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HOMEOMORPHIC EQUIVALENCE OF GROMOV AND INTERNAL BOUNDARIES

PÄIVI LAMMI

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These four years have been the most challenging time of my life. These years include several uphills and downhills, tears of joy but also tears of despair. I have banged my head against the wall many times and, by contrast, I have had moments of insight many times. But I guess each Ph.D. student of mathematics knows what I am talking about.

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Jyväskylä, January 2011

Päivi Lammi

List of included articles

This dissertation consists of an introductory part and the following publications:

[1] P. Koskela and P. Lammi, *Gehring–Hayman theorem for conformal deformations.*
Submitted.
Preprint 915, CRM Barcelona, 2009,
`http://www.crm.es/Publications/09/Pr915.pdf`.

[2] P. Lammi, *Compactness of a conformal boundary of the Euclidean unit ball.*
Ann. Acad. Sci. Fenn. Math. **36** (2011), 3–20.

[3] P. Lammi, *Quasihyperbolic boundary condition: compactness of the inner boundary.*
To appear in Illinois J. Math.

The author of this dissertation has actively taken part in the research of the joint paper [1].

Introduction

In this thesis we literally live on the edge. More precisely, we visit the same edge many times but we climb along different paths, survey the view from different angles, and ask weather or not the view looks similar and in what sense similar. We also visit different edges and ask, are these edges in some sense included in the same class.

In the language of mathematics, we consider a certain equivalence of boundaries of an abstract space determined by two different metrics. The goal of this thesis is to answer a very simple question: When is the original boundary of a metric space homeomorphic to the Gromov boundary of the same space? In this dissertation one finds both topological and analytic criteria for the boundaries to be homeomorphic to a given metric space. Motivation for this problem arises from the papers [BHK] and [HenK].

In this thesis we consider many different metrics, and for clarity, the usual metric notations will have an additional subscript that refers to the metric in use.

1 What is a Gromov boundary?

Before we can understand the homeomorphic equivalence of Gromov and original boundaries, we need to understand the concept of a Gromov boundary.

1.1 The unit disc

We start with a simple setting: the unit disc $\mathbb{B}^2 = B^2(0,1) \subset \mathbb{R}^2$ equipped with the *hyperbolic metric* h. For $x, y \in \mathbb{B}^2$ the hyperbolic distance h is

$$h(x,y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \frac{2}{1-|z|^2}\,|dz|, \tag{1.1}$$

where $\gamma_{xy} \subset \mathbb{B}^2$ is a curve joining points x and y, and $|dz|$ is the Euclidean length element. The *hyperbolic space* $(\mathbb{B}^2, h)$ is a *geodesic* space — that is, every pair of points x and y can be joined with a curve in $\mathbb{B}^2$ whose hyperbolic length is exactly the hyperbolic distance between the endpoints. We denote by $[x,y]$ such a curve, called a *geodesic.* In the Euclidean sense, hyperbolic geodesics are Euclidean line segments along radii or subarcs of circles that are orthogonal to the boundary circle $\mathbb{S}^1 = \partial\,\mathbb{B}^2$, see Figure 1.

This metric space, $(\mathbb{B}^2, h)$, is a *Gromov hyperbolic space.* Thus it is δ-*hyperbolic* for some $\delta \geq 0$. This means that hyperbolic triangles are δ-*thin* — that is, for all triples of geodesics $[x,y], [y,z], [z,x]$ every point in $[x,y]$ is within distance δ from $[y,z] \cup [z,x]$. In fact, here we can compute that $\delta = \log(\sqrt{2}+1)$.

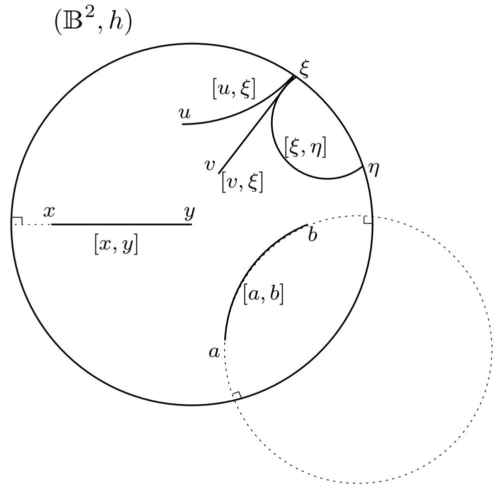

Figure 1: Example of hyperbolic geodesics $[a, b]$ and $[x, y]$, geodesic rays $[u, \xi]$ and $[v, \xi]$ representing ξ, and a geodesic line $[\xi, \eta]$

Does the metric space $(\mathbb{B}^2, h)$ have any kind of "boundary"? It is natural to think that the Euclidean boundary $\mathbb{S}^1$ of $\mathbb{B}^2$ is also the boundary in the hyperbolic sense, but is it, and does $(\mathbb{B}^2, h)$ even have a boundary? We can easily verify that the hyperbolic length of any radius of $\mathbb{B}^2$ is infinite, and therefore $(\mathbb{B}^2, h)$ is unbounded.

A *geodesic ray* in $(\mathbb{B}^2, h)$ is an isometric image of the interval $[0, \infty) \subset \mathbb{R}$. Two geodesic rays are *equivalent* if their Hausdorff distance in $(\mathbb{B}^2, h)$ is finite — that is, both rays have finite neighbourhoods such that the other ray is contained in the neighbourhood of the other ray. The *Gromov boundary* $\partial_G \mathbb{B}^2$ is the set of all equivalence classes of geodesic rays, and we say that a geodesic ray ends at $\xi \in \partial_G \mathbb{B}^2$ if it represents the point ξ, see Figure 1. For each $x \in \mathbb{B}^2$ and $\xi \in \partial_G \mathbb{B}^2$ there is a geodesic ray $[x, \xi]$ issuing from x and ending at ξ. Similarly, for every pair of points $\xi, \eta \in \partial_G \mathbb{B}^2$ there is a *geodesic line* $[\xi, \eta]$ from ξ to η that is an isometric image of $(-\infty, \infty)$ ending at ξ and η in the obvious sense ([BrHa, §III.H Lemma 3.1 and Lemma 3.2], see also Figure 1). Clearly, the boundaries $\partial_G \mathbb{B}^2$ and $\mathbb{S}^1$ can be identified as sets. That is, there exists an identity map $\mathbb{S}^1 \to \partial_G \mathbb{B}^2$ that is a bijection, at least.

