem of this section.

Theorem 1.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < \infty$ and $f \in W^{1,p}(\Omega)$ be the boundary data. Then there exists a unique p-harmonic function $u \in W^{1,p}(\Omega)$ such that $u - f \in W^{1,p}_0(\Omega)$, i.e., u is a weak solution to

$$\begin{cases} -\Delta_p u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The proof of this theorem is based on the direct method of calculus of variations. By that method we find a function $u_0 \in W_f^{1,p}(\Omega)$ that is a unique minimizer to a certain energy integral. After that we show that the minimizers of that energy integral are the same functions as the solutions of the problem (1.1). First we present the method separately, and then we use it for our purposes.

1.1 The direct method of calculus of variations

Let $(X, \|\cdot\|)$ be a reflexive Banach space, $K \subset X$ and $I : K \to \mathbb{R}$ a functional. The direct method of calculus of variations answers to the question how to find a minimizer u_0 for the functional I in K, if it is possible. The steps for this are the following:

1) Show that $\inf\limits_{u \in K} I(u)$ is finite.

2) By the definition of infimum, there exists a sequence $(u_j) \subset K$ such that $I(u_j) < \inf\limits_{u \in K} I(u) + \frac{1}{j}$.

3) Show that there exists $u_0 \in K$, to which the sequence (u_j) converges in a suitable sense.

4) Show that $I(u_0) = \inf\limits_{u \in K} I(u)$, i.e., $I(u_0) \leq I(v)$ for every $v \in K$.

If $\inf_{u \in K} I(u)$ is not finite, then the minimizer does not exist since I is real-valued. Step 2) can always be done since it is based only on the definition of infimum. How the steps 3) and 4) are done depends on the space X, on the set K and on the functional I.

1.2 Dirichlet energy integral

Let $X = W^{1,p}(\Omega)$, $K = W_f^{1,p}(\Omega)$ for given $f \in W^{1,p}(\Omega)$, and $I : K \to \mathbb{R}$,

$$I(u) = \int_\Omega |\nabla u(x)|^p \, dx.$$

The integral $I(u)$ is the so-called Dirichlet energy integral or p-energy integral. The functional I is weakly lower semicontinuous in K, i.e.,

$$I(u) \leq \liminf_{j \to \infty} I(u_j)$$

for every sequence $(u_j) \subset K$ for which $u_j \rightharpoonup u \in K$ weakly in K. This follows from the fact that the $\|\cdot\|_{L^p(\Omega;\mathbb{R}^n)}$-norm is weakly lower semicontinuous.

The next theorem shows that in this setting the direct method of calculus of variations works and we find a minimizer for the functional I in the set $W_f^{1,p}(\Omega)$. Furthermore, the minimizer can be proven to be unique by using the properties that this particular I has.

Theorem 1.4. Let $f \in W^{1,p}(\Omega)$. There exists a unique $u \in W_f^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^p \, dx \leq \int_\Omega |\nabla v(x)|^p \, dx$$

for every $v \in W_f^{1,p}(\Omega)$.

Proof. Let

$$I_0 = \inf_{v \in W_f^{1,p}(\Omega)} \int_\Omega |\nabla v(x)|^p \, dx.$$

Then $0 \le I_0 \le \int_\Omega |\nabla f(x)|^p \, dx < +\infty$ and we can choose functions $v_1, v_2, v_3, ... \in W_f^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla v_j(x)|^p \, dx < I_0 + \frac{1}{j} \tag{1.3}$$

for $j = 1, 2, 3, ...$ The Sobolev inequality holds for $v_j - f$ and hence

$$\begin{aligned} \|v_j\|_{L^p(\Omega)} &\le \|v_j - f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \\ &\le C\|\nabla v_j - \nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \\ &\le C\|\nabla v_j\|_{L^p(\Omega)} + C\|\nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \\ &\le C(I_0 + 1)^{\frac{1}{p}} + (C+1)\|f\|_{W^{1,p}(\Omega)}, \end{aligned}$$

where $C = C(n, p, \Omega)$ is the constant from the Sobolev inequality. This together with (1.3) implies that the sequence $(v_j)_{j=1}^\infty$ is bounded in $W^{1,p}(\Omega)$, and thus there exists a subsequence, still denoted as $(v_j)_{j=1}^\infty$, and a function $u \in W^{1,p}(\Omega)$ such that $v_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Since $W_f^{1,p}(\Omega)$ is weakly closed and nonempty, we deduce that $u \in W_f^{1,p}(\Omega)$. Then

$$I_0 \le \int_\Omega |\nabla u(x)|^p \, dx \le \liminf_{j \to \infty} \int_\Omega |\nabla v_j(x)|^p \, dx = I_0$$

by the weak lower semicontinuity of I. This proves the existence.

For the uniqueness, we use strict convexity ($p > 1$). Let u_1 and u_2 be two minimizers. Then for $u = \frac{u_1 + u_2}{2} \in W_f^{1,p}(\Omega)$ we have

$$\begin{aligned} \int_\Omega |\nabla u_1(x)|^p \, dx \le \int_\Omega |\nabla u(x)|^p \, dx &= \int_\Omega \left| \frac{\nabla u_1(x) + \nabla u_2(x)}{2} \right|^p \, dx \\ &\le \int_\Omega \frac{|\nabla u_1(x)|^p + |\nabla u_2(x)|^p}{2} \, dx \\ &= \frac{1}{2} \int_\Omega |\nabla u_1(x)|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u_2(x)|^p \, dx \\ &= \int_\Omega |\nabla u_1(x)|^p \, dx, \end{aligned}$$

since $\int_\Omega |\nabla u_1(x)|^p \, dx = \int_\Omega |\nabla u_2(x)|^p \, dx$. If $\nabla u_1(x) \ne \nabla u_2(x)$ in a set of positive measure, then the second inequality is strict by the strict convexity, which leads to a contradiction. Thus $\nabla u_1(x) = \nabla u_2(x)$ almost everywhere in Ω and hence $u_1 - u_2$ is constant. Since $u_1 - u_2 \in W_0^{1,p}(\Omega)$, the constant must be zero, which yields $u_1 = u_2$. This proves the uniqueness and the claim follows. □

Now we have proved that the functional $I : W_f^{1,p}(\Omega) \to \mathbb{R}$ has a unique minimizer. To prove Theorem 1.3, we show that there is a one to one correspondence between the minimizer of the functional I and the solution of problem (1.1). This can be directly seen from the next theorem.

Theorem 1.5. Let $f \in W^{1,p}(\Omega)$ be the boundary data. Then the following conditions are equivalent for $u \in W_f^{1,p}(\Omega)$:

(a) $-\Delta_p u = 0$ in Ω in the weak sense,

(b) $I(u) \le I(v)$ for every $v \in W_f^{1,p}(\Omega)$.

Proof. "(a) $\implies$ (b)": As discussed earlier, we recall that the test function class $\mathcal{C}_0^\infty(\Omega)$ can be replaced by $W_0^{1,p}(\Omega)$ when testing p-harmonicity, since $u \in W^{1,p}(\Omega)$ by the assumption. To start, let $v \in W_f^{1,p}(\Omega)$. By the convexity of $x \to |x|^p$, $p \ge 1$, we have

$$|\nabla v(x)|^p \ge |\nabla u(x)|^p + p|\nabla u(x)|^{p-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x)).$$

It follows that

$$\int_\Omega |\nabla v(x)|^p\,dx \ge \int_\Omega |\nabla u(x)|^p\,dx + p\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x))\,dx.$$

Since $v - u \in W_0^{1,p}(\Omega)$ is an admissible test function, the last integral is zero, and hence $I(u) \le I(v)$.

"(b) $\implies$ (a)": We assume that $u \in W_f^{1,p}(\Omega)$ is the energy minimizer. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be a test function. Then, by setting $u_t(x) = u(x) + t\varphi(x)$, we have $u_t \in W_f^{1,p}(\Omega)$ and $I(u) = I(u_0) \le I(u_t)$ for every $t \in \mathbb{R}$. Hence

$$\begin{aligned}
0 &= \lim_{t\to 0} \frac{I(u_t) - I(u)}{t} = \lim_{t\to 0} \int_\Omega \frac{|\nabla u_t(x)|^p - |\nabla u(x)|^p}{t}\,dx \\
&= \int_\Omega \lim_{t\to 0} \frac{|\nabla u_t(x)|^p - |\nabla u(x)|^p}{t}\,dx = \int_\Omega \frac{d}{dt}\Big[|\nabla u_t(x)|^p\Big]_{t=0}\,dx \\
&= \int_\Omega p\Big[(\nabla u(x) + t\nabla\varphi(x))|\nabla u(x) + t\varphi(x)|^{p-2} \cdot \nabla\varphi(x)\Big]_{t=0}\,dx \\
&= p\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\,dx.
\end{aligned}$$

In the third equality we use the mean value theorem and Hölder's inequality to see that $\int_\Omega \frac{|\nabla u_t(x)|^p - |\nabla u(x)|^p}{t}\,dx$ is bounded by a constant depending on p, φ, Ω, u, and then by Lebesgue's dominated convergence theorem we may take the limit inside the integral. □

Remark 1.6. According to the terminology of calculus of variations, the p-Laplace equation $-\Delta_p u = 0$ is the *Euler-Lagrange equation* for the variational integral I.

Proof of Theorem 1.3*.* The proof follows directly from Theorems 1.4 and 1.5. □

Another way to prove Theorem 1.3 is to use the theory of monotone operators for the existence and the maximum principle for the uniqueness. See, for example, [19].

2 Infinity harmonic functions, $p \equiv \infty$

At first we derive the infinity Laplace equation,

$$-\Delta_\infty u(x) = -\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = 0, \tag{2.1}$$

from the p-Laplace equation. The calculation is done formally for a p-harmonic function u_p with assumptions $u_p \in \mathcal{C}^2(\Omega)$ and that the gradient of u_p does not vanish. First compute that

$$\begin{aligned} 0 &= -\operatorname{div}\left(|\nabla u_p(x)|^{p-2}\nabla u_p(x)\right) \\ &= -|\nabla u_p(x)|^{p-4}\Big(|\nabla u_p(x)|^2 \Delta u_p(x) + (p-2)\Delta_\infty u_p(x)\Big) \end{aligned}$$

and then divide both sides by $(p-2)|\nabla u_p(x)|^{p-4}$ to get

$$0 = -\frac{|\nabla u_p(x)|^2 \Delta u_p(x)}{p-2} - \Delta_\infty u_p(x).$$

Here Δ is the usual Laplace operator. If $u_p \to u$ in $\mathcal{C}^2(\Omega)$ as $p \to \infty$, then

$$-\frac{|\nabla u_p(x)|^2 \Delta u_p(x)}{p-2} \to 0$$

as $p \to \infty$ and, by the standard theorems in the theory of viscosity solutions, see [8],

$$-\Delta_\infty u_p(x) \to -\Delta_\infty u(x)$$

as $p \to \infty$. Hence,

$$0 = -\Delta_\infty u(x),$$

that is, the limit function u satisfies the infinity Laplace equation in Ω.

For $u \in \mathcal{C}^2(\Omega)$ we can easily calculate $-\Delta_\infty u(x)$ pointwise. However, the class $\mathcal{C}^2(\Omega)$ is too small to solve the Dirichlet boundary value problem

$$\begin{cases} -\Delta_\infty u(x) = 0, & \text{if } x \in \Omega, \\ \quad u(x) = f(x), & \text{if } x \in \partial\Omega. \end{cases}$$

Indeed, in [5], Aronsson proved the uniqueness of smooth (classical) solutions to the Dirichlet problem above, but he also gave examples of cases when the existence could not be obtained. Here is one of his examples.

Example 2.1. This example is based on the result, which is also presented in [5], that if $u \in \mathcal{C}^2(\Omega)$ is a non-constant classical solution to $-\Delta_\infty u = 0$, then $|Du(x)| > 0$ for every $x \in \Omega$.

Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f : S(0,1) \to \mathbb{R}$, $f(x,y) = 2xy$. We assume that there exists a function $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $-\Delta_\infty u = 0$ in Ω and $u = f$ on $\partial\Omega$. Then, by symmetry, the function $(x,y) \mapsto u(-x,-y)$ is also a solution. The uniqueness of solutions then gives that $u(x,y) = u(-x,-y)$. Differentiating with respect to x yields

$$u_x(0,0) = \lim_{t\to 0} \frac{u(-te_1,0) - u(0,0)}{-t} = \lim_{t \to 0} \frac{u(te_1,0) - u(0,0)}{-t} = -u_x(0,0),$$

and hence $u_x(0,0) = 0$. Similarly, we calculate that $u_y(0,0) = 0$ and thus $Du(0,0) = 0$. This is a contradiction unless u is a constant function. But this is not the case since f is not constant. Thus the assumption that there exists a smooth solution u is wrong.

Since the classical solutions are not applicable, the next attempt would be to use weak solutions. After spending some moments with $-\Delta_\infty u = 0$ one finds out that it cannot be written in the divergence form. This is the reason why we cannot use weak solutions. Instead, we use viscosity solutions.

Definition 2.2. (i) A function $u \in USC(\Omega)$ ($u : \Omega \to \mathbb{R}$ is upper semicontinuous) is a viscosity subsolution of $-\Delta_\infty u = 0$ in Ω if for every local maximum point $\hat{x} \in \Omega$ of $u - \varphi$, where $\varphi \in \mathcal{C}^2(\Omega)$, we have $-\Delta_\infty \varphi(\hat{x}) \leq 0$.

(ii) A function $u \in LSC(\Omega)$ ($u : \Omega \to \mathbb{R}$ is lower semicontinuous) is a viscosity supersolution of $-\Delta_\infty u = 0$ in Ω if for every local minimum point $\hat{x} \in \Omega$ of $u - \varphi$, where $\varphi \in \mathcal{C}^2(\Omega)$, we have $-\Delta_\infty \varphi(\hat{x}) \geq 0$.

(iii) We say that a function $u \in \mathcal{C}(\Omega)$ is a viscosity solution of $-\Delta_\infty u = 0$ in Ω if it is both a viscosity sub- and supersolution in Ω.

Moreover, a viscosity subsolution (supersolution, solution) of $-\Delta_\infty u = 0$ is called an infinity subharmonic (superharmonic, harmonic) function.

Remark 2.3. a) Viscosity solutions may be defined similarly for various types of equations, for example, to $-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$ by only changing the operator. For the reference, see [8].

b) We did not require that $\hat{x}$ should be a *strict* local maximum (or minimum) point of $u - \varphi$. However, by considering the function $\varphi(x) + |x - \hat{x}|^4$ (or $\varphi(x) - |x - \hat{x}|^4$) instead of φ, we may assume the strictness if needed.

c) The word *local* can be replaced by the word *global*; all that matters is the behaviour of φ near $\hat{x}$.

We divide this section into three parts: existence, uniqueness and relationship with a certain minimizing problem. In the first two parts we prove the following theorem:

Theorem 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \partial\Omega \to \mathbb{R}$ a Lipschitz continuous function. Then there exists a unique infinity harmonic function $u \in \mathcal{C}(\Omega)$ such that $u(x) = f(x)$ for every $x \in \partial\Omega$, i.e., u is a viscosity solution to

$$\begin{cases} -\Delta_\infty u(x) = 0, & \text{in } \Omega, \\ \quad u(x) = f(x), & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

2.1 Existence of solutions

For given Lipschitz boundary data f, we define a function $F : \overline{\Omega} \to \mathbb{R}$ as

$$F(x) = \inf_{y \in \partial\Omega} \{f(y) + L|x - y|\}, \ x \in \overline{\Omega},$$

where $L = \operatorname{Lip}(f, \partial\Omega)$ is the Lipschitz constant of f. Then $\operatorname{Lip}(F, \overline{\Omega}) = \operatorname{Lip}(f, \partial\Omega)$ and $F(x) = f(x)$ for every $x \in \partial\Omega$. This is a so-called McShane-Whitney extension of f. If we had defined F as the supremum of $f(y) - L|x-y|$ over $\partial\Omega$, then we would have got another McShane-Whitney extension with the same properties. In any case, we have that $F \in W^{1,\infty}(\Omega)$.

Since $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ for every $p > 1$ (Ω is bounded), we may use F as a boundary data and solve the Dirichlet problem (1.1) for each $p > 1$. We get a family of p-harmonic functions $\{u_p\}_{p>1}$ with $u_p \in W^{1,p}_F(\Omega)$. To prove the existence, we show that the limit $\lim_{p\to\infty} u_p$ exists (up to a subsequence) and is a viscosity solution of (2.2).

We start with a lemma that gives a convergent subsequence u_{p_k}.

Lemma 2.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \partial\Omega \to \mathbb{R}$ a Lipschitz continuous function with F as above. If the functions $u_p \in W^{1,p}_F(\Omega)$ are weak solutions to $-\Delta_p u = 0$, then $\{u_p\}_{p\geq n+1}$ is a normal family.

Proof. Let $p \geq n+1$. Since $F \in W^{1,p}_F(\Omega)$, we know by the minimizing property of u_p that

$$\int_\Omega |\nabla u_p(x)|^p \, dx \leq \int_\Omega |\nabla F(x)|^p \, dx \leq \|\nabla F\|^p_{L^\infty(\Omega)}|\Omega|,$$

which yields

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^p \, dx\right)^{\frac{1}{p}} \leq \|\nabla F\|_{L^\infty(\Omega)}.$$

By Hölder's inequality we obtain

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^{n+1} \, dx\right)^{\frac{1}{n+1}} \leq \left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^p \, dx\right)^{\frac{1}{p}} \leq \|\nabla F\|_{L^\infty(\Omega)}.$$

Let $(\widetilde{u_p - F})$ be a zero extension of $u_p - F$ to $\mathbb{R}^n \setminus \Omega$. Then $(\widetilde{u_p - F}) \in W^{1,n+1}(\mathbb{R}^n)$, and by Morrey's inequality we have

$$|(\widetilde{u_p - F})(x) - (\widetilde{u_p - F})(y)| \leq C(n)|x-y|^{\frac{1}{n+1}}\|\nabla(\widetilde{u_p - F})\|_{L^{n+1}(\mathbb{R}^n)} \tag{2.3}$$

for every $x, y \in \mathbb{R}^n$. Here $C(n)$ is a constant from Morrey's inequality and $\|\nabla(\widetilde{u_p - F})\|_{L^{n+1}(\mathbb{R}^n)} = \|\nabla(u_p - F)\|_{L^{n+1}(\Omega)}$. In particular, (2.3) holds for $x, y \in \overline{\Omega}$, and for such x, y we have that

$$\begin{aligned}
|u_p(x) - u_p(y)| &\leq |F(x) - F(y)| + C(n)|x-y|^{\frac{1}{n+1}}\|\nabla u_p - F\|_{L^{n+1}(\Omega)} \\
&\leq L|x-y| + C(n)|x-y|^{\frac{1}{n+1}}\left(\|\nabla u_p\|_{L^{n+1}(\Omega)} + \|\nabla F\|_{L^{n+1}(\Omega)}\right) \\
&\leq L|x-y| + C(n)|x-y|^{\frac{1}{n+1}}\left(2\,\|\nabla F\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{n+1}}\right) \\
&= |x-y|^{\frac{1}{n+1}}\left(L|x-y|^{\frac{n}{n+1}} + \tilde{C}(n)\|\nabla F\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{n+1}}\right) \\
&\leq |x-y|^{\frac{1}{n+1}}\left(L(\operatorname{diam}\Omega)^{\frac{n}{n+1}} + \tilde{C}(n)\|\nabla F\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{n+1}}\right) \\
&= C(n, F, \Omega)|x-y|^{\frac{1}{n+1}}.
\end{aligned}$$

This guarantees that the family $\{u_p\}_{p\geq n+1}$ is equicontinuous in $\overline{\Omega}$.

Since $p > n$, it holds that $u_p \in \mathcal{C}(\overline{\Omega})$ and $u_p(x) = f(x)$ for every $x \in \partial\Omega$.

Then, by fixing some $y_0 \in \partial\Omega$, we have

$$\begin{aligned}|u_p(x)| &\le |u_p(y_0)| + C(n,F,\Omega)|x-y_0|^{\frac{1}{n+1}}\\ &= |f(y_0)| + C(n,F,\Omega)|x-y_0|^{\frac{1}{n+1}}\\ &\le \max_{\partial\Omega}|f| + C(n,F,\Omega)(\operatorname{diam}\Omega)^{\frac{1}{n+1}}\\ &= \tilde{C}(n,F,\Omega),\end{aligned}$$

which implies that the family $\{u_p\}_{p\ge n+1}$ is uniformly bounded in $\overline{\Omega}$. The claim follows from Arzelà-Ascoli's theorem. □

Corollary 2.6. There exists a subsequence $(p_k)_{k=1}^{\infty}$, where $p_k \to \infty$ as $k \to \infty$, and $u \in \mathcal{C}(\overline{\Omega})$ such that $u_{p_k} \to u$ uniformly in $\overline{\Omega}$ and $u(x) = f(x)$ for every $x \in \partial\Omega$.

The function $u \in \mathcal{C}(\overline{\Omega})$ in Corollary 2.6 is our candidate for a solution to (2.2). It already satisfies the boundary condition. Next we show that it satisfies the infinity Laplace equation in the viscosity sense. For this we need a little lemma which says that weak solutions to $-\Delta_p u = 0$ are also viscosity solutions to the same equation.

Lemma 2.7. Let $p \ge 2$ and $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of $-\Delta_p u = 0$. Then u is also a viscosity solution of $-\Delta_p u = 0$.

Proof. We prove by contradiction that u is a viscosity supersolution. Suppose that there exists $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that $u-\varphi$ attains its strict minimum at $\hat{x}$ but $-\Delta_p\varphi(\hat{x}) < 0$. Without loss of generality we may assume that $(u-\varphi)(\hat{x}) = 0$. Since the mapping $x \mapsto -\operatorname{div}\big(|\nabla\varphi(x)|^{p-2}\nabla\varphi(x)\big)$ is continuous, we find $r > 0$ such that $\overline{B}(\hat{x},r) \subset \Omega$ and $-\operatorname{div}\big(|\nabla\varphi(x)|^{p-2}\nabla\varphi(x)\big) < 0$ for every $x \in \overline{B}(\hat{x},r)$. Let

$$m := \inf\{u(x) - \varphi(x) : |x-\hat{x}| = r\} > 0$$

and define $\tilde{\varphi} \in \mathcal{C}^2(\Omega)$ such that $\tilde{\varphi}(x) = \varphi(x) + \frac{m}{2}$. Then $\tilde{\varphi}(\hat{x}) > u(\hat{x})$ and $u \ge \tilde{\varphi}$ on $S(\hat{x},r)$, which yields that $(\tilde{\varphi}-u)^+ \in W^{1,p}_0(B(\hat{x},r))$ and that the measure of $\{x \in B(\hat{x},r) : \tilde{\varphi}(x) - u(x) > 0\}$ is positive. Furthermore, $-\Delta_p\tilde{\varphi}(x) < 0$ holds in $\overline{B}(\hat{x},r)$. Then, after multiplification and integration by parts, we get

$$\begin{aligned}0 &> \int_{B(\hat{x},r)} |\nabla\tilde{\varphi}|^{p-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi}-u)^+\,dx\\ &= \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u\}} |\nabla\tilde{\varphi}|^{p-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi}-u)\,dx.\end{aligned}$$

On the other hand, by extending $(\tilde{\varphi}-u)^+$ as zero outside $B(\hat{x},r)$, we get by definition that

$$\begin{aligned}0 &= \int_{\Omega} |\nabla u|^{p-2}\nabla u\cdot\nabla(\tilde{\varphi}-u)^+\,dx\\ &= \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u\}} |\nabla u|^{p-2}\nabla u\cdot\nabla(\tilde{\varphi}-u)\,dx.\end{aligned}$$

Upon subtraction and using Lemma 6.1 from Appendix, we have

$$\begin{aligned}0 >& \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u\}} \left[|\nabla\tilde{\varphi}|^{p-2}\nabla\tilde{\varphi} - |\nabla u|^{p-2}\nabla u\right] \cdot \nabla(\tilde{\varphi} - u)\, dx \\ \geq& C(p) \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u\}} |\nabla\tilde{\varphi} - \nabla u|^p\, dx,\end{aligned}$$

which is a contradiction.

The proof that u is a viscosity subsolution is similar and we omit the details. □

Theorem 2.8. Let $\{u_{p_k}\}_{k=1}^{\infty}$ be a family of p_k-harmonic functions such that $u_{p_k} \to u$ locally uniformly in Ω as $p_k \to \infty$. Then u is infinity harmonic.

Proof. We show that the function u is a viscosity supersolution of $-\Delta_\infty u = 0$. The proof that u is a subsolution is similar.

Let $\varphi \in C^2(\Omega)$ and $\hat{x} \in \Omega$ such that $u-\varphi$ has a strict minimum at $\hat{x}$. Without loss of generality, we may assume that $(u - \varphi)(\hat{x}) = 0$. Let $r > 0$ be such that $B(\hat{x}, r) \subset \Omega$ and define

$$m_r := \min\{u(x) - \varphi(x) : |x - \hat{x}| = r\} > 0,$$

and

$$\varepsilon_{k,r} := \sup\{u_{p_k}(x) - u(x) : |x - \hat{x}| \leq r\} > 0$$

for $k \in \mathbb{N}$. Choose $k_1 = k_1(r) \in \mathbb{N}$ such that $\varepsilon_{k,r} < \frac{m_r}{2}$ when $k \geq k_1$. Now

$$\inf_{\overline{B}(\hat{x},r)} (u_{p_k} - \varphi) \leq u_{p_k}(\hat{x}) - \varphi(\hat{x}) = u_{p_k}(\hat{x}) - u(\hat{x}) \leq \varepsilon_{k,r} < \frac{m_r}{2}$$

and

$$\inf_{S(\hat{x},r)} (u_{p_k} - \varphi) \geq m_r - \varepsilon_{k,r} > m_r - \frac{m_r}{2} = \frac{m_r}{2}$$

holds for any $k \geq k_1$. This means that for large k the function $u_{p_k} - \varphi$ attains its local minimum *inside* the ball $B(\hat{x}, r)$. We fix such a point and denote it by x_k.

Next we show that $x_k \to \hat{x}$. This is done by repeating the previous procedure for smaller radii. For example, we find $k_2 = k_2(r/2)$ such that if $k \geq k_2$; then the function $u_{p_k} - \varphi$ attains its local minimum at $x_k \in B(\hat{x}, r/2)$. By choosing $x_k \in B(\hat{x}, r)$ for $k_1 \leq k < k_2$, $x_k \in B(\hat{x}, r/2)$ for $k_2 \leq k < k_3 = k_3(r/3)$ and so on, we find a sequence $(x_k)_{k=1}^{\infty}$ for which $x_k \to \hat{x}$.

Now, by Lemma 2.7, it holds that

$$-\Big(|\nabla\varphi(x_k)|^{p_k-2}\Delta\varphi(x_k) + (p_k - 2)|\nabla\varphi(x_k)|^{p_k-4}\Delta_\infty\varphi(x_k)\Big) \geq 0.$$

If $\nabla\varphi(\hat{x}) = 0$, then $-\Delta_\infty\varphi(\hat{x}) = 0$, and we are done. Thus we can assume that $\nabla\varphi(\hat{x}) \neq 0$. Then $\nabla\varphi(x_k) \neq 0$ for large k by the continuity of $\nabla\varphi$. We divide the previous inequality by $|\nabla\varphi(x_k)|^{p_k-4}(p_k - 2)$ and get

$$-\frac{|\nabla\varphi(x_k)|^2\Delta\varphi(x_k)}{p_k - 2} - \Delta_\infty\varphi(x_k) \geq 0.$$

Letting $k \to \infty$ and, consequently, $p_k \to \infty$, we obtain

$$-\Delta_\infty\varphi(\hat{x}) \geq 0,$$

which proves the claim. □

2.2 Uniqueness of solutions

The uniqueness of the solution of (2.2) follows immediately from the next theorem. It is due to Jensen [21], but we follow a recently published new proof by Armstrong and Smart [3].

Theorem 2.9. Let $u, v \in \mathcal{C}(\overline{\Omega})$ such that u is infinity subharmonic and v is infinity superharmonic. Then

$$\max_{\overline{\Omega}} (u - v) = \max_{\partial\Omega} (u - v).$$

We now introduce some notation. For $\varepsilon > 0$ we write $\Omega_\varepsilon := \{x \in \Omega : \overline{B}(x, \varepsilon) \subset \Omega\}$. If $u \in \mathcal{C}(\Omega)$ and $x \in \Omega_\varepsilon$, then we denote

$$S_\varepsilon^+ u(x) := \max_{y \in \overline{B}(x,\varepsilon)} \frac{u(y) - u(x)}{\varepsilon}$$

and

$$S_\varepsilon^- u(x) := \max_{y \in \overline{B}(x,\varepsilon)} \frac{u(x) - u(y)}{\varepsilon}.$$

By choosing $y = x$, we see that $S_\varepsilon^+ u(x), S_\varepsilon^- u(x) \geq 0$.

The next result is a comparison lemma for a finite difference equation and it is a first step towards the proof of Theorem 2.9.

Lemma 2.10. Suppose that $u, v \in \mathcal{C}(\Omega) \cap L^\infty(\Omega)$ and

$$S_\varepsilon^- u(x) - S_\varepsilon^+ u(x) \leq 0 \leq S_\varepsilon^- v(x) - S_\varepsilon^+ v(x) \tag{2.4}$$

holds for every $x \in \Omega_\varepsilon$. Then

$$\sup_{\Omega} (u - v) = \sup_{\Omega \setminus \Omega_\varepsilon} (u - v).$$

Proof. We prove by contradiction. Suppose that

$$\sup_{\Omega} (u - v) > \sup_{\Omega \setminus \Omega_\varepsilon} (u - v).$$

Then the set

$$E := \{x \in \Omega : (u - v)(x) = \sup_{\Omega} (u - v)\}$$

is nonempty, compact and $E \subset \Omega_\varepsilon$. Let

$$F := \{x \in E : u(x) = \max_{E} u\}$$

and choose a point $x_0 \in \partial F$. Then $x_0 \in E$, which means that $u - v$ attains its maximum at x_0. In particular, this means that for every $x \in \overline{B}(x_0, \varepsilon)$ it holds that $u(x) - v(x) \leq u(x_0) - v(x_0)$, and hence

$$\frac{v(x_0) - v(x)}{\varepsilon} \leq \frac{u(x_0) - u(x)}{\varepsilon} \leq S_\varepsilon^- u(x_0).$$

This implies that

$$S_\varepsilon^- v(x_0) \leq S_\varepsilon^- u(x_0). \tag{2.5}$$

If it was the case that $S_\varepsilon^+ u(x_0) = 0$, then by (2.4), (2.5) and the non-negativeness of S_ε^+ and S_ε^- it would hold that

$$S_\varepsilon^- u(x_0) = S_\varepsilon^+ u(x_0) = 0 = S_\varepsilon^- v(x_0) = S_\varepsilon^+ v(x_0).$$

Hence $u(x) \equiv u(x_0)$ and $v(x) \equiv v(x_0)$ in the ball $\overline{B}(x_0, \varepsilon)$. Since $x_0 \in \partial F$, there exists $y \in B(x_0, \varepsilon) \setminus F$. If $y \in E$, then $u(y) < \max_E u = u(x_0)$, which is a contradiction since u was supposed to be a constant in the ball $\overline{B}(x_0, \varepsilon)$. If $y \notin E$, then $(u-v)(y) < \sup_\Omega (u-v) = (u-v)(x_0)$, which is also a contradiction since $u - v$ was supposed to be a constant in $\overline{B}(x_0, \varepsilon)$. This yields that the case $S_\varepsilon^+ u(x_0) = 0$ is not possible.

Now we consider the case $S_\varepsilon^+ u(x_0) > 0$. Since u is continuous and $\overline{B}(x_0, \varepsilon)$ is compact, there exists $z \in \overline{B}(x_0, \varepsilon)$ such that

$$S_\varepsilon^+ u(x_0) = \frac{u(z) - u(x_0)}{\varepsilon}.$$

This implies that

$$u(z) = u(x_0) + \underbrace{\varepsilon S_\varepsilon^+ u(x_0)}_{>0} > u(x_0).$$

If $z \in E$, then $u(z) > u(x_0) = \max_E u$, which is a contradiction. Thus $z \notin E$, and it holds that $(u-v)(z) < (u-v)(x_0)$. From this we get that $v(z) - v(x_0) > u(z) - u(x_0)$, and thus

$$\varepsilon S_\varepsilon^+ v(x_0) \geq v(z) - v(x_0) > u(z) - u(x_0) = \varepsilon S_\varepsilon^+ u(x_0),$$

which yields

$$-S_\varepsilon^+ v(x_0) < -S_\varepsilon^+ u(x_0). \tag{2.6}$$

Combining (2.5) and (2.6), we have that

$$S_\varepsilon^- v(x_0) - S_\varepsilon^+ v(x_0) < S_\varepsilon^- u(x_0) - S_\varepsilon^+ u(x_0),$$

which contradicts (2.4), and the claim follows. □

We continue by presenting new notations. For $u \in \mathcal{C}(\Omega)$ and $x \in \Omega_\varepsilon$ we write

$$u^\varepsilon(x) := \max_{y \in \overline{B}(x,\varepsilon)} u(y)$$

and

$$u_\varepsilon(x) := \min_{y \in \overline{B}(x,\varepsilon)} u(y).$$

Then

$$\varepsilon S_\varepsilon^+ u(x) = \max_{y \in \overline{B}(x,\varepsilon)} (u(y) - u(x)) = u^\varepsilon(x) - u(x)$$

and

$$\varepsilon S_\varepsilon^- u(x) = \max_{y \in \overline{B}(x,\varepsilon)} (u(x) - u(y)) = u(x) - u_\varepsilon(x).$$

These notations are used in the next Lemma, which allows us to modify the solutions of the PDE (2.2) to get solutions that satisfy (2.4). Before we proceed to this Lemma, we present a property called *Comparison with cones*, which is tightly related to infinity harmonic functions and which is needed in the proof of the Lemma.

Definition 2.11. The function $u : \Omega \to \mathbb{R}$ enjoys comparison with cones from above in Ω, if for every open set $U \subset\subset \Omega$ and every $x_0 \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, for which

$$u(x) \leq C(x) = a + b|x - x_0| \tag{2.7}$$

holds for every $x \in \partial(U \setminus \{x_0\})$; one then has

$$u(x) \leq C(x)$$

also for every $x \in U$.

Comparison with cones from below is defined similarly, that is, " $\leq$ " is replaced by " $\geq$ ". Moreover, we say that u enjoys comparison with cones in Ω if u enjoys comparison with cones both from above and below.

Theorem 2.12. Assume that u is an infinity subharmonic (superharmonic, respectively) function in Ω. Then u enjoys comparison with cones from above (below) in Ω.

Proof. Let $U \subset\subset \Omega$ be open and $x_0 \in \mathbb{R}^n$, $a, b \in \mathbb{R}$ be such that $u \leq C$ on $\partial(U \setminus \{x_0\})$. Assume, on the contrary, that there exists $\hat{x} \in U \setminus \{x_0\}$ such that $u(\hat{x}) > C(\hat{x})$. Let $R > 0$ be so large that $x_0 \in B(x, R)$ for every $x \in \partial U$ and set

$$w(x) := a + b|x - x_0| + \varepsilon(R^2 - |x - x_0|^2),\ x \in U.$$

Then $u \leq w$ on $\partial(U \setminus \{x_0\})$ but $u(\hat{x}) > w(\hat{x})$ when $\varepsilon > 0$ is small enough. We may assume that $\hat{x}$ is the maximum point of $u - w$ in $U \setminus \{x_0\}$. One calculates that

$$-\Delta_\infty w(\hat{x}) = 2\varepsilon(2\varepsilon|\hat{x} - x_0|^2 - b)^2.$$

This is strictly positive if $b \leq 0$ or if $b > 0$ and ε is small enough. This is a contradiction to our assumption that u is infinity subharmonic, that is, $-\Delta_\infty u \leq 0$. □

Lemma 2.13. Suppose that u and v are infinity subharmonic and superharmonic functions in Ω, respectively. Then

$$S_\varepsilon^- u^\varepsilon(x) - S_\varepsilon^+ u^\varepsilon(x) \leq 0 \tag{2.8}$$

and

$$S_\varepsilon^- v_\varepsilon(x) - S_\varepsilon^+ v_\varepsilon(x) \geq 0 \tag{2.9}$$

for all $x \in \Omega_{2\varepsilon}$.