In order to study this identity map in more detail, we need a metric on $\partial_G \mathbb{B}^2$. It is clear how to measure Euclidean distances on $\mathbb{S}^1$, but the hyperbolic metric h does not extend to the boundary $\partial_G \mathbb{B}^2$. This is why we next give an alternate definition for the Gromov boundary which at the same time defines a metric on the Gromov boundary.

The Gromov boundary $\partial_G \mathbb{B}^2$ can be defined as the set of equivalence classes of sequences $(x_n) \subset \mathbb{B}^2$ which *tend to infinity* in the sense that

$$\lim_{n,m\to\infty} (x_n|x_m)_w = \infty,$$

where

$$(x|y)_w = \frac{1}{2}\{h(w,x) + h(w,y) - h(x,y)\} \tag{1.2}$$

is the *Gromov product* between points $x, y \in \mathbb{B}^2$ with respect to a base point $w \in \mathbb{B}^2$. Two sequences $(x_n), (y_n) \subset \mathbb{B}^2$, tending to infinity, are equivalent if

$$\lim_{n\to\infty} (x_n|y_n)_w = \infty.$$

The choice of the base point w does not affect the Gromov boundary $\partial_G \mathbb{B}^2$ as a set.

The Gromov product (1.2) extends to the Gromov boundary $\partial_G \mathbb{B}^2$ in a natural way. From the geometric point of view, the Gromov product has the following property:

$$\big|(x|y)_w - \operatorname{dist}_h(w, [x,y])\big| \le 8\delta \tag{1.3}$$

for any pair of points $x, y \in \mathbb{B}^2 \cup \partial_G \mathbb{B}^2$ and any hyperbolic geodesic $[x,y]$ between the points.

Now the function $d_w \colon \partial_G \mathbb{B}^2 \times \partial_G \mathbb{B}^2 \to \mathbb{R}$,

$$d_w(\xi,\eta) = \exp\{-(\xi|\eta)_w\}, \tag{1.4}$$

where $w \in \mathbb{B}^2$ is the base point, defines a metric. Taking the geometric property (1.3) of the Gromov product into account we obtain, up to a constant, a good approximation for the metric d_w :

$$d_w(\xi,\eta) \approx \exp\{-\operatorname{dist}_h(w, [\xi,\eta])\}, \qquad \xi,\, \eta \in \partial_G \mathbb{B}^2\,. \tag{1.5}$$

Again, the choice of the base point w is irrelevant because metrics with different base points, d_w and $d_{w'}$, are bi–Lipschitz equivalent.

Considering the metric d_w from the geometric point of view (1.5) it is easy to believe that the boundaries $\partial_G \mathbb{B}^2$ and $\mathbb{S}^1$ are homeomorphic. Actually, it is easy to verify that the boundaries are even bi–Lipschitz equivalent. Thus

$$d_w(\xi,\eta) \approx |\xi - \eta| \tag{1.6}$$

for each $\xi, \eta \in \partial_G \mathbb{B}^2$.

1.2 A general metric space

The unit disc $\mathbb{B}^2$ is a simple special case and easy to understand. One can generalize these concepts to a more general metric space (Ω, d). We first generalize the concept of the hyperbolic metric h to the *quasihyperbolic metric* k_d derived from the metric d.

Let (Ω, d) be a locally compact, rectifiably connected and non–complete metric space. The *boundary* $\partial_d\Omega$ of Ω is $\partial_d\Omega = \overline{\Omega} \setminus \Omega$, where $\overline{\Omega}$ is the metric completion of Ω. The boundary $\partial_d\Omega$ is nonempty, and for $z \in \Omega$ we denote

$$d(z) = \mathrm{dist}_d(z, \partial_d\Omega) = \inf\{d(z,x) : x \in \partial_d\Omega\}. \tag{1.7}$$

The quasihyperbolic metric k_d in Ω is defined to be

$$k_d(x,y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \frac{ds}{d(z)}, \qquad x, y \in \Omega, \tag{1.8}$$

where the infimum is taken over all rectifiable curves γ_{xy} joining points x and y in Ω, and ds is the length element with respect to the metric d. The relation between the hyperbolic metric h and the quasihyperbolic metric k_d in the unit disc $\mathbb{B}^2$ is $k_d(x,y) \le h(x,y) \le 2k_d(x,y)$.

There is also a third natural metric in Ω, denoted by $\ell_d(x,y)$ and defined as the infimum of the lengths (in the original d-metric) of all curves joining points x and y in Ω. If Ω is a domain in $\mathbb{R}^n$, and d is the Euclidean metric restricted to Ω, then $\ell = \ell_d$ is the Euclidean path metric.

Bonk, Heinonen and Koskela proved in [BHK, Proposition 2.8] that if the identity map $(\Omega, d) \to (\Omega, \ell_d)$ is a homeomorphism, then it is also a homeomorphism $(\Omega, d) \to (\Omega, k_d)$ and (Ω, k_d) is complete. Furthermore, as a complete locally compact length space, (Ω, k_d) is geodesic and proper (i.e. closed balls are compact), cf. [BrHa, §I.3]. From now on, depending on the context, $[x,y]$ denotes a hyperbolic or a quasihyperbolic geodesic between x and y.