Proof. We first prove (2.8). Let $x_0 \in \Omega_{2\varepsilon}$. Choose $y_0 \in \overline{B}(x_0, \varepsilon)$ and $z_0 \in \overline{B}(x_0, 2\varepsilon)$ such that $u(y_0) = u^\varepsilon(x_0)$ and $u(z_0) = u^{2\varepsilon}(x_0)$. Since

$$(u^\varepsilon)^\varepsilon(x_0) = \max_{y \in \overline{B}(x_0,\varepsilon)} u^\varepsilon(y) = \max_{y \in \overline{B}(x_0,\varepsilon)} \max_{z \in \overline{B}(y,\varepsilon)} u(z) = u^{2\varepsilon}(x_0)$$

and

$$(u^\varepsilon)_\varepsilon(x_0) = \min_{y \in \overline{B}(x_0,\varepsilon)} u^\varepsilon(y) = \min_{y \in \overline{B}(x_0,\varepsilon)} \max_{z \in \overline{B}(y,\varepsilon)} u(z) \geq u(x_0),$$

we have that

$$\begin{aligned}\varepsilon\left(S_\varepsilon^- u^\varepsilon(x_0) - S_\varepsilon^+ u^\varepsilon(x_0)\right) &= u^\varepsilon(x_0) - (u^\varepsilon)_\varepsilon(x_0) - ((u^\varepsilon)^\varepsilon(x_0) - u^\varepsilon(x_0)) \\ &= 2u^\varepsilon(x_0) - (u^\varepsilon)^\varepsilon(x_0) - (u^\varepsilon)_\varepsilon(x_0) \\ &\le 2u^\varepsilon(x_0) - u^{2\varepsilon}(x_0) - u(x_0) \qquad (2.10) \\ &= 2u(y_0) - u(z_0) - u(x_0).\end{aligned}$$

By the definition of z_0 it is easy to see that the inequality

$$u(w) \le u(x_0) + \frac{u(z_0) - u(x_0)}{2\varepsilon}|w - x_0| \qquad (2.11)$$

holds for every $w \in \partial(B(x_0, 2\varepsilon) \setminus \{x_0\})$, (that is, $|w - x_0| = 2\varepsilon$ or $w = x_0$). Since u is an infinity subharmonic function, Theorem 2.12 implies that (2.11) holds also for every $w \in B(x_0, 2\varepsilon)$. This allows to put $w = y_0$ to (2.11), and we get

$$\begin{aligned}u(y_0) &\le u(x_0) + \frac{u(z_0) - u(x_0)}{2\varepsilon}|y_0 - x_0| \\ &\le u(x_0) + \frac{u(z_0) - u(x_0)}{2\varepsilon}\varepsilon \\ &= \frac{1}{2}u(x_0) + \frac{1}{2}u(z_0).\end{aligned}$$

From this we deduce that $2u(y_0) - u(z_0) - u(x_0) \le 0$, which together with (2.10) yields

$$S_\varepsilon^- u^\varepsilon(x_0) - S_\varepsilon^+ u^\varepsilon(x_0) \le 0.$$

Since $x_0 \in \Omega_{2\varepsilon}$ was arbitrary, the proof of (2.8) is complete.

To prove (2.9), we substitute $-v$, which is an infinity subharmonic function in Ω, with (2.8) and use the facts that $S_\varepsilon^+(-v)(x) = S_\varepsilon^- v(x)$ and $(-v)^\varepsilon = -v_\varepsilon$. Indeed, for $x \in \Omega_{2\varepsilon}$, we calculate that

$$\begin{aligned}0 &\ge S_\varepsilon^-(-v)^\varepsilon(x) - S_\varepsilon^+(-v)^\varepsilon(x) \\ &= S_\varepsilon^-(-v_\varepsilon)(x) - S_\varepsilon^+(-v_\varepsilon)(x) \\ &= S_\varepsilon^+ v_\varepsilon(x) - S_\varepsilon^- v_\varepsilon(x),\end{aligned}$$

which is (2.9). □

Proof of Theorem 2.9. By Lemmas 2.10 and 2.13

$$\sup_{\Omega_\varepsilon}(u^\varepsilon - v_\varepsilon) = \sup_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}}(u^\varepsilon - v_\varepsilon)$$

for all $\varepsilon > 0$. Let $\varepsilon \to 0$, and the claim follows. □

Let us briefly discuss the necessity of the assumption that the boundary data is Lipschitz continuous. We used this assumption in the proof of existence, for simplicity, but it turns out that it is not crucial and it can be relaxed to continuity, see [21]. In the proof of uniqueness we did not use the Lipschitz continuity. Thus Theorem 2.4 is true if the boundary data is just continuous.

We also want to remark that Theorem 2.9 implies that the entire sequence $(u_p)_p$, which was used in the existence part, converges uniformly in $\overline{\Omega}$.

2.3 Minimizing property and related topics

Theorem 1.5 shows the equivalence of p-harmonic functions and the energy minimizers. In this section we discuss the analogous result for $p = \infty$.

First we define what it means to be a minimizer in the case $p = \infty$.

Definition 2.14. A function $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ satisfies the AMG (absolutely minimizing gradient) property in Ω if for every open $U \subset\subset \Omega$ and for every $v \in W^{1,\infty}_{\text{loc}}(\Omega)$ with $v = u$ on ∂U we have that

$$\|\nabla u\|_{L^\infty(U)} \leq \|\nabla v\|_{L^\infty(U)}.$$

As can be seen from [4], the need for a local minimizing property is justified. To be exact, it could be possible to have multiple functions with given boundary data that minimize the norm of the gradient in whole Ω. To gain uniqueness, we need to add an extra condition, which is that the minimizer must also be a local minimizer. In a p-harmonic case we do not need this extra condition, since a global minimizer is also a local minimizer, due to the set additivity of the integral. In fact, the AMG property is derived from this p-harmonicity by sending $p \to \infty$.

Now we show that infinity harmonic functions are the same as the functions that satisfy the AMG property. This is done by using comparison with cones, which was presented before. The next theorem says that all these properties are equivalent.

Theorem 2.15. Let $u \in W^{1,\infty}_{\text{loc}}(\Omega)$. Then the following conditions are equivalent:

(a) u satisfies the AMG property in Ω.

(b) u enjoys comparison with cones in Ω.

(c) u is infinity harmonic in Ω.

Proof. "(c) $\Longrightarrow$ (b)": This is Theorem 2.12. ⧅ "(b) $\Longrightarrow$ (a)": We omit the proof of this implication, since the proof is quite long and it is based on several technical lemmas, see [9, Theorem 3.2]. ⧅ "(b) $\Longrightarrow$ (c)": Suppose that (c) is not true. We may assume that u is not infinity subharmonic in Ω, that is, there exist $\varphi \in \mathcal{C}^2(\Omega)$ and $\hat{x} \in \Omega$ such that $u - \varphi$ attains its local zero maximum at $\hat{x}$, but $-\Delta_\infty \varphi(\hat{x}) > 0$.

By Lemma 6.3 in the Appendix, there exists a cone function C such that

(i) $C(\hat{x}) = \varphi(\hat{x})$

(ii) $\nabla C(\hat{x}) = \nabla \varphi(\hat{x})$

(iii) $D^2 C(\hat{x}) > D^2 \varphi(\hat{x})$,

where the third statement means that

$$D^2 C(\hat{x})\xi \cdot \xi > D^2 \varphi(\hat{x})\xi \cdot \xi \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

This means that $\hat{x}$ is a strict local maximum point for the function $\varphi - C$, and hence it is possible to find $r_1 > 0$ such that

$$\varphi(x) - C(x) < \varphi(\hat{x}) - C(\hat{x}) = 0$$

for every $x \in \overline{B}(\hat{x}, r_1) \setminus \{\hat{x}\}$. Further, by antithesis, there exists $r_2 > 0$ such that

$$u(x) - \varphi(x) \leq 0$$

in a ball $\overline{B}(\hat{x}, r_2)$. These inequalities together yield

$$u(x) - C(x) < 0$$

for all $x \in \overline{B}(\hat{x}, r) \setminus \{\hat{x}\}$, where $r = \min\{r_1, r_2\}$. This allows us to find $\varepsilon > 0$ such that

$$\max_{x \in \partial B(\hat{x}, r)} (u(x) - C(x)) < -\varepsilon.$$

Now for the cone $C - \varepsilon$ it holds that

$$u(x) \leq C(x) - \varepsilon \tag{2.12}$$

for every $x \in \partial B(\hat{x}, r)$. Since u enjoys comparison with cones, (2.12) holds also for every $x \in B(\hat{x}, r)$. This is a contradiction since $C(\hat{x}) - \varepsilon = u(\hat{x}) - \varepsilon < u(\hat{x})$. ⧅

"(a) $\implies$ (b)": Suppose that u does not enjoy comparison with cones from above. Then there exist $U \subset\subset \Omega$, $x_0 \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ such that $u(x) \leq C(x) := a + b|x - x_0|$ on $\partial\,(U \setminus \{x_0\})$ but $u(\hat{x}) > C(\hat{x})$ for some $\tilde{x} \in U$.

Consider a half-line that has endpoint at x_0 and which passes through $\hat{x}$, and pick two distinct points y_1 and y_2 from that half-line such that $y_1, y_2 \in \partial(U \setminus \{x_0\})$ and the segment $]y_1, y_2[$ lies in U. For identification, we assume that y_1 is the point which is located between x_0 and $\hat{x}$. Let $y_{j,\delta} = y_j + \delta(\hat{x} - y_j)$, $0 < \delta < 1$, for $j = 1, 2$. Since u satisfies the AMG property in Ω, we have that

$$\|\nabla u\|_{L^\infty(U)} \leq \|\nabla(a + b|x - x_0|)\|_{L^\infty(U)} = |b|,$$

and thus

$$|u(y_{j,\delta}) - u(y_j)| \leq |b||\hat{x} - y_j|\delta.$$

If $b \geq 0$, then

$$u(y_{1,\delta}) \leq u(y_1) + b|\hat{x} - y_1|\delta \leq a + b|y_1 - x_0| + b|\hat{x} - y_1|\delta.$$

As $\delta \to 1$, $u(y_{1,\delta}) \to u(\hat{x})$ and $a + b|y_1 - x_0| + b|\hat{x} - y_1|\delta \to C(\hat{x})$, which contradicts the assumption $u(\hat{x}) > C(\hat{x})$. If $b < 0$, then

$$u(y_{2,\delta}) \leq u(y_2) - b|\hat{x} - y_2|\delta \leq a + b|y_2 - x_0| - b|\hat{x} - y_2|\delta$$

When $\delta \to 1$, $u(y_{2,\delta}) \to u(\hat{x})$ and $a + b|y_2 - x_0| - b|\hat{x} - y_2|\delta \to C(\hat{x})$, which also contradicts with $u(\hat{x}) > C(\hat{x})$.

We can similarly prove that u enjoys comparison with cones from below. □

3 Variable $p(x)$, with $1 < \inf p(x) < \sup p(x) < +\infty$

In this section we let a (measurable) function $p : \Omega \to (1, \infty]$ vary in the set Ω. The function p is called a *variable exponent.* We set

$$p_- := \inf_{x \in \Omega} p(x) \text{ and } p^+ := \sup_{x \in \Omega} p(x)$$

and assume that

$$1 < p_- < p^+ < +\infty.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\int_\Omega |u(x)|^{p(x)} \, dx < +\infty.$$

The Luxembourg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. If p is a constant function, then the variable exponent Lebesgue space is just the standard Lebesgue space.

The following properties are needed later (for the proof, see [15]); for $u \in L^{p(\cdot)}(\Omega)$ it holds that

(i) if $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \int_\Omega |u(x)|^{p(x)} \, dx \leq \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$,

(ii) if $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, then $\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \leq \int_\Omega |u(x)|^{p(x)} \, dx \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$,

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} = 1$ if and only if $\int_\Omega |u(x)|^{p(x)} \, dx = 1$.

The outcome of these properties is

$$\min \left\{ \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \right\} \leq \int_\Omega |u(x)|^{p(x)} \, dx \leq \max \left\{ \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \right\}. \tag{3.1}$$

We also present a variable exponent version of Hölder's inequality:

$$\int_\Omega |fg| \, dx \leq C \|f\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}}$$

holds for $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, where $C = C(p_-, p^+)$ and $1/p(x) + 1/q(x) = 1$ for every $x \in \Omega$. Moreover, the dual space of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ and the space $L^{p(\cdot)}(\Omega)$ is reflexive.

We continue by defining a *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$. It consists of functions $u \in L^{p(\cdot)}(\Omega)$, whose weak gradient ∇u exists and belongs to $L^{p(\cdot)}(\Omega)$. The space $W^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The density of smooth functions in $W^{1,p(\cdot)}(\Omega)$ is not a trivial issue and it can happen that the smooth functions are not dense. However, if we assume that the variable exponent p is log-*Hölder*-continuous, that is,

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)} \tag{3.2}$$

for some $C > 0$ and for every $x, y \in \Omega$, then smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$. To the best of our knowledge, this is the weakest modulus of continuity for p that ensures the density of smooth functions in variable exponent Sobolev spaces, see [26] and [11, Chapter 9]. From now on we assume that (3.2) holds.

Under the assumption that $\mathcal{C}^\infty(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$, the *variable exponent Sobolev space with zero boundary values*, $W_0^{1,p(\cdot)}(\Omega)$, is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$, and it is reflexive. To check the basic properties of the variable exponent Lebesgue and Sobolev spaces, see [15] and [23].

We also want to present the variable exponent Sobolev-Poincaré inequality:

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C(\operatorname{diam}\Omega)\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds for functions $u \in W_0^{1,p(\cdot)}(\Omega)$. Here $C = C(n, p)$.

Remark 3.1. Generally, there are several sufficient assumptions for p so that the Sobolev-Poincaré inequality holds. These include

- p is log-Hölder continuous (our case)
- p satisfies a so-called *jump condition* in Ω: there exists $\delta > 0$ such that for every $x \in \Omega$ either

  $$\operatorname{ess\,inf}\{p(y) : y \in B(x, \delta)\} \geq n$$

  or

  $$\operatorname{ess\,sup}\{p(y) : y \in B(x, \delta)\} \leq \frac{n \cdot \operatorname{ess\,inf}\{p(y) : y \in B(x, \delta)\}}{n - \operatorname{ess\,inf}\{p(y) : y \in B(x, \delta)\}}.$$

  Note that if p is continuous up to the boundary, then p satisfies the jump-condition.

For the references, see [11] and [17].

We proceed as in the constant p-case. The methods are the same, only in the proofs we need to be more accurate. We start by defining a $p(x)$-harmonic function.

Definition 3.2. We say that a function $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$ is a weak solution (respectively, *subsolution*, *supersolution*) of

$$-\Delta_{p(x)} u = -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = 0 \tag{3.3}$$

in Ω, if

$$\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla\varphi(x)\, dx = 0 \text{ (respectively, } \leq 0, \geq 0)$$

for every test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (respectively, for every non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$).

A continuous weak solution of $-\Delta_{p(x)}u = 0$ is called a *$p(x)$-harmonic function.*

As in the constant p-case, the continuity requirement for a $p(x)$-harmonic function is redundant. For higher regularity results, see [1], [7] and the list of references given in [11, Chapter 13]. Furthermore, the test function space $\mathcal{C}_0^\infty(\Omega)$ can be extended to $W_0^{1,p(\cdot)}(\Omega)$ if we assume that the $p(x)$-harmonic function belongs to the class $W^{1,p(\cdot)}(\Omega)$. This is based on the fact that $\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and on the variable exponent version of Hölder's inequality. Calculations are the same as in Section 1 and will not be repeated.

The next theorem is the $p(x)$-version of Theorem 1.3. This is our main goal in this section.

Theorem 3.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, p a log-Hölder-continuous variable exponent with $1 < p_- < p^+ < +\infty$ and $f \in W^{1,p(\cdot)}(\Omega)$. Then there exists a unique $p(x)$-harmonic function $u \in W^{1,p(\cdot)}(\Omega)$ such that $u - f \in W_0^{1,p(\cdot)}(\Omega)$, that is, u is a weak solution of

$$\begin{cases} -\Delta_{p(x)}u = 0, & \text{in } \Omega, \\ \qquad\quad u = f, & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

For the proof we define a functional $I : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx.$$

Then (3.3) is the Euler-Lagrange equation of the functional I. The existence and uniqueness of solutions of (3.4) is obtained by the following two theorems, just as in the constant p-case.

Theorem 3.4. Let $f \in W^{1,p(\cdot)}(\Omega)$ be the boundary data. Then the following conditions are equivalent for $u \in W_f^{1,p(\cdot)}(\Omega)$:

(a) $-\Delta_{p(x)}u = 0$ in Ω,

(b) $I(u) \le I(v)$ for every $v \in W_f^{1,p(\cdot)}(\Omega)$.

Theorem 3.5. Let $f \in W^{1,p(\cdot)}(\Omega)$ be the boundary data. There exists a unique $u \in W_f^{1,p(\cdot)}(\Omega)$ such that

$$\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx \le \int_\Omega \frac{1}{p(x)} |\nabla v(x)|^{p(x)}\, dx$$

for every $v \in W_f^{1,p(\cdot)}(\Omega)$.

Now we try to modify the proofs from Section 1 to fit to the $p(x)$-case.

Proof of Theorem 3.4. "(a) $\implies$ (b)": Let $v \in W_f^{1,p(\cdot)}(\Omega)$. For any fixed $x_0 \in \Omega$ the function $y \to |y|^{p(x_0)}$ is convex, and thus

$$|\nabla v(x)|^{p(x_0)} \ge |\nabla u(x)|^{p(x_0)} + p(x_0)|\nabla u(x)|^{p(x_0)-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x))$$

holds for every $x \in \Omega$, especially for $x = x_0$. Since $x_0 \in \Omega$ was arbitrary, we have

$$|\nabla v(x)|^{p(x)} \geq |\nabla u(x)|^{p(x)} + p(x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x)).$$

We divide by $p(x)$ and integrate over Ω, and the result is

$$\begin{aligned}\int_\Omega \frac{1}{p(x)}|\nabla v(x)|^{p(x)}\,dx \geq \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx \\ + \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot (\nabla v(x) - \nabla u(x))\,dx.\end{aligned} \tag{3.5}$$

Since u is $p(x)$-harmonic and belongs to the class $W^{1,p(\cdot)}(\Omega)$, the function $v-u \in W_0^{1,p(\cdot)}(\Omega)$ is an admissible function to test the $p(x)$-harmonicity of u, and thus the last integral is zero. This implies that $I(u) \leq I(v)$.

"(b) $\implies$ (a)": Assume that $u \in W_f^{1,p(\cdot)}(\Omega)$ is the minimizer for the functional I. Fix $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and set $u_t(x) = u(x) + t\varphi(x)$. Then $u_t \in W_f^{1,p(\cdot)}(\Omega)$ and $I(u) = I(u_0) \leq I(u_t)$ for all $t \in \mathbb{R}$. Hence

$$\begin{aligned}0 &= \lim_{t\to 0}\frac{I(u_t)-I(u)}{t} = \lim_{t\to 0}\int_\Omega \frac{1}{p(x)}\frac{|\nabla u_t(x)|^{p(x)} - |\nabla u(x)|^{p(x)}}{t}\,dx \\ &= \int_\Omega \frac{1}{p(x)}\lim_{t\to 0}\frac{|\nabla u_t(x)|^{p(x)} - |\nabla u(x)|^{p(x)}}{t}\,dx \\ &= \int_\Omega \frac{1}{p(x)}\frac{d}{dt}\Big[|\nabla u_t(x)|^{p(x)}\Big]_{t=0}\,dx \\ &= \int_\Omega \frac{1}{p(x)}p(x)\Big[(\nabla u(x) + t\nabla\varphi(x))|\nabla u(x) + t\varphi(x)|^{p(x)-2}\cdot\nabla\varphi(x)\Big]_{t=0}\,dx \\ &= \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla\varphi(x)\,dx.\end{aligned}$$

If we can show why the third equality is true, the proof is complete since the other equalities are clear. To do this, we set

$$f_t(x) = \frac{|\nabla u_t(x)|^{p(x)} - |\nabla u_0(x)|^{p(x)}}{t}$$

and use the mean value theorem to get the following estimate: there exists $s \in \mathbb{R}$, $0 < |s| < |t| < 1$ such that

$$\begin{aligned}f_t(x) &= \frac{|\nabla u_0(x) + t\nabla\varphi(x)|^{p(x)} - |\nabla u_0(x)|^{p(x)}}{t} \\ &= p(x)|s\nabla\varphi(x) + \nabla u_0(x)|^{p(x)-2}\left(s\nabla\varphi(x) + \nabla u_0(x)\right)\cdot\nabla\varphi(x),\end{aligned}$$

and then

$$\begin{aligned}|f_t(x)| &\leq p^+\|\nabla\varphi\|_{L^\infty(\Omega)}|s\nabla\varphi(x) + \nabla u_0(x)|^{p(x)-1} \\ &\leq C(\varphi)p^+2^{p(x)-1}(s^{p(x)-1}|\nabla\varphi(x)|^{p(x)-1} + |\nabla u_0(x)|^{p(x)-1}) \\ &\leq C(\varphi)p^+2^{p^+-1}(\max\{1, \|\nabla\varphi\|_{L^\infty(\Omega)}^{p^+-1}\} + |\nabla u_0(x)|^{p(x)-1}) \\ &= C(\varphi,p)(\tilde{C}(\varphi,p) + |\nabla u_0(x)|^{p(x)-1}).\end{aligned}$$

Then by the variable exponent version of Hölder's inequality we have

$$\int_\Omega |f_t(x)|\,dx \le C(\varphi,p)\left(\tilde{C}(\varphi,p)|\Omega| + \int_\Omega |\nabla u_0(x)|^{p(x)-1}\cdot 1\,dx\right)$$
$$\le C(\varphi,p)\left(\tilde{C}(\varphi,p)|\Omega| + C(p)\||\nabla u_0|^{p(\cdot)-1}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)}\|1\|_{L^{p(\cdot)}(\Omega)}\right).$$

Since

$$\int_\Omega \left(|\nabla u_0(x)|^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\,dx = \int_\Omega |\nabla u_0(x)|^{p(x)}\,dx < +\infty,$$

we have $|\nabla u_0|^{p(\cdot)-1} \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$ and thus $\||\nabla u_0|^{p(\cdot)-1}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)} < +\infty$. It follows that f_t has an L^1-integrable majorant, and then, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to 0}\int_\Omega f_t(x)\,dx = \int_\Omega \lim_{t\to 0} f_t(x)\,dx.$$

This completes the proof. □

Remark 3.6. Using the same argument as at the beginning of the proof of Theorem 3.4, we have for $u, v \in L^{p(\cdot)}(\Omega)$ that

$$|tu(x) + (1-t)v(x)|^{p(x)} \le t|u(x)|^{p(x)} + (1-t)|v(x)|^{p(x)} \tag{3.6}$$

for every $0 \le t \le 1$ and every $x \in \Omega$. Since $p_- > 1$, the inequality is strict if $|\{x \in \Omega : u(x) \neq v(x)\}| > 0$.

Moreover, since the mapping $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$, where $F(x,\xi) = \frac{1}{p(x)}|\xi|^{p(x)}$, is $\mathcal{C}^1$-smooth and $\xi \mapsto F(x,\xi)$ is convex for every $x \in \Omega$ by (3.6), the functional I is weakly lower semicontinuous, see [16].

Although the next proof is mainly similar to the proof of Theorem 1.4, there are slight differences that come from the $p(x)$-term. To point this out, we repeat the proof.

Proof of Theorem 3.5. Let

$$I_0 = \inf_{v\in W_f^{1,p(\cdot)}(\Omega)} \int_\Omega \frac{1}{p(x)}|\nabla v(x)|^{p(x)}\,dx.$$

Then

$$0 \le I_0 \le \int_\Omega \frac{1}{p(x)}|\nabla f(x)|^{p(x)}\,dx \le \frac{1}{p_-}\int_\Omega |\nabla f(x)|^{p(x)}\,dx < +\infty.$$

This allows us to choose a sequence of functions $v_1, v_2, v_3, \ldots \in W_f^{1,p(\cdot)}(\Omega)$ such that

$$\int_\Omega \frac{1}{p(x)}|\nabla v_j(x)|^{p(x)}\,dx < I_0 + \frac{1}{j}$$

for $j = 1, 2, 3, \ldots$ Since $v_j - f \in W_0^{1,p(\cdot)}(\Omega)$, we find by the Sobolev-Poincaré inequality that

$$
\begin{aligned}
\|v_j\|_{L^{p(\cdot)}(\Omega)} &\le \|v_j - f\|_{L^{p(\cdot)}(\Omega)} + \|f\|_{L^{p(\cdot)}(\Omega)} \\
&\le C(n,p)\|\nabla v_j - \nabla f\|_{L^{p(\cdot)}(\Omega)} + \|f\|_{L^{p(\cdot)}(\Omega)} \\
&\le C(n,p)\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} + C(n,p)\|\nabla f\|_{L^{p(\cdot)}(\Omega)} + \|f\|_{L^{p(\cdot)}(\Omega)} \\
&\le C(n,p)\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} + \tilde{C}(n,p)\|f\|_{W^{1,p(\cdot)}(\Omega)}.
\end{aligned}
\tag{3.7}
$$

We next estimate the term $\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)}$. If $\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} > 1$, then

$$
\begin{aligned}
\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} &= \left(\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)}^{p_-}\right)^{\frac{1}{p_-}} \le \left(\int_\Omega |\nabla v_j(x)|^{p(x)}\,dx\right)^{\frac{1}{p_-}} \\
&\le \left(p^+ \int_\Omega \frac{1}{p(x)}|\nabla v_j(x)|^{p(x)}\,dx\right)^{\frac{1}{p_-}} \\
&\le \left(p^+(I_0 + \frac{1}{j})\right)^{\frac{1}{p_-}} \le \left(p^+(I_0+1)\right)^{\frac{1}{p_-}}.
\end{aligned}
$$

If $\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} \le 1$, then no estimates are needed. Hence we get

$$
\|\nabla v_j\|_{L^{p(\cdot)}(\Omega)} \le 1 + \left(p^+(I_0+1)\right)^{\frac{1}{p_-}} \tag{3.8}
$$

for every $j = 1, 2, 3, \ldots$ Now (3.7) together with (3.8) gives that the sequence $(v_j)_{j=1}^\infty$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Then there exists a subsequence, still denoted as $(v_j)_{j=1}^\infty$, and $u_0 \in W_f^{1,p(\cdot)}(\Omega)$ such that $v_j \rightharpoonup u_0$ weakly in $W^{1,p(\cdot)}(\Omega)$. The weak lower semicontinuity of the functional I implies that

$$
I_0 \le I(u_0) \le \liminf_{j\to\infty} I(v_j) = I_0.
$$

This proves the existence.

To prove the uniqueness, we assume that u_1 and u_2 are two minimizers. Then $u = \frac{u_1+u_2}{2} \in W_f^{1,p(\cdot)}(\Omega)$, and we have

$$
\begin{aligned}
I_0 &\le \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx = \int_\Omega \frac{1}{p(x)}\left|\frac{\nabla u_1(x) + \nabla u_2(x)}{2}\right|^{p(x)}\,dx \\
(\star) &\le \int_\Omega \frac{1}{p(x)}\frac{|\nabla u_1(x)|^{p(x)} + |\nabla u_2(x)|^{p(x)}}{2}\,dx \\
&= \frac{1}{2}\int_\Omega \frac{1}{p(x)}|\nabla u_1(x)|^{p(x)}\,dx + \frac{1}{2}\int_\Omega \frac{1}{p(x)}|\nabla u_2(x)|^{p(x)}\,dx \\
&= \frac{1}{2}I_0 + \frac{1}{2}I_0 = I_0.
\end{aligned}
$$

In $(\star)$ we used convexity. If the measure of the set $\{x \in \Omega : \nabla u_1(x) \ne \nabla u_2(x)\}$ is positive, then the inequality $(\star)$ is strict by strict convexity, which leads to a contradiction. Thus $\nabla u_1(x) = \nabla u_2(x)$ almost everywhere in Ω, and hence the Sobolev-Poincaré inequality implies that

$$
\|u_1 - u_2\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u_1 - \nabla u_2\|_{L^{p(\cdot)}(\Omega)} = 0.
$$

Then $u_1 = u_2$, and the uniqueness follows. □

4 Variable $p(x)$ with $p(\cdot) \equiv +\infty$ in a subdomain

This section is based on [25] (an article by Manfredi, Rossi and Urbano), which was the first attempt to analyse the Dirichlet problem (0.1) in the case where the exponent $p(\cdot)$ becomes infinity in some part of the domain.

Let us fix the setting for this section. We assume that $\Omega \subset \mathbb{R}^n$ and $D \subset \Omega$ are both bounded and convex domains with smooth boundaries, at least of class $\mathcal{C}^1$. The variable exponent $p : \Omega \to (1, \infty]$ satisfies

$$p(x) = +\infty \quad \text{for every } x \in D \tag{4.1}$$

and is assumed to be continuously differentiable in $\Omega \setminus \overline{D}$. We also assume that both functions p and ∇p have a continuous extension from $\Omega \setminus \overline{D}$ to $\partial D \cap \Omega$ and that

$$p_- = \inf_{x \in \Omega} p(x) > n \tag{4.2}$$

and

$$p^+ := \sup_{x \in \Omega \setminus \overline{D}} p(x) < +\infty. \tag{4.3}$$

In a way, the problem considered in this section is a combination of the problems studied in the two previous sections. In the set D we have infinity harmonic functions (as in Section 2) and in the set $\Omega \setminus \overline{D}$ we have $p(x)$-harmonic functions (as in Section 3). However, we have extra assumptions for p, which we explain next.

Let us first compute how $-\Delta_{p(x)}$ (with $p(x) < +\infty$ everywhere) evaluates on $\mathcal{C}^2(\Omega)$ functions. For $\varphi \in \mathcal{C}^2(\Omega)$ we find

$$\begin{aligned} -\Delta_{p(x)}\varphi(x) &= -\operatorname{div}\left(|\nabla\varphi(x)|^{p(x)}\nabla\varphi(x)\right) \\ &= -|\nabla\varphi(x)|^{p(x)-2} \\ &\quad \times \left(\Delta\varphi(x) + \nabla p(x)\cdot\nabla\varphi(x)\log|\nabla\varphi(x)| + (p(x)-2)\frac{\Delta_\infty\varphi(x)}{|\nabla\varphi(x)|^2}\right). \end{aligned} \tag{4.4}$$

In order to use the theory of viscosity solutions, the mapping $x \mapsto -\Delta_{p(x)}\varphi(x)$ should be continuous. This is needed, for example, in Lemma 2.7. Since ∇p appears in (4.4), it is natural to assume that $p \in \mathcal{C}^1(\Omega \setminus \overline{D})$. This also ensures the density of smooth functions in $W^{1,p(\cdot)}(\Omega \setminus \overline{D})$. That the functions p and ∇p can be extended continuously from $\Omega \setminus \overline{D}$ to $\partial D \cap \Omega$ is needed to guarantee the continuity of the function $x \mapsto -\Delta_{p(x)}\varphi(x)$ in $\Omega \setminus D$ in the last theorem on this section. The assumption (4.3) is there in order for us to use the theory from Section 3, and the assumption (4.2) guarantees the continuous embedding; for the variable exponent $q : \Omega \to [q_-, q^+]$, $n < q_- < q^+ < +\infty$, it holds that

$$W^{1,q(\cdot)}(\Omega) \hookrightarrow W^{1,q_-}(\Omega) \subset \mathcal{C}(\overline{\Omega}). \tag{4.5}$$

The variable exponent q can even be discontinuous, and still (4.5) holds. We shall use this embedding property later to bounded functions $p_k(x)$, where $p_k(x) = \min\{p(x), k\}$, for which $n < p_- < p_k(x) < p^+ < +\infty$ holds in Ω for large k. To check (4.5), see [23].

Remark 4.1. The convexity of Ω and D guarantees that the Lipschitz constant of $W^{1,\infty}$-functions coincides with the L^∞-norm of their gradients. This property is needed in some of the proofs. The boundary smoothness assumption guarantees the existence of the exterior unit normal vector to ∂D in Ω, which is also needed later.

The first step in trying to solve the Dirichlet boundary value problem,

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & \text{if } x \in \Omega, \\ \qquad u(x) = f(x), & \text{if } x \in \partial\Omega, \end{cases} \tag{4.6}$$

is to replace $p(x)$ by a sequence of bounded functions $p_k(x)$ such that $p_k(x)$ is increasing in k and converges to $p(x)$ as $k \to \infty$. In this work we use the sequence where

$$p_k(x) := \min\{p(x), k\}.$$

We shall use the notation $(4.6)_k$ to refer to the problem (4.6) for the variable exponents $p_k(x)$.

Since $p(x)$ is bounded in $\Omega \setminus D$, we have for large k that

$$p_k(x) = \begin{cases} p(x), & \text{if } x \in \Omega \setminus D, \\ k, & \text{if } x \in D. \end{cases}$$

Moreover, the boundary of the set $\{x \in \Omega : p(x) > k\}$ is ∂D for large k, and thus it is independent of k. This property is needed later in the proofs.

The next step is to solve $(4.6)_k$. Since $p_k(x)$ is not log-Hölder continuous, the existence and uniqueness do not follow from Theorem 3.3. However, Lemma 4.3 below yields a solution to $(4.6)_k$ that we call u_k. If we could show that the limit $\lim_{k\to\infty} u_k$ exists, then it would be a natural candidate for a solution to (4.6) with the original $p(x)$. The uniqueness is achieved quite easily; we combine the results from Sections 2 and 3.

Before we formulate the main result, we introduce the sets in which the minimization process is done. For $k \in \mathbb{N}$ we define

$$S_k = \{u \in W^{1,p_k(\cdot)}(\Omega) : u|_{\partial\Omega} = f\}$$

and

$$S = \{u \in W^{1,p_-}(\Omega) : u|_{\Omega\setminus\overline{D}} \in W^{1,p(\cdot)}(\Omega \setminus \overline{D}), \|\nabla u\|_{L^\infty(D)} \leq 1 \text{ and } u|_{\partial\Omega} = f\}.$$

Observe that $S \subset S_k$ for every $k \in \mathbb{N}$.

Theorem 4.2. Let $p(\cdot)$ be the variable exponent with the properties defined above and $f : \partial\Omega \to \mathbb{R}$ a Lipschitz function. Then the following three claims are true:

1) There exists a unique solution u_k to $(4.6)_k$.

2) If $S \neq \emptyset$, then the uniform limit $u_\infty := \lim\limits_{k\to\infty} u_k$ exists and it is a minimizer of the variational integral

$$\int_{\Omega\setminus\overline{D}} \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx$$

in S. The function u_∞ is the only minimizer amongst those functions in S that are also infinity harmonic in D in the viscosity sense. Moreover, u_∞ is a viscosity solution of

$$\begin{cases} -\Delta_{p(x)}u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ sgn\left(|\nabla u|(x) - 1\right) sgn\left(\frac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where ν is the exterior unit normal vector to ∂D in Ω.

3) If $\partial D \cap \partial\Omega \neq \emptyset$ and the Lipschitz constant of $f|_{\partial D \cap \partial \Omega}$ is strictly greater than one, then $S = \emptyset$ and

$$\liminf_{k\to\infty} \left(\int_D \frac{|\nabla u_k(x)|^k}{k}\, dx \right)^{\frac{1}{k}} > 1,$$

and hence the natural energy associated to the sequence (u_k) is unbounded.

4.1 Approximate solutions u_k

We first prove the existence and uniqueness of solutions to $(4.6)_k$.

Lemma 4.3. Let p be the variable exponent as defined earlier and $f : \partial\Omega \to \mathbb{R}$ a Lipschitz function. Then, for k large enough, there exists a unique weak solution $u_k \in W^{1,p_k(\cdot)}(\Omega)$ to $(4.6)_k$ such that $u_k(x) = f(x)$ for every $x \in \partial\Omega$. Moreover, u_k is the unique minimizer of the functional

$$I_k(u) = \int_\Omega \frac{|\nabla u(x)|^{p_k(x)}}{p_k(x)}\, dx = \int_{\Omega\setminus\overline{D}} \frac{|\nabla u(x)|^{p(x)}}{p(x)}\, dx + \int_D \frac{|\nabla u(x)|^k}{k}\, dx$$

in S_k.

Proof. Suppose that $k > p^+$. Then

$$p_k(x) = \min\{k, p(x)\} \geq p_- > n$$

for every $x \in \Omega$. This ensures that p_k satisfies the jump condition in Ω (see Remark 3.1), and hence the Sobolev-Poincaré inequality is applicable. This means that the proof of Theorem 3.5 goes through with $p_k(x)$ and we find a unique minimizer u_k for the functional I_k in S_k. By (4.5), u_k is continuous up to the boundary and we can take the boundary condition pointwise.