When (Ω, k_d) is geodesic, the definition of the Gromov hyperbolicity of the space (Ω, k_d) is given using triangles exactly as described in section 1.1 for $(\mathbb{B}^2, h)$. If (Ω, k_d) is a geodesic and proper Gromov hyperbolic space, we define the Gromov boundary $\partial_G\Omega$ similarly as in the hyperbolic case $(\mathbb{B}^2, h)$, using geodesic rays. Also the Gromov product (1.2), its geometric property (1.3) and the definition of the Gromov boundary $\partial_G\Omega$ through it are similar. By contrast (cf. (1.4)), the function $d_w \colon \partial_G\Omega \times \partial_G\Omega \to \mathbb{R}$,

$$d_w(\xi, \eta) = \exp\{-(\xi|\eta)_w\}, \tag{1.9}$$

where $w \in \Omega$ is a base point, does not, in general, define a metric because (1.9) does not necessarily satisfy the triangle inequality. However, there is a constant $\epsilon(\delta) > 0$ such that for $0 < \epsilon < \epsilon(\delta)$ one finds a metric $d_{w,\epsilon}$ on $\partial_G\Omega$ satisfying

$$\frac{1}{2}\exp\{-\epsilon(\xi|\eta)_w\} \le d_{w,\epsilon}(\xi,\eta) \le \exp\{-\epsilon(\xi|\eta)_w\} \tag{1.10}$$

for $\xi, \eta \in \partial_G\Omega$. Combining (1.3), (1.9) and (1.10), we obtain, similarly to (1.5),

$$d_{w,\epsilon}(\xi,\eta) \approx \exp\{-\epsilon\, \mathrm{dist}_{k_d}(w,[\xi,\eta])\}, \tag{1.11}$$

whenever $0 < \epsilon < \epsilon(\delta)$ and $\xi, \eta \in \partial_G\Omega$. Notice that the choices of the base point w and $0 < \epsilon < \epsilon(\delta)$ are irrelevant, because metrics $d_{w,\epsilon}$ and $d_{w',\epsilon'}$ are equivalent. In fact, the metrics are equivalent in the following way:

$$d_{w,\epsilon}(\xi,\eta) \approx d_{w',\epsilon'}(\xi,\eta)^{\frac{\epsilon}{\epsilon'}} \tag{1.12}$$

for all ξ, $\eta \in \partial_G\Omega$. The Gromov boundary equipped with this metric $d_{w,\epsilon}$ is always compact (cf. [Bo] and [GhHa]).

Going back to the case of the unit disc $(\mathbb{B}^2, h)$ and equipping the Gromov boundary $\partial_G\,\mathbb{B}^2$ with the metric $d_{w,\epsilon}$ for $0 < \epsilon < 1$ we observe that the boundaries $\mathbb{S}^1$ and $\partial_G\,\mathbb{B}^2$ are not bi-Lipschitz anymore (cf. inequality (1.6)). Instead, we find that

$$d_{w,\epsilon}(\xi,\eta) \approx |\xi-\eta|^{\epsilon} \tag{1.13}$$

for all $\xi, \eta \in \partial_G\,\mathbb{B}^2$.

1.3 Equivalence of Gromov and original boundaries

When we study a more general Euclidean metric space than $\mathbb{B}^2$, inequality (1.13) is not necessarily true. However, Bonk, Heinonen and Koskela proved in [BHK, Theorem 1.11] that for a certain class of Euclidean spaces the boundaries are *quasisymmetric.* That is, for some homeomorphism $f : [0,\infty) \to [0,\infty)$ and for all triples of distinct points ξ, η, ζ we have

$$\frac{d_{w,\epsilon}(\zeta,\xi)}{d_{w,\epsilon}(\zeta,\eta)} \le f\Big(\frac{|\zeta-\xi|}{|\zeta-\eta|}\Big). \tag{1.14}$$

In particular, they proved that a bounded domain in $\mathbb{R}^n$ is uniform if and only if it is both Gromov hyperbolic with respect to the quasihyperbolic metric and its Euclidean boundary is quasisymmetrically equivalent to the Gromov boundary. Here by the *uniformity* of a metric space (Ω, d) we mean that Ω is *quasi–convex* (inequality (1.15)) and satisfies the *twisted cone condition* (inequality (1.16)). That is, for some $D \ge 1$ every pair points $x, y \in \Omega$ can be joined with a curve γ_{xy} in Ω such that

$$\mathit{length}_d(\gamma_{xy}) \le D d(x,y), \tag{1.15}$$

and for every $a \in \gamma_{xy}$

$$\min\{\mathit{length}_d(\gamma_{xy}(x,a)), \mathit{length}_d(\gamma_{xy}(a,y))\} \le D d(a), \tag{1.16}$$

where $\gamma_{xy}(x,a)$ is the subcurve of γ_{xy} from x to a and $\gamma_{xy}(a,y)$ is the rest of the curve.

Now the *question* is, how much can we relax the assumptions on the domain $\Omega \subset \mathbb{R}^n$ so that the Euclidean boundary and the Gromov boundary are still homeomorphic? More generally, can one do something like this also in more general metric spaces? We continue with a couple of instructive examples.

The upper half plane is uniform, but the quasihyperbolic geodesic ray $[w, \infty)$, where w is any point in the upper half plane, cannot be identified with any point in the Euclidean boundary $\mathbb{R} \times \{0\}$ of the half plane.

The unit disc excluding the radius $\mathbb{B}^2 \setminus ([0,1] \times \{0\})$ equipped with the Euclidean metric is not uniform and one Euclidean boundary point on the radius may define two distinct points on the Gromov boundary, see Figure 2. Again, the Euclidean and the Gromov boundaries cannot be identified even as sets. However, if we change the Euclidean metric to the Euclidean path metric ℓ, the boundary defined by the metric ℓ and the Gromov boundary are even quasisymmetric (this fact follows from the proof of [BHK, Theorem 1.11]). Thus in attempting to answer our question it is better to consider bounded quasi–convex metric spaces (see the definition in (1.15)).