Next we show that Theorem 3.4 holds for $p_k(x)$. The only difficulty we face is the density of $\mathcal{C}_0^\infty(\Omega)$ functions in $W_0^{1,p_k(\cdot)}(\Omega)$. Since D is convex, it is a so-called *exterior cone domain* and thus by the results of [12] and [20], the density of smooth functions in $W^{1,p_k(\cdot)}(\Omega)$ follows from the corresponding densities in $W^{1,k}(D)$ and $W^{1,p(\cdot)}(\Omega \setminus \overline{D})$ which are known. We deduce that u_k is also a unique weak solution of $(4.6)_k$. □

The problem $(4.6)_k$ is considered in the weak sense since the solutions need not be smooth. Next we derive an equivalent formulation to the problem $(4.6)_k$. Let us first suppose that $u = u_k \in \mathcal{C}^2(\Omega)$. We set $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $F(x,\xi) =$

$|\xi|^{p_k(x)-2}\xi$ and denote $F_k := F|_{D\times\mathbb{R}^n}$ and $F_{p(x)} := F|_{\Omega\setminus\overline{D}\times\mathbb{R}^n}$. We take $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and calculate

$$\operatorname{div}(F(x,\nabla u)\varphi) = [\operatorname{div} F(x,\nabla u)]\varphi + F(x,\nabla u)\cdot\nabla\varphi.$$

Then by the divergence theorem it holds that

$$\begin{aligned}
&\int_D F(x,\nabla u)\cdot\nabla\varphi\,dx \\
&\quad = \int_D \operatorname{div}[F_k(x,\nabla u)\varphi]\,dx - \int_D [\operatorname{div} F_k(x,\nabla u)]\varphi\,dx \\
&\quad = \int_{\operatorname{spt}\varphi\cap D} \operatorname{div}[F_k(x,\nabla u)\varphi]\,dx - \int_D [\operatorname{div} F_k(x,\nabla u)]\varphi\,dx \\
&\quad = \int_{\partial(\operatorname{spt}\varphi\cap D)} [F_k(x,\nabla u)\cdot\nu]\varphi\,dS - \int_D [\operatorname{div} F(x,\nabla u)]\varphi\,dx \\
&\quad = \int_{\partial D\cap\Omega} [F_k(x,\nabla u)\cdot\nu]\varphi\,dS - \int_D [\operatorname{div} F_k(x,\nabla u)]\varphi\,dx,
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\int_{\Omega\setminus\overline{D}} F(x,\nabla u)\cdot\nabla\varphi\,dx \\
&\quad = \int_{\Omega\setminus\overline{D}} \operatorname{div}[F_{p(x)}(x,\nabla u)\varphi]\,dx - \int_{\Omega\setminus\overline{D}} [\operatorname{div} F_{p(x)}(x,\nabla u)]\varphi\,dx \\
&\quad = -\int_{\partial D\cap\Omega} [F_{p(x)}(x,\nabla u)\cdot\nu]\varphi\,dS - \int_{\Omega\setminus\overline{D}} [\operatorname{div} F_{p(x)}(x,\nabla u)]\varphi\,dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_\Omega F(x,\nabla u)\cdot\nabla\varphi\,dx &= \int_{\partial D\cap\Omega} [F_k(x,\nabla u)\cdot\nu]\varphi\,dS - \int_{\partial D\cap\Omega} [F_{p(x)}(x,\nabla u)\cdot\nu]\varphi\,dS \\
&\quad - \int_D [\operatorname{div} F_k(x,\nabla u)]\varphi\,dx - \int_{\Omega\setminus\overline{D}} [\operatorname{div} F_{p(x)}(x,\nabla u)]\varphi\,dx.
\end{aligned} \tag{4.7}$$

Since u is a smooth solution, we have that

$$\operatorname{div} F_k(x,\nabla u) = 0$$

in D and

$$\operatorname{div} F_{p(x)}(x,\nabla u) = 0$$

in $\Omega\setminus\overline{D}$, and thus

$$\int_\Omega F(x,\nabla u)\cdot\nabla\varphi\,dx = \int_{\partial D\cap\Omega} [F_k(x,\nabla u)\cdot\nu]\varphi\,dS - \int_{\partial D\cap\Omega} [F_{p(x)}(x,\nabla u)\cdot\nu]\varphi\,dS. \tag{4.8}$$

Generally, u need not be smooth. Then the natural interpretation (in the weak sense) of (4.7) is (4.8) for a solution u. This yields that

$$\int_\Omega F(x,\nabla u)\cdot\nabla\varphi\,dx = 0$$

if and only if

$$\int_{\partial D\cap\Omega}[F_k(x,\nabla u)\cdot\nu]\varphi\,dS=\int_{\partial D\cap\Omega}[F_{p(x)}(x,\nabla u)\cdot\nu]\varphi\,dS$$

for every $\varphi\in\mathcal{C}_0^\infty(\Omega)$. The latter equality is the interpretation of

$$|\nabla u(x)|^{k-2}\frac{\partial u}{\partial\nu}(x)=|\nabla u(x)|^{p(x)-2}\frac{\partial u}{\partial\nu}(x)\ \text{ for } x\in\partial D\cap\Omega$$

in the weak sense.

We get the following lemma.

Lemma 4.4. Problem $(4.6)_k$ is equivalent to the problem

$$\begin{cases}-\Delta_{p(x)}u_k(x)=0, & x\in\Omega\setminus\overline{D},\\ -\Delta_k u_k(x)=0, & x\in D,\\ |\nabla u_k(x)|^{k-2}\,\frac{\partial u_k}{\partial\nu}(x)=|\nabla u_k(x)|^{p(x)-2}\,\frac{\partial u_k}{\partial\nu}(x), & x\in\partial D\cap\Omega,\\ u_k(x)=f(x), & x\in\partial\Omega.\end{cases}\tag{4.9}$$

Here ν is the exterior unit normal vector to ∂D in Ω.

Remark 4.5. From now on, if u is smooth, we may write $-\Delta_{p_k(x)}u(x)$ as

- $-\Delta_{p(x)}u(x)$, if $x\in\Omega\setminus\overline{D}$
- $-\Delta_k u(x)$, if $x\in D$
- $|\nabla u_k(x)|^{k-2}\,\frac{\partial u_k}{\partial\nu}(x)-|\nabla u_k(x)|^{p(x)-2}\,\frac{\partial u_k}{\partial\nu}(x)$, if $x\in\partial D\cap\Omega$.

The next step is to show that weak solutions u_k of $(4.6)_k$ are also viscosity solutions of the same problem. This result is similar to Lemma 2.7, but we need to pay extra attention when considering the equation on $\partial D\cap\Omega$. Before that we define viscosity solutions of the equation $-\Delta_{p_k(\cdot)}u=0$.

Definition 4.6. i) An upper semicontinuous function $u:\Omega\to\mathbb{R}$ is a viscosity subsolution of $-\Delta_{p_k(\cdot)}u=0$ in Ω if for every local maximum point $\hat{x}\in\Omega$ of $u-\varphi$, where $\varphi\in\mathcal{C}^2(\Omega)$, we have $-\Delta_{p(\hat{x})}\varphi(\hat{x})\le 0$ if $\hat{x}\in\Omega\setminus\overline{D}$, $-\Delta_k\varphi(\hat{x})\le 0$ if $\hat{x}\in D$, and the minimum of $\{-\Delta_{p(\hat{x})}\varphi(\hat{x}),\ -\Delta_k\varphi(\hat{x}),\ |\nabla\varphi(\hat{x})|^{k-2}\frac{\partial\varphi}{\partial\nu}(\hat{x})-|\nabla\varphi(\hat{x})|^{p(\hat{x})-2}\frac{\partial\varphi}{\partial\nu}(\hat{x})\}\le 0$ if $\hat{x}\in\partial D\cap\Omega$.

ii) A lower semicontinuous function $u:\Omega\to\mathbb{R}$ is a viscosity supersolution of $-\Delta_{p_k(\cdot)}u=0$ in Ω if for every local minimum point $\hat{x}\in\Omega$ of $u-\varphi$, where $\varphi\in\mathcal{C}^2(\Omega)$, we have $-\Delta_{p(\hat{x})}\varphi(\hat{x})\ge 0$ if $\hat{x}\in\Omega\setminus\overline{D}$, $-\Delta_k\varphi(\hat{x})\ge 0$ if $\hat{x}\in D$, and the maximum of $\{-\Delta_{p(\hat{x})}\varphi(\hat{x}),\ -\Delta_k\varphi(\hat{x}),\ |\nabla\varphi(\hat{x})|^{k-2}\frac{\partial\varphi}{\partial\nu}(\hat{x})-|\nabla\varphi(\hat{x})|^{p(\hat{x})-2}\frac{\partial\varphi}{\partial\nu}(\hat{x})\}\ge 0$ if $\hat{x}\in\partial D\cap\Omega$.

iii) We say that a function $u\in\mathcal{C}(\Omega)$ is a viscosity solution of $-\Delta_{p_k(\cdot)}u=0$ in Ω if it is both a viscosity sub- and supersolution in Ω.

At this point we want to remark that, as in Remark 2.3, the local maxima and minima of $u-\varphi$ can be assumed to be *strict* and/or *global* maxima and minima.

Lemma 4.7. Let u_k be a continuous weak solution of $(4.6)_k$. Then u_k is also a viscosity solution of $(4.6)_k$.

Proof. To prove the lemma, we need to show that u_k is a viscosity solution of $-\Delta_{p_k(\cdot)}u = 0$ in Ω and that $u_k(x) = f(x)$ for every $x \in \partial\Omega$. First we notice that the boundary condition is true by Lemma 4.3. We prove that u_k is a viscosity supersolution of $-\Delta_{p_k(\cdot)}u = 0$ in Ω. The proof that u_k is a viscosity subsolution is similar, and we omit the details.

To check that the condition in Definition 4.6 ii) holds, we argue by contradiction. Suppose that there exist $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that $u_k - \varphi$ attains its strict minimum at $\hat{x}$ and $-\Delta_{p_k(\hat{x})}\varphi(\hat{x}) < 0$. Without loss of generality we may assume that $(u_k - \varphi)(\hat{x}) = 0$. Depending on the location of $\hat{x}$, we have three cases.

Case 1. $\hat{x} \in \Omega \setminus \overline{D}$. Then the counter-assumption becomes

$$
\begin{aligned}
-\Delta_{p(\hat{x})}\varphi(\hat{x}) =& - |\nabla\varphi(\hat{x})|^{p(\hat{x})-2}\Delta\varphi(\hat{x}) - (p(\hat{x})-2)|\nabla\varphi(\hat{x})|^{p(\hat{x})-4}\Delta_\infty\varphi(\hat{x}) \\
&- |\nabla\varphi(\hat{x})|^{p(\hat{x})-2}\log|\nabla\varphi(\hat{x})|\nabla p(\hat{x})\cdot\nabla\varphi(\hat{x}) \\
<&0.
\end{aligned}
$$

Since the mapping $x \mapsto -\Delta_{p(x)}\varphi(x)$ is continuous in $\Omega \setminus \overline{D}$, we find $r > 0$ such that $B(\hat{x}, r) \subset \Omega \setminus \overline{D}$ and $-\Delta_{p(x)}\varphi(x) < 0$ for every $x \in \overline{B}(\hat{x}, r)$. Let

$$m := \inf\{u(x) - \varphi(x) : |x - \hat{x}| = r\} > 0$$

and define $\tilde{\varphi} \in \mathcal{C}^2(\overline{\Omega})$ such that $\tilde{\varphi}(x) = \varphi(x) + \frac{m}{2}$. Then $u_k \geq \tilde{\varphi}$ on $S(\hat{x}, r)$, $\tilde{\varphi}(\hat{x}) > u_k(\hat{x})$ and

$$-\Delta_{p(x)}\tilde{\varphi}(x) = -\Delta_{p(x)}\varphi(x) < 0 \tag{4.10}$$

for all $x \in \overline{B}(\hat{x}, r)$. Next we multiply (4.10) by $(\tilde{\varphi} - u_k)^+ \in W_0^{1,p(\cdot)}(B(\hat{x}, r))$, which is an admissible function for the integration by parts formula, and we get

$$
\begin{aligned}
0 >& \int_{B(\hat{x},r)} -\Delta_{p(x)}\tilde{\varphi}(x)(\tilde{\varphi} - u_k)^+ \, dx \\
=& \int_{B(\hat{x},r)} |\nabla\tilde{\varphi}|^{p(x)-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi} - u_k)^+ \, dx \qquad (4.11)\\
=& \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u_k\}} |\nabla\tilde{\varphi}|^{p(x)-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi} - u_k) \, dx.
\end{aligned}
$$

On the other hand, we may extend $(\tilde{\varphi} - u_k)^+$ as zero outside $B(\hat{x}, r)$ and use it as a test funtion in the weak formulation of $-\Delta_{p(x)}u_k = 0$ to get

$$
\begin{aligned}
0 =& \int_{\Omega\setminus\overline{D}} |\nabla u_k|^{p(x)-2}\nabla u_k\cdot\nabla(\tilde{\varphi} - u_k)^+ \, dx \qquad (4.12)\\
=& \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u_k\}} |\nabla u_k|^{p(x)-2}\nabla u_k\cdot\nabla(\tilde{\varphi} - u_k) \, dx.
\end{aligned}
$$

By subtracting (4.12) from (4.11) and using Corollary 6.2 from Appendix, we get

$$
\begin{aligned}
0 >& \int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u_k\}} \Big[|\nabla\tilde{\varphi}|^{p(x)-2}\nabla\tilde{\varphi} - |\nabla u_k|^{p(x)-2}\nabla u_k\Big]\cdot\nabla(\tilde{\varphi} - u_k) \, dx \\
\geq& C\int_{B(\hat{x},r)\cap\{\tilde{\varphi}>u_k\}} |\nabla\tilde{\varphi} - \nabla u_k|^{p(x)} \, dx,
\end{aligned}
$$

where C is a constant depending on $\sup\{p(x) : x \in B(\hat{x}, r)\}$. This is a contradiction and Case 1 is done.

Case 2. $\hat{x} \in D$. The proof of this case is exactly the same as in Lemma 2.7 and it will not be repeated.

Case 3. $\hat{x} \in \partial D \cap \Omega$. Now the counter-assumption is that each quantity $-\Delta_k \varphi(\hat{x})$, $-\Delta_{p(\hat{x})}\varphi(\hat{x})$ and $|\nabla\varphi(\hat{x})|^{k-2}\frac{\partial\varphi}{\partial\nu}(\hat{x}) - |\nabla\varphi(\hat{x})|^{p(\hat{x})-2}\frac{\partial\varphi}{\partial\nu}(\hat{x})$ is strictly negative. Then, by the $\mathcal{C}^2$-smoothness of φ, there exists $r > 0$ such that

$$-\Delta_k \varphi(x) < 0 \text{ and } -\Delta_{p(x)}\varphi(x) < 0$$

for every $x \in \overline{B}(\hat{x}, r)$, and

$$|\nabla\varphi(x)|^{k-2}\frac{\partial\varphi}{\partial\nu}(x) - |\nabla\varphi(x)|^{p(x)-2}\frac{\partial\varphi}{\partial\nu}(x) < 0$$

for every $x \in \overline{B}(\hat{x}, r) \cap \partial D$. Let

$$m := \inf\{u_k(x) - \varphi(x) : |x - \hat{x}| = r\} > 0$$

and define $\tilde{\varphi} \in \mathcal{C}^2(\overline{\Omega})$ such that $\tilde{\varphi}(x) = \varphi(x) + \frac{m}{2}$. Then $u_k \geq \tilde{\varphi}$ on $S(\hat{x}, r)$, $\tilde{\varphi}(\hat{x}) > u_k(\hat{x})$, and we get

$$-\Delta_k \tilde{\varphi}(x) < 0 \text{ in } B(\hat{x}, r) \cap D, \tag{4.13}$$

$$-\Delta_{p(x)}\tilde{\varphi}(x) < 0 \text{ in } B(\hat{x}, r) \cap \big(\Omega \setminus \overline{D}\big) \tag{4.14}$$

and

$$|\nabla\tilde{\varphi}(x)|^{k-2}\frac{\partial\tilde{\varphi}}{\partial\nu}(x) - |\nabla\tilde{\varphi}(x)|^{p(x)-2}\frac{\partial\tilde{\varphi}}{\partial\nu}(x) < 0 \tag{4.15}$$

in $B(\hat{x}, r) \cap \partial D$. Next we multiply (4.13) and (4.14) by $(\tilde{\varphi} - u)^+$, and we have

$$\begin{aligned} 0 &> \int_{B(\hat{x},r)\cap D} -\Delta_k \tilde{\varphi}(x)(\tilde{\varphi} - u_k)^+ \, dx \\ &= \int_{B(\hat{x},r)\cap D} |\nabla\tilde{\varphi}|^{k-2}\nabla\tilde{\varphi} \cdot \nabla(\tilde{\varphi} - u_k)^+ \, dx \\ &\quad - \int_{B(\hat{x},r)\cap \partial D} |\nabla\tilde{\varphi}|^{k-2}\frac{\partial\tilde{\varphi}}{\partial\nu}(\tilde{\varphi} - u_k)^+ \, dS \end{aligned}$$

and

$$\begin{aligned} 0 &> \int_{B(\hat{x},r)\cap(\Omega\setminus\overline{D})} -\Delta_{p(x)}\tilde{\varphi}(x)(\tilde{\varphi} - u_k)^+ \, dx \\ &= \int_{B(\hat{x},r)\cap(\Omega\setminus\overline{D})} |\nabla\tilde{\varphi}|^{p(x)-2}\nabla\tilde{\varphi} \cdot \nabla(\tilde{\varphi} - u_k)^+ \, dx \\ &\quad + \int_{B(\hat{x},r)\cap \partial D} |\nabla\tilde{\varphi}|^{p(x)-2}\frac{\partial\tilde{\varphi}}{\partial\nu}(\tilde{\varphi} - u_k)^+ \, dS, \end{aligned}$$

which yield, after adding them together,

$$
\begin{aligned}
&\int_{B(\hat{x},r)\cap D} |\nabla\tilde{\varphi}|^{k-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi}-u_k)^+\,dx \\
&+\int_{B(\hat{x},r)\cap\left(\Omega\setminus\overline{D}\right)} |\nabla\tilde{\varphi}|^{p(x)-2}\nabla\tilde{\varphi}\cdot\nabla(\tilde{\varphi}-u_k)^+\,dx \\
&<\int_{B(\hat{x},r)\cap\partial D} \underbrace{\left[|\nabla\tilde{\varphi}|^{k-2}\frac{\partial\tilde{\varphi}}{\partial\nu}-|\nabla\tilde{\varphi}|^{p(x)-2}\frac{\partial\tilde{\varphi}}{\partial\nu}\right]}_{<0,\text{ by (4.15)}}(\tilde{\varphi}-u_k)^+\,dS \\
&<0.
\end{aligned}
\tag{4.16}
$$

On the other hand, we extend $(\tilde{\varphi}-u_k)^+$ as a zero outside $B(\hat{x},r)$ and use it as a test function in the weak formulation of $-\Delta_{p_k(x)}u=0$ to get

$$
0=\int_{B(\hat{x},r)} |\nabla u_k|^{p_k(x)-2}\nabla u_k\cdot\nabla(\tilde{\varphi}-u_k)^+\,dx.
$$

By splitting the integral above over the sets $B(\hat{x},r)\cap D$ and $B(\hat{x},r)\cap\left(\Omega\setminus\overline{D}\right)$, subtracting it from (4.16) and using the monotonity argument separately for both integrals, like we did in Case 1, we get a contradiction and Case 3 is done. $\square$

So far we have showed that the problem $(4.6)_k$ has a unique weak solution u_k for every k large enough and that they are also viscosity solutions. The next thing is to study the limit $\lim_{k\to\infty}u_k$, if there exists one. In the next theorem we get uniform estimates for the sequence (u_k). At this point we ask the reader to recall the definitions of the sets S, S_k and the functional I_k. Recall also Lemma 2.5, in which the Lipschitz-continuation of the boundary data $f:\partial\Omega\to\mathbb{R}$ plays an important role but which is now replaced by the assumption that $S\neq\emptyset$ since we do not know if the Lipschitz constant of f is at most one.