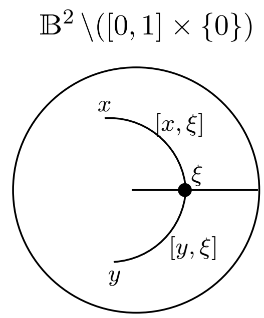

Figure 2: One Euclidean boundary point defines two distinct Gromov boundary points

Boundedness and quasiconvexity of a metric space are not enough to guarantee that the two different boundaries are homeomorphic. Indeed, let

$$\Omega = (0,1) \times (0,1) \setminus \bigcup_{j=1}^{\infty} \Big(\Big\{ \frac{1}{2^j} \times [0, \frac{1}{2}] \Big\} \Big) \tag{1.17}$$

be as pictured in Figure 3 and let us equip Ω with the Euclidean path metric ℓ. In order that the boundaries $\partial_\ell \Omega$ and $\partial_G \Omega$ be homeomorphic, the boundary $\partial_\ell \Omega$ should be compact. It is closed and bounded but not, however, compact. Indeed, let $(x_j) \subset \partial_\ell \Omega$ be the sequence of the "midpoints",

$$x_j = \Big(\frac{3}{2^{j+2}}, 0 \Big).$$

This sequence does not have a convergent subsequence because

$$\ell(x_j, x_{j+1}) \geq 1$$

for every $j \in \mathbb{N}$.

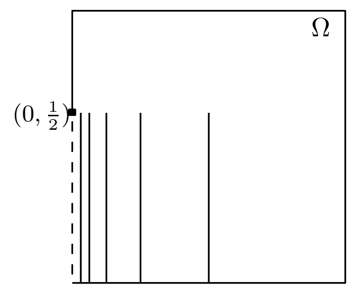

Figure 3: A bounded quasi–convex domain Ω with $\partial_\ell \Omega$ not compact

All the above–mentioned spaces equipped with the quasihyperbolic metric really are Gromov hyperbolic spaces. It is not always the case that an abstract quasihyperbolic metric space is Gromov hyperbolic, and this is something we have to take care of, when we are studying Gromov boundaries. In [BB], Balogh and Buckley studied geometric characterizations of Gromov hyperbolicity of bounded domains in $\mathbb{R}^n$ equipped with the quasihyperbolic metric. They discovered that a bounded domain in $\mathbb{R}^n$ equipped with the Euclidean path metric satisfies the *Gehring–Hayman theorem* if the domain is Gromov hyperbolic in the quasihyperbolic metric. Thus it is natural to study spaces that satisfy the Gehring–Hayman theorem. This property has turned out to be a central tool for proving theorems related to Gromov hyperbolic spaces (e.g. [BHK]).

2 The Gehring–Hayman theorem

Let us take a brief look at the Gehring–Hayman theorem and its history.

2.1 The unit disc

Given $x, y \in \mathbb{B}^2$, the hyperbolic geodesic $[x, y]$ is, in the Euclidean sense, essentially the shortest curve joining x to y in $\mathbb{B}^2$. More precisely,

$$\mathit{length}([x, y]) \leq \frac{\pi}{2} \mathit{length}(\gamma_{xy}),$$

whenever γ_{xy} is a curve joining points x and y in $\mathbb{B}^2$. Gehring and Hayman proved in [GH] that hyperbolic geodesics are also essentially the shortest curves in any conformal image of the unit disc. They proved the following theorem

Theorem 2.1. [GH, Theorem 2] *If* $f\colon \mathbb{B}^2 \to \Omega \subset \mathbb{C}$ *is a conformal mapping, and* $[u, v]$ *is a hyperbolic geodesic and* γ_{uv} *is any other curve joining the points* $u, v \in \Omega$*, then*

$$\mathit{length}([u, v]) \leq C\,\mathit{length}(\gamma_{uv}),$$

where $C \geq 1$ *is an absolute constant.*

Because hyperbolic geodesics are, by definition, conformally invariant, Theorem 2.1 says that

$$\mathit{length}(f([x, y])) \leq C\,\mathit{length}(f(\gamma_{xy})),$$

where $x = f^{-1}(u)$ and $y = f^{-1}(v)$. Furthermore, because f is a conformal mapping, Theorem 2.1 really says that

$$\int_{[x,y]} |f'(z)|\,|dz| \leq C \int_{\gamma_{xy}} |f'(z)|\,|dz| \tag{2.1}$$

for every x and y in $\mathbb{B}^2$. Hence, in the unit disc, a hyperbolic geodesic is essentially the shortest curve also with respect to the *density* $\rho(z) = |f'(z)|$, in the sense of the deformed metric d_ρ, where

$$d_\rho(x, y) = \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \rho(z)\,|dz| \tag{2.2}$$

and the infimum is taken over all curves γ_{xy} in $\mathbb{B}^2$ with endpoints x and y.

2.2 A general metric space

The Gehring–Hayman theorem has been studied quite a bit. See for instance [HeiN] and [HeiR]. Also in [BKR] Bonk, Koskela and Rohde generalized the Gehring–Hayman inequality (2.1) to conformal deformations of the unit ball $\mathbb{B}^n$, $n \geq 2$, and they found the critical properties of $|f'(z)|$ in inequality (2.1) that appear to be essential for such a generalization.

The density $\rho(z) = |f'(z)|$, where $f\colon \mathbb{B}^2 \to \Omega$ is a conformal mapping, satisfies a *Harnack inequality* (with the constant e^{12}, cf. [Po, p. 10])

$$e^{-12}\rho(x) \leq \rho(z) \leq e^{12}\rho(x)$$

for all $z \in B(x, (1-|x|)/2)$ and all $x \in \mathbb{B}^2$. Balls $B(x, (1-|x|)/2)$ are called *Whitney type balls.* The density $\rho(z) = |f'(z)|$ also satisfies the *area growth* estimate (with the constant π),

$$\int_{B_\rho(x,r)} \rho^2(z)\,dA(z) \leq \pi r^2$$

for all $x \in \mathbb{B}^2$, where $B_\rho(x, r)$ refers to the ball with centre x and radius r in the metric d_ρ, see the definition in expression (2.2).

These two properties of the density ρ are the critical ones, and Bonk, Koskela and Rohde put this into a more abstract form: If a continuous function $\rho\colon \mathbb{B}^n \to (0,\infty)$ satisfies a Harnack inequality with a constant $A \geq 1$,

$$A^{-1}\rho(x) \leq \rho(z) \leq A\rho(x) \tag{HI(A)}$$

for all $z \in B(x,(1-|x|)/2)$ and all $x \in \mathbb{B}^n$, and a Euclidean volume growth condition with a constant $B > 0$,

$$\int_{B_\rho(x,r)} \rho^n \, dm_n \leq Br^n \tag{VG(B)}$$

for all $x \in \mathbb{B}^n$ and all $r > 0$, where m_n denotes n-dimensional Lebesgue measure, then ρ is called a *conformal deformation.* In [BKR, Theorem 3.1] they proved that a hyperbolic geodesic is essentially the shortest curve in the unit ball $\mathbb{B}^n$, also with respect to conformal deformations.

Subsequently, Herron showed in [Her1] that $\mathbb{B}^n$ can be replaced with any uniform metric measure space (Ω, d, μ) with bounded geometry. We proved in [1] that one can relax the assumption of "bounded geometry" of a uniform metric measure space, and still a quasihyperbolic geodesic is essentially the shortest curve with respect to conformal deformations.

Theorem 2.2. [1, Theorem 1.1] *Let $Q > 1$ and let (Ω, d, μ) be a non–complete uniform space equipped with a measure that is Q-regular on balls of Whitney type. If $\rho\colon \Omega \to (0,\infty)$ is a conformal deformation on Ω, then there is a constant $C \geq 1$ that depends only on the data associated with Ω and ρ such that*

$$\int_{[x,y]} \rho \, ds \leq C \int_{\gamma_{xy}} \rho \, ds$$

whenever $[x,y]$ is a quasihyperbolic geodesic and γ_{xy} is a curve joining x to y in Ω.

From now on the version of Gehring–Hayman theorem we are using in the metric space (Ω, d) says that for some $C \geq 1$ and for all $x, y \in \Omega$ it holds that

$$\mathit{length}_d([x,y]) \leq C\, \mathit{length}_d(\gamma_{xy}) \tag{2.3}$$

whenever $[x,y]$ is a quasihyperbolic geodesic and γ_{xy} is a curve joining x to y in Ω.

3 Homeomorphic equivalence of Gromov and original boundaries: A topological condition

3.1 Conformal deformations of the Euclidean unit ball

Let us consider the unit ball $\mathbb{B}^n$ equipped with a conformal metric d_ρ as a general metric space. What is its original boundary and what is then the

Gromov boundary? Are they homeomorphic and are they related to the Euclidean boundary? For simplicity, the metric notations which refer to the metric d_ρ will have just the additional subscript ρ.

Suppose the density ρ is a conformal deformation such that $\mathbb{B}^n_\rho := (\mathbb{B}^n, d_\rho)$ is bounded. The *ρ-boundary* $\partial_\rho \mathbb{B}^n$ of $\mathbb{B}^n_\rho$ is $\partial_\rho \mathbb{B}^n = \overline{\mathbb{B}^n_\rho} \setminus \mathbb{B}^n_\rho$, where $\overline{\mathbb{B}^n_\rho}$ is the metric completion of $\mathbb{B}^n_\rho$.

The quasihyperbolic metric space $(\mathbb{B}^n, k_\rho)$ derived from the metric d_ρ is complete, proper and geodesic. Bonk, Koskela and Rohde showed in [BKR, Proposition 6.2] that there is a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\rho(x)(1-|x|)}{\mathrm{dist}_\rho(x, \partial_\rho \mathbb{B}^n)} \leq C \qquad \text{for each } x \in \mathbb{B}^n,$$

and therefore the spaces $(\mathbb{B}^n, h)$ and $(\mathbb{B}^n, k_\rho)$ are bi–Lipschitz equivalent. Thus $(\mathbb{B}^n, k_\rho)$ is Gromov hyperbolic (cf. [BrHa, §III.H]). Furthermore, for the Gromov boundaries $(\partial_G \mathbb{B}^n, d_{w,\epsilon})$ and $(\partial_G \mathbb{B}^n_\rho, d'_{w,\epsilon})$ we obtain by comparing inequalities (1.4), (1.10) and (1.2) that there are constants $C \geq 1$ and $0 < \alpha \leq 1$ such that

$$C^{-1} d'_{w,\epsilon}(\xi,\eta)^{\frac{1}{\alpha}} \leq d_{w,\epsilon}(\xi,\eta) \leq C d'_{w,\epsilon}(\xi,\eta)^{\alpha}$$

for all $\xi, \eta \in \partial_G \mathbb{B}^n$, where ϵ has been chosen such that $d_{w,\epsilon}$, $d'_{w,\epsilon}$ are metrics (remember also inequality (1.12)). In particular, the Gromov boundaries are homeomorphic. We already know that the boundaries $\mathbb{S}^{n-1}$ and $\partial_G \mathbb{B}^n$ are homeomorphic (inequalities (1.6) and (1.13)) and hence the boundaries $\mathbb{S}^{n-1}$ and $\partial_G \mathbb{B}^n_\rho$ are homeomorphic.