Theorem 4.8. Let u_k be the minimizer of the functional I_k in S_k (i.e., the solution of $(4.6)_k$). Suppose also that $S\neq\emptyset$. Then the sequence (u_k) is uniformly bounded in $W^{1,p_-}(\Omega)$, equicontinuous and uniformly bounded in $\overline{\Omega}$.

Proof. Let $k>p^+$ and $v\in S$. We denote by $F:\overline{\Omega}\to\mathbb{R}$ a McShane-Whitney extension of f. By the embedding property (4.5) we know that $u_k=F$ on $\partial\Omega$ and $u_k\in W^{1,p_-}(\Omega)$. Then we use Sobolev's inequality to see that

$$
\begin{aligned}
\|u_k\|_{L^{p_-}(\Omega)} &\le \|u_k-F\|_{L^{p_-}(\Omega)}+\|F\|_{L^{p_-}(\Omega)} \\
&\le C\|\nabla u_k-\nabla F\|_{L^{p_-}(\Omega)}+\|F\|_{L^{p_-}(\Omega)} \\
&\le C\|\nabla u_k\|_{L^{p_-}(\Omega)}+(C+1)\|F\|_{W^{1,\infty}(\Omega)},
\end{aligned}
\tag{4.17}
$$

where constant $C=C(n,p_-,\Omega)$ is a constant from the Sobolev inequality. Next we estimate the norm of the gradient of u_k as follows:

$$
\begin{aligned}
&\|\nabla u_k\|_{L^{p_-}(\Omega)} \\
&\quad =\left(\int_\Omega |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\
&\quad =\left(\int_D |\nabla u_k|^{p_-}\,dx + \int_{\Omega\setminus\overline{D}} |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\
&\quad \le 2^{\frac{1}{p_-}}\left(\int_D |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} + 2^{\frac{1}{p_-}}\left(\int_{\Omega\setminus\overline{D}} |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\
&\quad \le 2^{\frac{1}{p_-}}\left(\int_D |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} + 2^{\frac{2}{p_-}}\left(\int_{(\Omega\setminus\overline{D})\cap\{|\nabla u_k|\le 1\}} |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\
&\qquad + 2^{\frac{2}{p_-}}\left(\int_{(\Omega\setminus\overline{D})\cap\{|\nabla u_k|> 1\}} |\nabla u_k|^{p_-}\,dx\right)^{\frac{1}{p_-}} \qquad (4.18)\\
&\quad \le 2^{\frac{1}{p_-}}|D|^{\frac{1}{p_-}-\frac{1}{k}}\left(\int_D |\nabla u_k|^{k}\,dx\right)^{\frac{1}{k}} + 2^{\frac{2}{p_-}}\left|(\Omega\setminus\overline{D})\cap\{|\nabla u_k|\le 1\}\right|^{\frac{1}{p_-}} \\
&\qquad + 2^{\frac{2}{p_-}}\left(\int_{(\Omega\setminus\overline{D})\cap\{|\nabla u_k|> 1\}} |\nabla u_k|^{p(x)}\,dx\right)^{\frac{1}{p_-}} \\
&\quad \le 2^{1+\frac{1}{p_-}}|D|^{\frac{1}{p_-}}\left(\int_D |\nabla u_k|^{k}\,dx\right)^{\frac{1}{k}} + 2^{\frac{2}{p_-}}\max\{|\Omega|,1\} \\
&\qquad + 2^{\frac{2}{p_-}}\left(\int_{\Omega\setminus\overline{D}} |\nabla u_k|^{p(x)}\,dx\right)^{\frac{1}{p_-}}.
\end{aligned}
$$

In the last inequality we assumed that k is so large that $D^{\frac{1}{p_-}-\frac{1}{k}} \le 2|D|^{\frac{1}{p_-}}$. On the other hand, since u_k is a minimizer of I_k in S_k and v is an element of S, we deduce that

$$
\begin{aligned}
I_k(u_k) &= \int_D \frac{|\nabla u_k|^k}{k}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla u_k|^{p(x)}}{p(x)}\,dx \\
&\le \int_D \frac{|\nabla v|^k}{k}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx \\
&\le \frac{|D|}{k} + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx \qquad (4.19)\\
&\le |D| + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx =: C_1,
\end{aligned}
$$

where C_1 does not depend on k. Using this estimate we get

$$
\left(\int_D |\nabla u_k|^k\,dx\right)^{\frac{1}{k}} = k^{\frac{1}{k}}\left(\int_D \frac{|\nabla u_k|^k}{k}\,dx\right)^{\frac{1}{k}} \le 2C_1^{\frac{1}{k}} \le 2\max\{C_1^{\frac{1}{p^+}},1\} \qquad (4.20)
$$

and

$$\left(\int_{\Omega\setminus\overline{D}} |\nabla u_k|^{p(x)}\,dx\right)^{\frac{1}{p_-}} \le \left(p^+\int_{\Omega\setminus\overline{D}} \frac{|\nabla u_k|^{p(x)}}{p(x)}\,dx\right)^{\frac{1}{p_-}} \le (p^+)^{\frac{1}{p_-}} C_1^{\frac{1}{p_-}}. \quad (4.21)$$

Finally we put (4.20) and (4.21) to (4.18) to see that $\|\nabla u_k\|_{L^{p_-}(\Omega)}$ is bounded above by a constant that does not depend on k. This observation, together with (4.17), gives the uniformly boundedness of the sequence (u_k) in $W^{1,p_-}(\Omega)$.

The proof that the sequence (u_k) is equicontinuous in $\overline{\Omega}$ runs as follows. First extend $u_k - F$ as zero outside Ω and then use Morrey's inequality to $u_k - F \in W^{1,p_-}(\mathbb{R}^n)$ to get the pointwise estimate for $|u_k(x) - u_k(y)|$. The calculations are exactly the same as in the proof of Lemma 2.5. You will also need the fact that $\|\nabla u_k\|_{L^{p_-}(\Omega)}$ is bounded from above by a constant not depending on k. Fortunately, this is what we just proved above.

The uniform boundedness is also proved as in Lemma 2.5. □

4.2 Passing to the limit

It is time to let $k \to \infty$ and prove Theorem 4.2. At first we study what the location of D in Ω tells us about the set S. Recall that

$$S = \{u \in W^{1,p_-}(\Omega) : u|_{\Omega\setminus\overline{D}} \in W^{1,p(\cdot)}(\Omega\setminus\overline{D}), \|\nabla u\|_{L^\infty(D)} \le 1 \text{ and } u|_{\partial\Omega} = f\}.$$

1) If D is compactly supported in Ω, that is, $D \subset\subset \Omega$, then $S \neq \emptyset$. Indeed, let $F : \overline{\Omega} \to \mathbb{R}$ be a McShane-Whitney extension of the boundary data f to Ω. Using convolution it is easy to construct a smooth function $\psi : \overline{\Omega} \to \mathbb{R}$ such that

- $\psi \equiv 0$ in an open set $U \supset\supset D$
- $\psi \equiv 1$ in $\Omega_\varepsilon = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$ for small $\varepsilon > 0$
- $\delta := \operatorname{dist}(U, \Omega_\varepsilon) > 0$
- $|\nabla\psi(x)| \le \frac{7}{\delta}$ for every $x \in \Omega_\varepsilon$.

Then the product function $F\psi$ is an element of S.

2) If $\partial D \cap \partial\Omega \neq \emptyset$ and the Lipschitz constant of $f : \partial\Omega \to \mathbb{R}$ is at most one, then $S \neq \emptyset$ since the McShane-Whitney extension of f to Ω does not increase the Lipschitz constant, and hence it is an element of S.

3) If $\partial D \cap \partial\Omega \neq \emptyset$ and $\operatorname{Lip}(f, \partial D \cap \partial\Omega) > 1$, then $S = \emptyset$, since any extension u of $f|_{\partial D\cap\partial\Omega}$ to D verifies $\|\nabla u\|_{L^\infty(D)} > 1$.

4) The case where $\partial D \cap \partial\Omega \neq \emptyset$, $\operatorname{Lip}(f, \partial D \cap \partial\Omega) \le 1$ and $\operatorname{Lip}(f, \partial\Omega) > 1$, is complicated. Under these assumptions, it is possible to construct an example where $S = \emptyset$ as well as an example that leads to $S \neq \emptyset$.

Note added after the completion of the manuscript: Related to the case 4), there is a very recent paper, see [28], that considers the non-emptiness (and the emptiness) of the set S more closely.

Case $S = \emptyset$

In the next theorem we show that if $\partial D \cap \partial\Omega \neq \emptyset$ and $\mathrm{Lip}\,(f, \partial D \cap \partial\Omega) > 1$, then the natural energy associated to the sequence (u_k) is unbounded. If $\mathrm{Lip}\,(f, \partial D \cap \partial\Omega) \leq 1$, then the proof fails and we do not know if the theorem is true or not.

Theorem 4.9. Assume that $\partial D \cap \partial\Omega \neq \emptyset$ and $\mathrm{Lip}\,(f, \partial D \cap \partial\Omega) > 1$. Then

$$\liminf_{k\to\infty} \left(I_k(u_k)\right)^{\frac{1}{k}} > 1,$$

which means that $I_k(u_k) \to +\infty$ and the natural energy associated to the sequence (u_k) is unbounded.

Proof. Assume, on the contrary, that

$$\liminf_{k\to\infty} \left(I_k(u_k)\right)^{\frac{1}{k}} = \beta \leq 1.$$

Let $m > p^+$ and take $k > m$. By imitating the proof of Theorem 4.8 and noticing that β plays the role of the constant C_1 (i.e., the counter-assumption plays the role of $S \neq \emptyset$), we find out that (u_k) is a bounded sequence in $W^{1,m}(\Omega)$. Then there exists a subsequence, still denoted as (u_k), and $u_\infty \in W^{1,m}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ with $u_\infty = f$ on $\partial\Omega$ such that $u_k \rightharpoonup u_\infty$ weakly in $W^{1,m}(\Omega)$. By restricting the functions to D we have that $u_k \rightharpoonup u_\infty$ in $W^{1,m}(D)$, where $u_\infty \in W^{1,m}(D)$ and $u_\infty = f$ on $\partial D \cap \partial\Omega$. Then the weak lower semicontinuity of the $\|\cdot\|_{L^m(D;\mathbb{R}^n)}$-norm and Hölder's inequality imply that

$$\begin{aligned}
\left(\int_D |\nabla u_\infty|^m \, dx\right)^{\frac{1}{m}} &\leq \liminf_{k\to\infty} \left(\int_D |\nabla u_k|^m \, dx\right)^{\frac{1}{m}} \\
&\leq \liminf_{k\to\infty} \left[|D|^{\frac{1}{m}-\frac{1}{k}} \left(\int_D |\nabla u_k|^k \, dx\right)^{\frac{1}{k}} \right] \\
&= \liminf_{k\to\infty} \left[|D|^{\frac{1}{m}-\frac{1}{k}} k^{\frac{1}{k}} \underbrace{\left(\int_D \frac{|\nabla u_k|^k}{k} \, dx\right)^{\frac{1}{k}}}_{\leq I_k(u_k)^{\frac{1}{k}}} \right] \\
&\leq |D|^{\frac{1}{m}} \beta.
\end{aligned}$$

This estimate is independent of k and so it is true for every m. Thus, taking the limit $m \to \infty$, we get that $u_\infty \in W^{1,\infty}(D)$ and

$$|\nabla u_\infty(x)| \leq \beta$$

for almost every $x \in D$. Thus we have found a Lipschitz extension of $f|_{\partial D \cap \partial\Omega}$ to D that decreases the Lipschitz constant. This is a contradiction since for every such Lipschitz extension u it holds that $\mathrm{Lip}\,(u, D) \geq \mathrm{Lip}\,(f, \partial D \cap \partial\Omega)$. □

Case $S \neq \emptyset$

This is the more interesting case. We remind that for large k, the solutions u_k of $(4.6)_k$ are minimizers of the functional

$$I_k(u) = \int_\Omega \frac{|\nabla u|^{p_k(x)}}{p_k(x)}\,dx = \int_D \frac{|\nabla u|^k}{k}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\,dx$$

in

$$S_k = \{u \in W^{1,p_k(\cdot)}(\Omega) : u = f \text{ on } \partial\Omega\}.$$

For an element v of S, where

$$S = \{u \in W^{1,p_-}(\Omega) : u|_{\Omega\setminus\overline{D}} \in W^{1,p(\cdot)}(\Omega\setminus\overline{D}), \|\nabla u\|_{L^\infty(D)} \leq 1 \text{ and } u|_{\partial\Omega} = f\},$$

we have

$$\int_D \frac{|\nabla v|^k}{k}\,dx \leq \int_D \frac{1}{k}\,dx = \frac{|D|}{k} \to 0$$

as $k \to \infty$, which leads to

$$I_k(v) \to \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx =: I_\infty(v).$$

Theorem 4.10. Let u_k be minimizers of I_k in S_k and assume that $S \neq \emptyset$. Then there exists $u_\infty \in S$ such that, along subsequences,

1) (u_k) converges uniformly in $\overline{\Omega}$
2) (u_k) converges weakly in $W^{1,m}(D)$ for every $m > p^+$, and
3) (u_k) converges weakly in $W^{1,p(\cdot)}(\Omega\setminus\overline{D})$

to u_∞. Moreover, u_∞ is a minimizer of I_∞ in S and also an infinity harmonic function in D.

Proof. By Theorem 4.8, the sequence (u_k) is equicontinuous and uniformly bounded in $\overline{\Omega}$. Then the Arzelà-Ascoli theorem says that there exists a function $u_\infty \in \mathcal{C}(\overline{\Omega})$ and a subsequence $(k_j)_j$ such that $(u_{k_j})_j$ converges uniformly in $\overline{\Omega}$ to u_∞. Since $u_k = f$ on $\partial\Omega$, also $u_\infty = f$ on $\partial\Omega$. Moreover, since $(u_{k_j})_j$ is bounded in $W^{1,m}(D)$ for every $m > p^+$ by the proof of Theorem 4.9, we get both $u_\infty \in W^{1,m}(D)$ and the weak convergence (upon subsequence, notation remains the same) $u_{k_j} \rightharpoonup u_\infty$ in $W^{1,m}(D)$ for every $m > p^+$.

To obtain the weak convergence in $W^{1,p(\cdot)}(\Omega\setminus\overline{D})$, we use the estimate (4.21) and the Sobolev-Poincaré inequality (as in (3.7)) to show that $(u_{k_j})_j$ is bounded in $W^{1,p(\cdot)}(\Omega\setminus\overline{D})$. This fact together with the pointwise convergence of $(u_{k_j})_j$ gives that $u_\infty \in W^{1,p(\cdot)}(\Omega\setminus\overline{D})$ and $u_{k_j} \rightharpoonup u_\infty$ weakly in $W^{1,p(\cdot)}(\Omega\setminus\overline{D})$ as $j \to \infty$.

To see that u_∞ is really an element of S, we have to show that $\|\nabla u_\infty\|_{L^\infty(D)} \leq 1$. Choosing some $v \in S$, we get

$$I_k(u_k) \leq I_k(v) \leq C$$

by the minimizing property and (4.19), and, consequently,

$$\liminf_{k\to\infty} I_k(u_k)^{\frac{1}{k}} \leq \liminf_{k\to\infty} C^{\frac{1}{k}} = 1.$$

Now the setting is similar to the proof of Theorem 4.9 with $\beta = 1$, and by that we get $u_\infty \in W^{1,\infty}(D)$ and $|\nabla u_\infty(x)| \leq 1$ for almost every $x \in D$. This finally guarantees that $u_\infty \in S$.

Next we show that u_∞ is a minimizer of I_∞. Indeed, since $u_{k_j} \rightharpoonup u_\infty$ weakly in $W^{1,p(\cdot)}(\Omega \setminus \overline{D})$ and I_∞ is weakly lower semicontinuous (see Remark 3.6), we have for every $v \in S$ that

$$I_\infty(u_\infty) \leq \liminf_{j\to\infty} I_\infty(u_{k_j}) \leq \liminf_{j\to\infty} I_{k_j}(u_{k_j}) \leq \liminf_{j\to\infty} I_{k_j}(v) = I_\infty(v).$$

That u_∞ is infinity harmonic in D is clear by Theorem 2.8, since each u_{k_j} is k_j-harmonic in D and u_{k_j} converges uniformly to u_∞. □

The next interesting question is the uniqueness of the minimizer. Unfortunately, this cannot be obtained. The reason for this is that I_∞ sees only what happens in $\Omega \setminus \overline{D}$. If we modify the minimizer u_∞ inside D such that $\|\nabla \tilde{u}_\infty\|_{L^\infty(D)} \leq 1$ still holds for the modified function $\tilde{u}_\infty$, we have found another minimizer.

Example 4.11. Let $D = B(0,1)$ and $\Omega = B(0,2)$, $f : S(0,2) \to \mathbb{R}$ such that $f \equiv 0$. Then the function $u \equiv 0$, which is now the same as u_∞, is an element of S and is clearly a minimizer of I_∞. If we define a new fuction v such that $v(x) = 0$ for every $1 \leq |x| \leq 2$ and $v(x) = 1 - |x|$ for every $0 \leq |x| \leq 1$, then v is also a minimizer of I_∞ and an element of S. The difference between these two functions is that v is not infinity harmonic in D.

By demanding that the minimizer is also infinity harmonic in D, we can gain uniqueness.

Theorem 4.12. Let $u_1, u_2 \in S$ be two minimizers of I_∞. Suppose also that both of them are infinity harmonic in D in the viscosity sense. Then $u_1 = u_2$.

Proof. Clearly $u_1 = u_2 = f$ on $\partial\Omega$. By Remark 3.6, I_∞ is a strictly convex functional in S, and thus $u_1 = u_2$ in $\Omega \setminus \overline{D}$. This implies that $u_1 = u_2$ on ∂D by continuity. Since both u_1 and u_2 are infinity harmonic in D with the same boundary values on ∂D, we deduce that $u_1 = u_2$ also in D (see Theorem 2.9). □

Remark 4.13. Since u_∞ is infinity harmonic in D and it is the only minimizer of I_∞ in S, the whole sequence (u_k) converges uniformly in $\overline{\Omega}$.

The final step in proving Theorem 4.2 is to show that the uniform limit u_∞ of (u_k) is a viscosity solution to the limit problem "$\lim_{k\to\infty}(4.6)_k$". We formulate the limit problem with the help of the equation (4.9).

Theorem 4.14. Let u_k be a weak solution of $(4.6)_k$ and denote by u_∞ the uniform limit of $(u_k)_k$. Then u_∞ is a viscosity solution of

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ sgn\left(|\nabla u(x)| - 1\right) sgn\left(\frac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \partial\Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where ν stands for an exterior unit normal vector to ∂D in Ω.

Proof. We show that u_∞ is a viscosity supersolution. The subsolution case runs similarly and thus it will be omitted.

First notice that $u_\infty(x) = f(x)$ for every $x \in \partial\Omega$, since $u_k(x) = f(x)$ for every $x \in \partial\Omega$ and for all k. Let $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\overline{\Omega})$ such that $u_\infty - \varphi$ attains its strict minimum at $\hat{x}$. Without loss of generality we may assume that $(u_\infty - \varphi)(\hat{x}) = 0$. Depending on where the point $\hat{x}$ locates, we have three different cases.

Case 1. $\hat{x} \in \Omega \setminus \overline{D}$. We use the same method as in the proof of Theorem 2.8, to find points $x_k \in \Omega \setminus \overline{D}$ for which $x_k \to \hat{x}$ and $u_k - \varphi$ attains its local minimum at x_k. Since u_k is a viscosity solution of (4.9) and every x_k lies in $\Omega \setminus \overline{D}$, we have that

$$-\Delta_{p(x_k)}\varphi(x_k) \geq 0$$

for every $k > p^+$. Since the mapping $x \mapsto -\Delta_{p(x)}\varphi(x)$ is continuous in $\Omega \setminus D$, we get

$$-\Delta_{p(\hat{x})}\varphi(\hat{x}) \geq 0$$

as a limit when $k \to \infty$.

Case 2. $\hat{x} \in D$. Since u_∞ is a uniform limit of k-harmonic functions in D, we have by Theorem 2.8 that $-\Delta_\infty\varphi(\hat{x}) \geq 0$.

Case 3. $\hat{x} \in \partial D \cap \Omega$. In this case we need to show that

$$\max\left\{-\Delta_{p(\hat{x})}\varphi(\hat{x}), -\Delta_\infty\varphi(\hat{x}), sgn\left(|\nabla\varphi(\hat{x})| - 1\right) sgn\left(\frac{\partial\varphi}{\partial\nu}(\hat{x})\right)\right\} \geq 0. \quad (4.22)$$

Using again the same method as in the proof of Theorem 2.8, we find points $x_k \in \Omega$ such that $x_k \to x$, and $u_k - \varphi$ attains its local minimum at x_k. There are three subcases depending on the location of the points x_k.

Case 3.1. Infinitely many x_k locate in $\Omega \setminus \overline{D}$. Then for these x_k it holds, by Lemma 4.7, that

$$-\Delta_{p(x_k)}\varphi(x_k) \geq 0$$

for every $k > p^+$, and, consequently, by the continuity of $x \mapsto -\Delta_{p(x)}\varphi(x)$ in $\Omega \setminus D$, that

$$-\Delta_{p(\hat{x})}\varphi(\hat{x}) \geq 0.$$

This proves that (4.22) is true.

Case 3.2. Infinitely many x_k locate in D. Since u_k is k-harmonic in D also in the viscosity sense, see Lemma 2.7, we then have

$$-\Delta_k\varphi(x_k) = -|\nabla\varphi(x_k)|^{k-2}\Delta\varphi(x_k) - (k-2)|\nabla\varphi(x_k)|^{k-4}\Delta_\infty\varphi(x_k) \geq 0.$$

If $\nabla\varphi(\hat{x}) = 0$, then trivially $-\Delta_\infty\varphi(\hat{x}) = 0$, and we are done. Otherwise, $\nabla\varphi(\hat{x}) \neq 0$, which implies that $\nabla\varphi(x_k) \neq 0$ for large k. Then, after dividing, we get

$$-\frac{|\nabla\varphi(x_k)|^2\Delta\varphi(x_k)}{k-2} - \Delta_\infty\varphi(x_k) \geq 0,$$

and, by taking the limit as $k \to \infty$, we conclude that

$$-\Delta_\infty\varphi(\hat{x}) \geq 0$$

by the continuity of $x \mapsto -\Delta_\infty\varphi(x)$ in Ω. Thus (4.22) is true.

Case 3.3. Infinitely many x_k locate on $\partial D \cap \Omega$. By Lemma 4.7 we know that the maximum of

$$\left\{-\Delta_{p(x_k)}\varphi(x_k), -\Delta_k\varphi(x_k), |\nabla\varphi(x_k)|^{k-2}\frac{\partial\varphi}{\partial\nu}(x_k) - |\nabla\varphi(x_k)|^{p(x_k)-2}\frac{\partial\varphi}{\partial\nu}(x_k)\right\}$$

is non-negative for every k. If $-\Delta_{p(x_k)}\varphi(x_k) \geq 0$ or $-\Delta_k\varphi(x_k) \geq 0$ for infinitely many k, then, along such subsequences, we get in the limit that $-\Delta_{p(\hat{x})}\varphi(\hat{x}) \geq 0$, or $-\Delta_\infty\varphi(\hat{x}) \geq 0$, respectively, and (4.22) holds. Thus we may assume that

$$|\nabla\varphi(x_k)|^{k-2}\frac{\partial\varphi}{\partial\nu}(x_k) - |\nabla\varphi(x_k)|^{p(x_k)-2}\frac{\partial\varphi}{\partial\nu}(x_k) \geq 0$$

for infinitely many k, let us say for every k without losing the generality. We may also assume that $\nabla\varphi(\hat{x}) \neq 0$, since otherwise we would have that $-\Delta_\infty\varphi(\hat{x}) = 0$, and thus (4.22) holds. Then, for large k, $\nabla\varphi(x_k) \neq 0$, and we get

$$\frac{\partial\varphi}{\partial\nu}(x_k)\left[|\nabla\varphi(x_k)|^{k-p(x_k)} - 1\right] \geq 0.$$

From this we deduce, by taking the limit $k \to \infty$, that

- $|\nabla\varphi(\hat{x})| > 1$ implies $\dfrac{\partial\varphi}{\partial\nu}(\hat{x}) \geq 0$

  and

- $|\nabla\varphi(\hat{x})| < 1$ implies $\dfrac{\partial\varphi}{\partial\nu}(\hat{x}) \leq 0$.

This is the same thing as

$$sgn\,(|\nabla\varphi(\hat{x})| - 1)\, sgn\left(\frac{\partial\varphi}{\partial\nu}(\hat{x})\right) \geq 0,$$

and thus (4.22) holds. This completes the proof that u_∞ is a viscosity supersolution. □

5 One-dimensional case, where p is continuous and $\sup p(x) = +\infty$

In this section we consider a one-dimensional case where the variable exponent p is continuous and unbounded. The properties of p, besides the dimension, differ from the situation we had in Section 4, where the function p was not continuous in Ω and "jumped" to infinity at D without reaching the values between p^+ and $+\infty$. If we had allowed $p|_{\Omega\setminus\overline{D}}$ to be unbounded, then $p_k(x) = \min\{k, p(x)\}$ would have no longer equalled to $p(x)$ for large k. This would have caused big problems, not to mention the negative contribution that the unboundedness of p has on the Sobolev space $W^{1,p(\cdot)}(\Omega\setminus\overline{D})$. By studying the initial problem (0.1) for continuous and unbounded p, we will understand the nature of the problem better if we embark on the analysis on the real line.

We work under the following assumptions. We assume that $\Omega = (a, b) \subset \mathbb{R}$ is an open and bounded interval and $p : (a, b) \to (1, \infty)$ is a continuously differentiable function with the following two properties:

- $p_- := \inf\limits_{x\in(a,b)} p(x) > 1$
- $\lim\limits_{x\to a} p(x) = +\infty$ or $\lim\limits_{x\to b} p(x) = +\infty$.

In this section we try to understand the one-dimensional Dirichlet boundary value problem

$$\begin{cases} -\dfrac{d}{dx}\left[|u'(x)|^{p(x)-2}u'(x)\right] = 0, \text{ if } x \in (a, b), \\ u(a) = f(a), \\ u(b) = f(b), \end{cases} \tag{5.1}$$

where the function $f : \{a, b\} \to \mathbb{R}$ prescribes the boundary values. We do not try to fully cover the problem. Our aim is to see what difficulties one may face, what needs to be taken into consideration and, of course, to get some preliminary results. In the last part of this section we consider the case in which p is allowed to be infinity in a set of positive measure.

The $p(x)$-harmonic functions on the real line have been studied in many papers. For these, see [11, Chapter 13] and the references therein.

5.1 Discussion

In what follows, the case $p_- = 1$ is also allowed until further notice. We first define a modular on the set $\{u : (a, b) \to \mathbb{R} : u \text{ is measurable}\}$ by setting

$$\rho_{p(\cdot)}(u) := \int_{(a,b)} |u(x)|^{p(x)}\, dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}((a, b))$ consists of all measurable functions u on (a, b) for which $\rho_{p(\cdot)}(\lambda u)$ is finite for some $\lambda > 0$. Notice that this definition is equivalent to the definition in Section 3, since in that definition $p(x)$ was bounded and thus the term $\lambda^{p(x)}$ would not affect to the finiteness of $\rho(\lambda u)$. The Luxembourg norm on this space is defined as

$$\|u\| := \|u\|_{L^{p(\cdot)}((a,b))} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \le 1\right\}.$$

Next we consider the relation between the modular and the norm. In Section 3, where the function p was bounded, we could estimate $\rho_{p(\cdot)}(u)$ from above and below in terms of $\|u\|$, see (3.1). In the basic theory of variable exponent Lebesgue and Sobolev spaces, the formula (3.1) is useful since one can easily switch between the modular and the norm. But this is not the case with unbounded p. Only the estimate

$$\|u\| \leq 1 \implies \rho_{p(\cdot)}(u) \leq \|u\|^{p_-}$$

is true, see [23], where this estimate is deduced from the basic properties of $\rho_{p(\cdot)}(u)$ and from Fatou's lemma. Next we will give counterexamples to show the lack of general two-sided estimates for $\rho_{p(\cdot)}(u)$ in terms of $\|u\|$.

Example 5.1. a) Let $\Omega = (0,1)$ and $p(x) = \frac{1}{x}$. Then for a constant function $u \equiv 2$ it holds that

$$\rho_{p(\cdot)}(u) = \int_{(0,1)} 2^{\frac{1}{x}}\, dx = +\infty.$$

Since

$$\rho_{p(\cdot)}(u/\lambda) = \int_{(0,1)} \left|\frac{2}{\lambda}\right|^{p(x)} dx \leq 1$$

if and only if $\lambda \geq 2$, we get by definition that $\|u\| = 2$.

b) Let $\Omega = (0,1)$ and $p(x) = \frac{1}{x}$. For every $j = 1, 2, 3, ...$ we define a function u_j such that

$$u_j(x) = \left(\frac{1}{2}\right)^{\frac{j}{p(x)}}.$$

Then we calculate that

$$\rho_{p(\cdot)}(u_j) = \int_{(0,1)} |u_j(x)|^{p(x)}\, dx = \int_{(0,1)} \frac{1}{2^j}\, dx = \frac{1}{2^j},$$

and thus $\rho_{p(\cdot)}(u_j) \to 0$ as $j \to \infty$. On the other hand, since $\rho_{p(\cdot)}(u_j/1) = \rho_{p(\cdot)}(u_j) \leq 1$, we have that $\|u_j\| \leq 1$, and moreover, since

$$\rho_{p(\cdot)}\left(\frac{u_j}{\lambda}\right) = \int_{(0,1)} \frac{1}{\lambda^{p(x)}} |u_j(x)|^{p(x)}\, dx = 2^{-j} \int_{(0,1)} \frac{1}{\lambda^{p(x)}}\, dx = +\infty$$

for every $0 < \lambda < 1$ and every j, we deduce that $\|u_j\| = 1$ for all j.

The unboundedness of p causes also another problem. We do not even know if $\mathcal{C}^\infty((a,b)) \cap L^{p(\cdot)}((a,b))$ is dense in $L^{p(\cdot)}((a,b))$, much less that $\mathcal{C}^\infty((a,b)) \cap W^{1,p(\cdot)}((a,b))$ is dense in $W^{1,p(\cdot)}((a,b))$. Then we cannot define $W_0^{1,p(\cdot)}((a,b))$ as a closure of $\mathcal{C}_0^\infty((a,b))$-functions in $W^{1,p(\cdot)}((a,b))$. This could lead to troubles if we wanted to use weak solutions and move from compactly supported smooth test functions to $W_0^{1,p(\cdot)}$-functions by using the method we used earlier in this thesis. On the other hand, it might be possible to do this in some other way, but we do not know how to do it.

Earlier in this thesis, when we proved that the Dirichlet boundary value problem has a unique solution, we did that in two steps. First we showed that being a solution is equivalent to being a minimizer to a certain variational integral. Then we proved that the variational integral has a unique minimizer by using the direct method of calculus of variations. These together implied the result. By the discussion above, this path is rocky for unbounded p. It may not be impossible, but it needs plenty of effort and new results. This is the reason why we consider the problem (5.1) only by using smooth solutions that verify the conditions pointwise.

5.2 Preliminary results

Let (a,b) be an open interval in $\mathbb{R}$ and let $p:(a,b)\to(1,\infty)$ be a continuously differentiable function such that $\lim_{x\to b}p(x)=+\infty$ and $p_->1$ (equal to one is not allowed anymore). For the boundary data $f:\{a,b\}\to\mathbb{R}$, we assume that $f(a)<f(b)$. Our aim is to solve (5.1), that is, to find a twice continuously differentiable function u such that

$$-\frac{d}{dx}\left[|u'(x)|^{p(x)-2}u'(x)\right]=0$$

for every $x\in(a,b)$, $u(a)=f(a)$ and $u(b)=f(b)$.

Since we assumed that $f(a)<f(b)$, we are looking for an increasing solution (decreasing for $f(a)>f(b)$ and constant for $f(a)=f(b)$), see Remark 5.2 later. For an increasing function u it holds that $u'\geq 0$. Then the equation reads as

$$-\frac{d}{dx}\left[u'(x)^{p(x)-1}\right]=0.$$

After integrating and raising both sides to the power of $\frac{1}{p(x)-1}$, we get that

$$u'(x)=C^{\frac{1}{p(x)-1}}$$

for some $C>0$. Hence

$$u(x)=E+\int_a^x C^{\frac{1}{p(t)-1}}\,dt, \tag{5.2}$$

where $E\in\mathbb{R}$ is a constant.

Remark 5.2. We will show why the assumption $f(a)<f(b)$ implies that $u'>0$ on the whole interval (a,b). First, since $u(a)=f(a)<f(b)=u(b)$, there must be points where $u'>0$. If x is such a point, then

$$(\star)\qquad u'(x)=C^{\frac{1}{p(x)-1}}\geq\begin{cases}1, & \text{if } C\geq 1,\\ C^{\frac{1}{p_--1}}, & \text{if } 0<C<1.\end{cases}$$

If there was a point $z\in(a,b)$ such that $u'(z)=0$, then, by continuity, u' should attain all the values between 0 and $\min\{1,C^{\frac{1}{p_--1}}\}$. This contradicts $(\star)$.

Similarly, we may deduce that $f(a)>f(b)$ implies $u'<0$ and $f(a)=f(b)$ implies $u'=0$ on the whole interval (a,b).

To find a function u that solves (5.1) uniquely, we need to determine C and E. Since $u(a)=E$, we have to choose $E=f(a)$ so that the left boundary value is correct, and therefore E is uniquely determined. Next we put $u(b)=f(b)$, which leads to

$$u(b)-u(a)=f(b)-f(a)=\int_a^b C^{\frac{1}{p(t)-1}}\,dt. \tag{5.3}$$

The question is: is it possible to find unique $C > 0$ such that (5.3) holds? The answer is positive by the next lemma.

Lemma 5.3. Let $p : (a, b) \to (1, \infty)$ be a continuous function such that $p_- > 1$. Then the function $F : [0, \infty) \to [0, \infty)$, where

$$F(c) = \int_a^b c^{\frac{1}{p(t)-1}}\, dt,$$

is a homeomorphism, that is, the function F has the following properties:

1) $F(0) = 0$
2) $\lim_{c\to\infty} F(c) = +\infty$
3) F is continuous and strictly increasing.

Proof. The proof is an easy exercise and is omitted. □

Thus we have proved that the problem (5.1) has a unique smooth solution if $f(a) < f(b)$. Similarly, the same result holds if $f(a) > f(b)$. If $f(a) = f(b)$, then by Remark 5.2 we have $u' = 0$, and hence u must be a constant function, $u \equiv f(a)$. We gather these results under the following theorem.

Theorem 5.4. The problem (5.1) has a unique smooth solution.

5.3 Measure of $\{p(x) = +\infty\}$ is positive

At the end of this section we say something about the case where p is extended as ∞ on $[b, c)$ for some $c > b$, while the assumptions for $p|_{(a,b)}$ remain the same. Then the $p(x)$-Laplace equation reads as

$$\begin{cases} -\frac{d}{dx}\left[|u'(x)|^{p(x)-2}u'(x)\right] = 0, & \text{if } a < x < b,\\ -u''(x)u'(x)^2 = 0, & \text{if } b < x < c,\\ \text{some transmission condition}, & \text{if } x = b. \end{cases} \tag{5.4}$$

On the interval (a, b) the solutions of this equation are functions of the form (5.2), and on the interval (b, c) we have one-dimensional (classical) infinity harmonic functions, that is, affine functions which are of the form $a_1 + a_2x$.