But what about the boundaries $\partial_G \mathbb{B}^n_\rho$ and $\partial_\rho \mathbb{B}^n$? Let us go back to one of the previous examples to see that these boundaries are not always homeomorphic. Let $\rho(z) = |f'(z)|$, where $f\colon \mathbb{B}^2 \to \Omega$ is a conformal mapping onto the simply connected domain Ω in (1.17) and pictured in Figure 3. When the metric ℓ in Ω is the Euclidean path metric, we can identify $(\mathbb{B}^2, d_\rho)$ with (Ω, ℓ) and the boundary $\partial_\rho \mathbb{B}^2$ with $\partial_\ell \Omega$. We already discovered that the boundary $\partial_\ell \Omega$ is not compact and thus the boundary $\partial_\rho \mathbb{B}^2$ is not compact. Hence, $\partial_\rho \mathbb{B}^2$ cannot be homeomorphic to $\mathbb{S}^1$ or to $\partial_G \mathbb{B}^2_\rho$.

It seems that in order to show that $\partial_\rho \mathbb{B}^n$ is homeomorphic to $\mathbb{S}^{n-1}$, and thus to $\partial_G \mathbb{B}^n_\rho$, we have to assume the compactness of the boundary $\partial_\rho \mathbb{B}^n$. This turns out to be both necessary and sufficient condition, and the next theorem (that is a consequence of Theorem 3.3 below) shows that the following conditions are equivalent:

$$\begin{aligned}
&\text{(i) } (\partial_\rho \mathbb{B}^n, d_\rho) \text{ is compact}\\
&\text{(ii) } (\mathbb{B}^n \cup \partial_\rho \mathbb{B}^n, d_\rho) \text{ is compact}\\
&\text{(iii) The identity map } \mathrm{id}\colon (\mathbb{B}^n, d_\rho) \to (\mathbb{B}^n, |\cdot|) \text{ has a homeomorphic}\\
&\qquad \text{extension to } i\colon (\mathbb{B}^n \cup \partial_\rho \mathbb{B}^n, d_\rho) \to (\mathbb{B}^n \cup \mathbb{S}^{n-1}, |\cdot|).
\end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose that* $\rho\colon \mathbb{B}^n \to (0,\infty)$ *is a conformal deformation. Then the three conditions in* (3.1) *are equivalent.*

Combining this theorem with the previous discussion we obtain a corollary:

Corollary 3.2. *Let* $\rho\colon \mathbb{B}^n \to (0,\infty)$ *be a conformal deformation. Then* $\partial_\rho \mathbb{B}^n$ *is compact if and only if the Gromov boundary* $\partial_G \mathbb{B}^n_\rho$ *is homeomorphic to the* ρ*-boundary* $\partial_\rho \mathbb{B}^n$.

In the classical situation Theorem 3.1 really says that a conformal mapping $f\colon \mathbb{B}^2 \to \Omega \subset \mathbb{C}$ has a homeomorphic extension from $\overline{\mathbb{B}^2}$ onto $\Omega \cup \partial_\ell \Omega$, where Ω is equipped with the Euclidean path metric ℓ, if and only if the boundary $\partial_\ell \Omega$ is compact (cf. [Po]).

We can actually show that Theorem 3.1 holds in a more general setting, and this is the main theorem in the paper [2]:

Theorem 3.3. [2, Theorem 1.1] *Suppose that* $\rho\colon \mathbb{B}^n \to (0,\infty)$ *is a density that satisfies either*

$$\int_{B_\rho(x,r)} \rho^n \, dm_n \le \varphi(r) \qquad \textit{for all } x \in \mathbb{B}^n \textit{ and } r>0,$$

where φ *is an increasing homeomorphism of* $(0,\infty)$ *so that* $\varphi(r) = Br^n |\log r|^{n-1}$ *for some* $B>0$ *and for every* $0<r<e^{-1}$, *or both a Harnack inequality HI(A) and for some* $B>0$ *it holds that*

$$\int_{B_\rho(x,r)} \rho^n \, dm_n \le Br^{n-\epsilon} \qquad \textit{for all } x \in \mathbb{B}^n \textit{ and } r>0,$$

where ϵ *is sufficiently small, depending on the constant in HI(A). Then the three conditions in* (3.1) *are equivalent.*
In addition, when $n=2$ *these three conditions are also equivalent to*

(iv) $\mathbb{B}^n_\rho$ *is bounded and* $\partial_\rho \mathbb{B}^n$ *is locally connected.*

However, when $n \ge 3$, *there are examples where* (iv) *holds but none of* (i,ii,iii) *in* (3.1) *are true.*

We do not know, whether or not Corollary 3.2 holds in the setting in Theorem 3.3. Firstly, we do not know the precise criteria for $(\mathbb{B}^n, k_\rho)$ to be Gromov hyperbolic. For example, if $\rho(z) = \frac{1}{1-|z|}$, then the metric space $(\mathbb{B}^n, d_\rho)$ is complete and the boundary $\partial_\rho \mathbb{B}^n$ is empty. In this case we are not able to define the quasihyperbolic metric k_ρ. If $\rho(z) = 1-|z|$, then the metric space $(\mathbb{B}^n, d_\rho)$ is bounded and it has a single boundary point. Furthermore, $(\mathbb{B}^n, k_\rho)$ is Gromov hyperbolic because quasihyperbolic triangles are π-thin. By suitably gluing versions of the above densities one further obtains an unbounded space whose boundary is a singleton, and whose quasihyperbolization is Gromov hyperbolic. Secondly, even though $(\mathbb{B}^n, k_\rho)$ is Gromov hyperbolic in the setting of Theorem 3.3, we do not know, if the Gromov boundary $\partial_G \mathbb{B}^n_\rho$ is homeomorphic to $\mathbb{S}^{n-1}$.

3.2 A general metric space

Let us prove a metric space version of Theorem 3.1 (and Corollary 3.2).