Suppose that $u : (a, b) \to \mathbb{R}$ is a given increasing function of the form (5.2) (i.e., $C > 0$ and $E \in \mathbb{R}$). Then we have the following limit:

$$\lim_{x\to b-} u'(x) = \lim_{x\to b-} C^{\frac{1}{p(x)-1}} = 1.$$

We choose the constants a_1 and a_2 such that $a_2 = 1$ and

$$a_1 = u(b) - b = E + \int_a^b C^{\frac{1}{p(t)-1}}\, dt - b.$$

Then the function $v : (b, c) \to \mathbb{R}$, where $v(x) = a_1 + a_2x$, can be glued to the function u at c such that the function $w : (a, c) \to \mathbb{R}$,

$$w(x) = \begin{cases} u(x), & \text{if } a < x \le b,\\ v(x), & \text{if } b < x < c, \end{cases}$$

is a member of the class $\mathcal{C}^1((a,c))$. If we had chosen $a_2 \neq 1$ and $a_1 = u(b) - a_2 b$, then we would have got only a continuous extension.

Now we study how the function w fits to the boundary value problem. Since w is an increasing function, the boundary data f is fixed such that $f(c) > f(a)$. Then

- $w(a) = f(a) \implies u(a) = f(a) \implies E = f(a)$
- $w(c) = f(c) \implies v(c) = f(c) \implies u(b) = v(b) = f(c) - (c-b)$.

Then we must choose $C > 0$ such that

$$\int_a^b C^{\frac{1}{p(t)-1}}\, dt = u(b) - u(a) = f(c) - f(a) - (c-b).$$

This is possible if and only if $f(c) - f(a) > c - b$.

We have worked out the following theorem.

Theorem 5.5. Let p be as above and $f : \{a, c\} \to \mathbb{R}$ be the boundary data. If $f(c) - f(a) > c - b$, then there exists a unique function $w : [a, c] \to \mathbb{R}$ such that

1) $w \in \mathcal{C}^1((a,c))$
2) $w \in \mathcal{C}^2((a,b))$ and $-\frac{d}{dx}\left[|w'(x)|^{p(x)-2} w'(x)\right] = 0$ pointwise on (a,b)
3) w is infinity harmonic on (b,c) in the classical sense
4) $w(a) = f(a)$ and $w(c) = f(c)$.

By studying the second derivative of u, which is

$$u''(x) = -\frac{p'(x)}{(p(x)-1)^2} C^{\frac{1}{p(x)-1}} \log C,$$

we can deduce the following two facts.

1) $w \in \mathcal{C}^2((a,b))$ if and only if $\lim_{x \to b-} u''(x) = 0$, which is equivalent to
$$\lim_{x \to b-} -\frac{p'(x)}{(p(x)-1)^2} = 0.$$
2) If the function p is increasing on (a,b), that is, $p' \geq 0$, then $u'' \leq 0$ if $C \geq 1$, and $u'' \geq 0$ if $0 < C < 1$. This means that u is concave or convex on (a,b). Then the part a) of Lemma 6.5 in Appendix says that w is infinity harmonic at b in the viscosity sense (v is affine and thus both concave and convex). On the other hand, the part b) of that lemma says that if we glue the functions u and v together differently from w, then we do not merely lose the differentiability at b, but also the infinity harmonicity at b.

To emphasize the second item above, we write it down as a part of the next remark.

Remark 5.6. a) If p is increasing on (a, b), then item 1) in Theorem 5.5 may be changed to "w is infinity harmonic at b in the viscosity sense". This is a proper transmission condition for (5.4) at b.

b) If $f(c) - f(a) < b - c$, then we naturally get similar results for $-w$. If again $f(a) = f(c)$, then we end up with a constant function $w \equiv f(a)$.

c) The results we have gained are easy to extend for certain types of p, which are the unions of our model case. An example of this type of p is

$$p(x) = \begin{cases} -\frac{1}{x} + 1, & \text{if } -1 < x < 0, \\ +\infty, & \text{if } 0 \le x \le 1, \\ \frac{1}{x-1} + 1, & \text{if } 1 < x < 2. \end{cases} .$$

The next example is the last statement in this section. It considers the Hölder continuity of w' with certain exponent $p(x)$.

Example 5.7. Let $0 < \alpha < \infty$ and $p : (-1, 1) \to (1, \infty]$ be such that

$$p(x) = \begin{cases} 1 + |x|^{-\alpha}, & \text{if } -1 < x < 0, \\ +\infty, & \text{if } 0 \le x < 1. \end{cases}$$

Let w be any $\mathcal{C}^1((-1, 1))$-solution to (5.4) (no fixed boundary data). If $-1 < x < a < 0$, then

$$\begin{aligned} |u'(x) - u'(a)| &= \left| \int_a^x u''(t)\, dt \right| = \left| \int_a^x -\frac{p'(t)}{(p(t)-1)^2} C^{\frac{1}{p(t)-1}} \log C \, dt \right| \\ &\le \max\{1, C^{\frac{1}{p(-1)-1}}\} |\log C| \int_a^x -\frac{p'(t)}{(p(t)-1)^2}\, dt \\ &= \tilde{C} \left[\frac{1}{p(t)-1} \right]_{t=a}^{t=x} = \tilde{C} \left[\frac{1}{p(x)-1} - \frac{1}{p(a)-1} \right], \end{aligned}$$

where $\tilde{C} = \max\{1, C\} |\log C|$ since $p(-1) - 1 = 1$. Hence, by the continuity of w' at zero, we get for $-1 < x < 0$ that

$$\begin{aligned} |w'(x) - w'(0)| &= \lim_{a \to 0-} |w'(x) - w'(a)| \le \lim_{a \to 0-} \tilde{C} \left[\frac{1}{p(x)-1} - \frac{1}{p(a)-1} \right] \\ &= \frac{C}{p(x)-1} = C|x|^{\alpha}. \end{aligned}$$

If $x > 0$, then $|w'(x) - w'(0)| = |1 - 1| = 0$. From this, and from the fact that w is smooth on $(-1, 0)$ and on $(0, 1)$, we may draw the following conclusions.

- $0 < \alpha \le 1 \implies w'$ is α-Hölder continuous on $(-1, 1)$
- $\alpha > 1 \implies w \in \mathcal{C}^2((-1, 1))$. This can be seen directly from the estimate we get for $|w'(x) - w'(0)|$ or from the condition we got earlier:

$$\lim_{x \to 0-} -\frac{p'(x)}{(p(x)-1)^2} = \lim_{x \to 0-} -\frac{-\alpha|x|^{-\alpha-1}}{|x|^{-2\alpha}} = \lim_{x \to 0-} \alpha|x|^{\alpha-1} = 0.$$

6 Appendix

In this section, we present and prove some auxiliary results that are needed in this thesis.

Lemma 6.1. For all $x, y \in \mathbb{R}^n$ and $p \geq 2$, we have

$$(x|x|^{p-2} - y|y|^{p-2}) \cdot (x-y) \geq 2^{1-p}|x-y|^p.$$

Proof. Let $x, y \in \mathbb{R}^n$. First, the equality

$$|x-y|^2 = (x-y)\cdot(x-y) = |x|^2 - 2x\cdot y + |y|^2$$

implies that

$$x \cdot y = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{2}|x-y|^2.$$

Then we have

$$\begin{aligned}
&(x|x|^{p-2} - y|y|^{p-2}) \cdot (x-y) \\
=&|x|^p + |y|^p - x\cdot y|y|^{p-2} - y \cdot x|x|^{p-2} \\
=&|x|^p + |y|^p - \frac{1}{2}\left(|x|^2 + |y|^2 - |x-y|^2\right)|y|^{p-2} \\
&- \frac{1}{2}\left(|x|^2 + |y|^2 - |x-y|^2\right)|x|^{p-2} \\
=&\frac{1}{2}\Big(|x|^p + |y|^p - |x|^2|y|^{p-2} - |y|^2|x|^{p-2} + |x-y|^2(|x|^{p-2} + |y|^{p-2}\Big) \\
=&\frac{1}{2}\left[\underbrace{\left(|x|^2 - |y|^2\right)\left(|x|^{p-2} - |y|^{p-2}\right)}_{\text{same sign, product}\geq 0} + |x-y|^2\left(|x|^{p-2} + |y|^{p-2}\right)\right] \\
\geq&\frac{1}{2}|x-y|^2\left(|x|^{p-2} + |y|^{p-2}\right) \\
(\star)\ \geq&\frac{1}{2}|x-y|^2 2^{2-p}|x-y|^{p-2} \\
=&2^{1-p}|x-y|^p.
\end{aligned}$$

In $(\star)$ we used the inequality

$$|x-y|^{p-2} \leq (|x| + |y|)^{p-2} \leq 2^{p-2}(|x|^{p-2} + |y|^{p-2}).$$

□

By the previous Lemma we immediately get the following corollary.

Corollary 6.2. Let $A \subset \mathbb{R}^n$ and $p : A \to [2, \infty)$ be such that $p_A^+ := \sup\{p(z) : z \in A\} < +\infty$. Then for every $x, y \in \mathbb{R}^n$ and $z \in A$ it holds that

$$(x|x|^{p(z)-2} - y|y|^{p(z)-2}) \cdot (x-y) \geq 2^{1-p_A^+}|x-y|^{p(z)}.$$

Now we prove the result that we needed in the proof of Theorem 2.15. Without loss of generality we may consider the case where $x_0 = 0$. In the proof we will use the tensor product " $\otimes$ ", which is defined as follows: for $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$, the product $v \otimes w$ is $n \times n$ matrix with

$$(v \otimes w)_{i,j} = v_i w_j.$$

Recall also that for symmetric $n \times n$ matrices A and B, we denote $A > B$ if $A\xi \cdot \xi > B\xi \cdot \xi$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$.

Lemma 6.3. Suppose that $0 \in \Omega$ and $v \in \mathcal{C}^2(\Omega)$ is such that $-\Delta_\infty v(0) > 0$. Then there exists a cone function $C(x) = a|x - z|$, where $a \geq 0$ and $z \in \mathbb{R}^n \setminus \{0\}$, which satisfies

$$\nabla C(0) = \nabla v(0) \text{ and } D^2 C(0) > D^2 v(0).$$

Proof. Let $\varepsilon > 0$ be so small that

$$\frac{1}{|\nabla v(0)|} \Delta_\infty v(0) + \varepsilon \left| D^2 v(0) \frac{\nabla v(0)}{|\nabla v(0)|} \right| < 0.$$

This can be done because $\Delta_\infty v(0) = D^2 v(0) \nabla v(0) \cdot \nabla v(0) < 0$, and thus $\nabla v(0) \neq 0$. Then we set $a := |\nabla v(0)|$ and choose z such that $\frac{z}{|z|} = -\frac{\nabla v(0)}{|\nabla v(0)|}$ and

$$\frac{1}{\varepsilon} + \|D^2 v(0)\| < \frac{|\nabla v(0)|}{|z|}.$$

Here $\|D^2 v(0)\|$ denotes the operator norm of the matrix $D^2 v(0)$. This implies that for $C(x) := a|x - z|$ we have $\nabla C(0) = -a\frac{z}{|z|} = \nabla v(0)$. To show the right order of the Hessian matrices we fix $y \in \mathbb{R}^n \setminus \{0\}$ and write it in the form $y = \alpha \frac{\nabla v(0)}{|\nabla v(0)|} + y^\perp$ where $\nabla v(0) \cdot y^\perp = 0$. We calculate

$$\begin{aligned}
D^2 C(0) y \cdot y =& \frac{|\nabla v(0)|}{|z|} \left(I - \frac{z}{|z|} \otimes \frac{z}{|z|} \right) y \cdot y \\
=& \frac{|\nabla v(0)|}{|z|} \Bigg\{ |y|^2 - \alpha^2 \underbrace{\left[\left(\frac{\nabla v(0)}{|\nabla v(0)|} \otimes \frac{\nabla v(0)}{|\nabla v(0)|} \right) \frac{\nabla v(0)}{|\nabla v(0)|} \right] \cdot \frac{\nabla v(0)}{|\nabla v(0)|}}_{=1} \\
& - 2\alpha \underbrace{\left[\left(\frac{\nabla v(0)}{|\nabla v(0)|} \otimes \frac{\nabla v(0)}{|\nabla v(0)|} \right) \frac{\nabla v(0)}{|\nabla v(0)|} \right] \cdot y^\perp}_{=0} \\
& - \underbrace{\left[\left(\frac{\nabla v(0)}{|\nabla v(0)|} \otimes \frac{\nabla v(0)}{|\nabla v(0)|} \right) y^\perp \right] \cdot y^\perp}_{=0} \Bigg\} \\
=& \frac{|\nabla v(0)|}{|z|} \left(|y|^2 - \alpha^2 \right) \\
=& \frac{|\nabla v(0)|}{|z|} |y^\perp|^2
\end{aligned}$$

and

$$
\begin{aligned}
D^2v(0)y\cdot y =&\alpha^2\frac{1}{|\nabla v(0)|^2}\Delta_\infty v(0)+2\alpha\left(D^2v(0)\frac{\nabla v(0)}{|\nabla v(0)|}\right)\cdot y^\perp\\
&+D^2v(0)y^\perp\cdot y^\perp\\
\le&\alpha^2\frac{1}{|\nabla v(0)|^2}\Delta_\infty v(0)+2\left|\alpha D^2v(0)\frac{\nabla v(0)}{|\nabla v(0)|}\right||y^\perp|\\
&+\left|D^2v(0)y^\perp\right||y^\perp|\\
\le&\alpha^2\frac{1}{|\nabla v(0)|^2}\Delta_\infty v(0)+2\left(\frac{\varepsilon}{2}\alpha^2\left|D^2v(0)\frac{\nabla v(0)}{|\nabla v(0)|}\right|^2+\frac{1}{2\varepsilon}|y^\perp|^2\right)\\
&+\|D^2v(0)\||y^\perp|^2\\
\le&\underbrace{\alpha^2\left(\frac{1}{|\nabla v(0)|}\Delta_\infty v(0)+\varepsilon\big|D^2v(0)\frac{\nabla v(0)}{|\nabla v(0)|}\big|^2\right)}_{<0}\\
&+\underbrace{\left(\frac{1}{\varepsilon}+\|D^2v(0)\|\right)}_{<\frac{|\nabla v(0)|}{|z|}}|y^\perp|^2\\
<&\frac{|\nabla v(0)|}{|z|}|y^\perp|^2.
\end{aligned}
$$

Thus $D^2C(0)>D^2v(0)$. □

The next example helps us in the upcoming lemma.

Example 6.4. a) The function $f(x)=1-|x|$ is not infinity subharmonic at zero, since

$$-\phi''(0)\phi'(0)^2=-(-2)(-1/2)^2=1/2>0$$

for the admissible test function $\phi(x)=1-\frac{1}{2}x-x^2$.

b) Similarly, by using only second order polynomials, we can show that no "real" (i.e., non-constant) cone function is infinity harmonic at the vertex point.

c) Let

$$f(x)=\begin{cases}-\sqrt{1-x^2}, & \text{if } x\in(0,1),\\ \sqrt{1-(x-1)^2}, & \text{if } x\in[1,2).\end{cases}$$

Then f is continuous, convex on $(0,1)$, concave on $(1,2)$ and differentiable at $x\neq 1$. The function f is infinity harmonic at 1, since it cannot be tested by any $\mathcal{C}^2$-function from above or below at that point. However, the derivative $f'(1)$ does not exist, since $\lim_{x\to 1} f'(x)=\infty$.

Lemma 6.5. Let $(a,c)\subset\mathbb{R}$ be an open interval, $x_0\in(a,c)$ and let $f:(a,c)\to\mathbb{R}$ be a continuous function such that f is convex (concave) on (a,x_0) and concave (convex) on (x_0,c).

a) If f is differentiable at x_0, then f is infinity harmonic at x_0.

b) Suppose that f is infinity harmonic at x_0, $f'(x)$ exists for every $x\neq x_0$ and that the limits $\lim_{x\to x_0+}f'(x)$ and $\lim_{x\to x_0-}f'(x)$ exist and are finite. Then f is differentiable at x_0.

Proof. a) Let $\phi \in \mathcal{C}^2((a,c))$ be such that $\phi(x_0) = f(x_0)$ and $\phi(x) > f(x)$ when $x \neq x_0$. This means that the function $f - \phi$ has a local maximum at x_0 and so $\phi'(x_0) = f'(x_0)$. Using Taylor's formula, we have

$$\phi(x) = \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{1}{2}\phi''(x_0)(x - x_0)^2 + o(|x - x_0|^2), \text{when } x \neq x_0.$$

When $x < x_0$, by the convexity of f we deduce that

$$\phi(x) \geq f(x) \geq f(x_0) + f'(x_0)(x - x_0) = \phi(x_0) + \phi'(x_0)(x - x_0).$$

Thus

$$\frac{\phi''(x_0)}{2}(x - x_0)^2 + o(|x - x_0|^2) \geq 0.$$

Dividing above by $|x - x_0|^2$ and letting $x \to x_0-$, we have $\phi''(x_0) \geq 0$. Hence

$$-\phi''(x_0)(\phi'(x_0))^2 \leq 0,$$

and so f is infinity subharmonic at x_0. For superharmonicity we use a similar argument and the concavity of f on (x_0, c).

b) Suppose that f is not differentiable at x_0. Then

$$m := \lim_{x \to x_0-} f'(x) \neq \lim_{x \to x_0+} f'(x) = M.$$

Since the cone function C, where

$$C(x) := \begin{cases} mx + f(x_0) - mx_0, & \text{if } x \leq x_0, \\ Mx + f(x_0) - Mx_0, & \text{if } x > x_0, \end{cases}$$

is not infinity harmonic at x_0 (see Example 6.4 b)), neither f is that. Indeed, without loss of generality we may assume that C is not infinity subharmonic at x_0. Then there exists a second order polynomial ϕ such that $\phi(x) > C(x)$ for $x \neq x_0$, $\phi(x_0) = C(x_0)$ and $-\phi''(x_0)\phi'(x_0)^2 > 0$. From the definitions of m and M and from the convexity of $f|_{(a,x_0)}$ and concavity of $f|_{(x_0,c)}$, we deduce that f behaves such as C near x_0, and thus $\phi(x) > f(x)$ near x_0. Since $\phi(x_0) = f(x_0)$, ϕ is a proper test function to test the infinity harmonicity of f at x_0. But we already had that $-\phi''(x_0)\phi'(x_0)^2 > 0$, which means that f is not infinity subharmonic at x_0. This contradicts with the initial assumption and the proof is complete. □

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