Theorem 3.4. *Let (Ω, d) be locally compact, non–complete, bounded, and quasi–convex metric space. Assume that (Ω, k_d) is Gromov hyperbolic and that (Ω, d) satisfies the Gehring–Hayman theorem. Let $w \in \Omega$ be a base point. Then the following three conditions are equivalent:*

(i) *$(\partial_d\Omega, d)$ is compact*

(ii) *$(\Omega \cup \partial_d\Omega, d)$ is compact*

(iii) *The identity map* $\mathrm{id}\colon (\partial_G\Omega, d_{w,\epsilon}) \to (\partial_d\Omega, d)$ *is a homeomorphism.*

Proof. The implications (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i) are clear because $(\partial_G\Omega, d_{w,\epsilon})$ is always compact and $\partial_d\Omega \subset \overline{\Omega}$ is closed. The implication (i) $\Rightarrow$ (ii) follows by adapting the proof of the same assertion for Theorem 3.3:

Let $\{A_\alpha\}_{\alpha\in I}$ be a d-open cover of the set $\Omega\cup\partial_d\Omega$. Let $J \subset I$ be a maximal index set such that for every $\alpha \in J$ it holds that $A_\alpha \cap \partial_d\Omega \neq \emptyset$. Because $\partial_d\Omega$ is compact, there exists a finite index set $J' \subset J$ such that

$$\partial_d\Omega \subset \bigcup_{\alpha\in J'} A_\alpha.$$

Now $A := (\Omega \cup \partial_d\Omega) \setminus \bigcup_{\alpha\in J'} A_\alpha$ is d-closed and $A \subset \Omega$. Because the identity map $(\Omega, d) \to (\Omega, k_d)$ is a homeomorphism, A is closed also in the sense of k_d. Because $\partial_d\Omega$ is compact and A is d-closed, it holds that there exist $c > 0$ such that $d(z) \geq c$ for every $z \in A$. Thus, because (Ω, d) satisfies the Gehring–Hayman theorem and it is quasi–convex and bounded, A is also bounded in the sense of k_d. Hence, there is $x \in \Omega$ and $M > 0$ such that

$$A \subset \overline{B}_{k_d}(x, M),$$

and because of the properness of the space (Ω, k_d) the set A is compact. Consequently, A is compact in the metric d. Moreover, there is a finite index set $I' \subset I$ so that

$$A \subset \bigcup_{\alpha\in I'} A_\alpha,$$

and hence we find a finite index set $I' \cup J'$ such that

$$\Omega \cup \partial_d\Omega \subset \bigcup_{\alpha\in I'\cup J'} A_\alpha.$$

Therefore $(\Omega \cup \partial_d\Omega, d)$ is compact.

The last step is to prove the implication (ii) $\Rightarrow$ (iii). We show only that the identity map $\mathrm{id}\colon (\partial_G\Omega, d_{w,\epsilon}) \to (\partial_d\Omega, d)$ is well–defined because the rest of the proof follows direct from other results: The same arguments as in

[3, Theorem 1.1] apply to prove that the map id is bijective and that id^{-1} is continuous. Therefore, the well–known topological result (see e.g. [Mu]) implies that id^{-1} is a homeomorphism, hence id is as well.

Thus let $\xi \in \partial_G\Omega$. Then there is a geodesic ray $[w,\xi]$. Let $(x_j) \subset [w,\xi]$ be a sequence such that $k_d(w,x_j) \to \infty$, when $j \to \infty$. Because $(\Omega \cup \partial_d\Omega, d)$ is compact, in the metric d the sequence (x_j) has a convergent subsequence $(x_{j_k})_k$. From the Gehring–Hayman theorem and quasi–convexity we obtain that for every $r > 0$ there is $N_r > 0$ such that for each $p \geq 1$,

$$\ell_d([x_{j_k}, x_{j_{k+p}}]) < r,$$

whenever $k > N_r$. Thus the limit $\xi' := \lim_{j\to\infty} x_j$ exists in the metric d and it has to be on the boundary $\partial_d\Omega$.

Let $[y,\xi]$ be another geodesic ray that represents the point $\xi \in \partial_G\Omega$. In the sense of d the ray $[y,\xi]$ ends at the point $\xi'' \in \partial_d\Omega$. Let us show that $\xi' = \xi''$. Let $u \in \{w,y\}$, and let $[u,\xi](t)$ be the image of $t \in [0,\infty) \subset \mathbb{R}$, where $[u,\xi]\colon [0,\infty) \to \Omega$ is a mapping parametrized by the arc length with respect to the metric k_d. Then because the geodesic rays $[w,\xi]$ and $[y,\xi]$ are equivalent, there exists a constant $M > 0$ such that

$$k_d([w,\xi](t), [y,\xi](t)) < M \qquad \text{for each } t \geq 0$$

(for details see the proof of [3, Theorem 1.1]). Furthermore, by the elementary inequality (see [GP, Lemma 2.1] and [BHK, Inequality (2.4)]) it follows that for each $t \geq 0$,

$$\log\Big(1 + \frac{d([w,\xi](t), [y,\xi](t))}{\min\{d([w,\xi](t)), d([y,\xi](t))\}}\Big) \leq k([w,\xi](t), [y,\xi](t)) < M.$$

Therefore because $d([w,\xi](t)) \to 0$ and $d([y,\xi](t)) \to 0$, when $t \to \infty$, it must be that

$$d([w,\xi](t), [y,\xi](t)) \to 0,$$

when $t \to \infty$, and thus $\xi' = \xi''$.

$\square$

4 Homeomorphic equivalence of Gromov and original boundaries: An analytic condition

We have given a partial, topological answer to our question: "When is the original boundary of a metric space homeomorphic to the Gromov boundary of the same space?" We now turn to the analytic aspects, following [3]. Here the starting point is also [BHK, Theorem 1.11] with the paper [HenK].

Based on [BHK], a bounded domain in $\mathbb{R}^n$ is uniform if and only if it is Gromov hyperbolic in the quasihyperbolic metric and its Euclidean boundary is quasisymmetrically equivalent to the Gromov boundary. It is also well–known that in a bounded uniform domain $\Omega \subset \mathbb{R}^n$ the quasihyperbolic metric k satisfies a logarithmic growth condition

$$k(w,x) \leq C \log\Big(\frac{d(w)}{d(x)}\Big) + C' \tag{4.1}$$

where w is a fixed base point in Ω and constants $C \geq 1$ and $C' < \infty$ depend on the constant of uniformity and the diameter of the domain (cf. [GM]). Here $d(x)$ is an abbreviation for the Euclidean distance from the point x to the Euclidean boundary $\partial\Omega$.

However, this growth condition (4.1) does not necessarily guarantee that the boundaries are quasisymmetric. Indeed, let us construct a simply connected planar domain Ω for which the quasihyperbolic metric satisfies the growth condition (4.1), but the boundary $\partial_\ell\Omega$ in the Euclidean path metric is not quasisymmetrically equivalent to the Gromov boundary $\partial_G\Omega$. To construct Ω, we "weld" the sequence of squares $Q_j = (a_j - l_j, a_j) \times (1, l_j)$, where $a_j = 1 - 2^{-j}$ and $l_j = 2^{-j}$, $j = 0, 1, 2, \ldots$, to the square $(-1,1)^2$ via the intervals $(a_j - l_j/2 - l_j^2, a_j - l_j/2 + l_j^2) \times \{1\}$, $j = 2, 3, 4, \ldots$, as in Figure 4. Let the origin be the base point. The quasihyperbolic metric k satisfies the condition (4.1), but taking two boundary points from the "throat" of the small square and the third from the top middle of the same square shows that the Gromov boundary cannot be quasisymmetric to $\partial_\ell\Omega$. Nevertheless, the boundary $\partial_\ell\Omega$ is homeomorphic to the Gromov boundary $\partial_G\Omega$, see [BP] and [KOT].

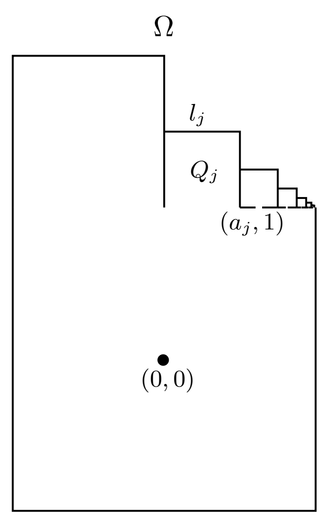

Figure 4: Example of a domain Ω, whose boundary in the Euclidean path metric is not quasisymmetric to the Gromov boundary

Moreover, not every growth condition guarantees that the boundaries can be identified even as sets. For example, let

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq -1,\ |y| < \exp\{-x\}\},$$

and let the origin be the base point. Now the quasihyperbolic metric k satisfies the growth condition

$$k(0, z) \leq C\frac{d(0)}{d(z)} + C \tag{4.2}$$

for some $C \geq 1$, but the the ray $[0, \infty)$ cannot be identified with any point in the Euclidean boundary $\partial\Omega$.

Thus, in order that the boundary $\partial_\ell\Omega$ and the Gromov boundary $\partial_G\Omega$ be homeomorphic, we need a condition stronger than (4.2). Suppose that we are given the growth condition

$$k_d(w, x) \leq \phi\Big(\frac{d(w)}{d(x)}\Big), \tag{4.3}$$

where $\phi\colon (0, \infty) \to (0, \infty)$ is an increasing function and w is a fixed base point in a metric space (Ω, d), cf. [HenK]. It is more convenient to write

$$d(x) \leq \frac{d(w)}{\phi^{-1}(k_d(w, x))}, \tag{4.4}$$

and let us assume that the function ϕ satisfies

$$\sum_{j=1}^{\infty} \frac{1}{\phi^{-1}(j)} < \infty. \tag{4.5}$$

Condition (4.5) is sufficient for the original boundary and the Gromov boundary to be homeomorphic in a rather general setting. This is the content of our main theorem in [3].

Theorem 4.1. [3, Theorem 1.1] *Let (Ω, d) be a locally compact, and non–complete quasi–convex space. Assume that (Ω, k_d) is Gromov hyperbolic and that the Gehring–Hayman theorem holds in (Ω, d). Let $w \in \Omega$ be a base point and suppose that the quasihyperbolic metric k_d satisfies* (4.3) *and* (4.5). *Under these assumptions, the identity map* $\mathrm{id}\colon (\partial_G\Omega, d_{w,\epsilon}) \to (\partial_d\Omega, d)$ *is a homeomorphism, and, moreover, $\partial_d\Omega$ is compact.*

5 Final remarks

Even though we have now given two partial answers to our problem which considered, when is the original boundary of a metric space homeomorphic to the Gromov boundary of the same space, there are still some parts missing in the theory. When is the quasihyperbolic metric space (Ω, k_d) Gromov hyperbolic? Is the Gehring–Hayman theorem always true in (Ω, d) if (Ω, k_d) is Gromov hyperbolic? What are the minimal assumptions on (Ω, d) so that geometric characterizations of Gromov hyperbolicity can be done as in [BB]?

The Gehring–Hayman theorem for conformal deformations also raises further questions. We already stated in the introduction of [1] that the Gehring–Hayman theorem is a central tool in many papers, especially in [Her1] and [Her2]. We expect that Theorem 2.2 will allow one to relax the assumption of "bounded geometry" and thus extend large parts of the papers [Her1] and [Her2] to a much more general setting.

In conclusion, this field is not totally studied and there is place for further research.

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