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EXISTENCE, UNIQUENESS AND QUALITATIVE PROPERTIES OF ABSOLUTE MINIMIZERS

VESA JULIN

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Jyväskylä, July 2010

Vesa Julin

List of included articles

[A] V. Julin, *Solutions of Aronsson equation near isolated points*, Calculus of Variations and Partial Differential Equations **37** (2010), 303–328.

[B] S. N. Armstrong, M. G. Crandall, V. Julin, C. K. Smart, *Convexity criteria and uniqueness of absolutely minimizing functions*, to appear in Archive for Rational Mechanics and Analysis, preprint arXiv:1003.3171.

[C] V. Julin, *Existence of an absolute minimizer via Perron's method*, to appear in Journal of Convex Analysis, preprint arXiv:1004.5008.

Author's contribution in the paper [B] was altogether strong. In particular, in the proofs of [B, Theorem 2.1] and in [B, Proposition 3.2], [B, Lemma 5.1] and [B, Proposition A,1]. Originally the idea of finite difference scheme is due to Le Gruyer and Archer [26] and Armstrong and Smart [1].

Introduction

This thesis deals with problems which arise when we minimize the functional

$$F(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} H(x,u(x),Du(x)), \tag{0.1}$$

where u is locally Lipschitz, $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ for short, and $Du(x)$ denotes the gradient of u at x. By Rademacher's theorem, any locally Lipschitz continuous function is differentiable at almost every point, and therefore (0.1) makes sense. We say that a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is *an absolute minimizer of (0.1) in Ω* or *a solution of the minimization problem in Ω* if for every $V \subset\subset \Omega$ it holds that

$$F(v,V) \geq F(u,V) \quad \text{for all } v \in \mathrm{Lip}_{\mathrm{loc}}(V)\cap C(\bar V),\ v = u \,\text{on}\, \partial V. \tag{0.2}$$

Definition (0.2) originates from Aronsson [2], [3], who was the first to study the minimization problem (0.1). There are many applications for this L^∞ calculus of variation, for example, in studying density-dependent Bingham fluids (such as land-sliding) [21] and dielectric breakdown [20]. These applications typically involve materials with a certain threshhold which determines how the material behaves. Let us for instance consider the following: If the electric field Du in a given body Ω satisfies $|Du(x)| \leq M$ for all $x \in \Omega$, the body behaves like an insulator, but as soon as this condition is violated, the body starts to conduct. It is thus reasonable to model this by functional of the form (0.1) since the dielectric breakdown occurs due to pointwise excessive energy.

Article [C] deals with the existence of an absolute minimizer of (0.1) under Dirichlet boundary conditions. In the other two papers we consider a special case of (0.1) by assuming that $H(x,s,p) = H(p)$. In [A] we study the behavior of absolute minimizers near an isolated singular point, and in [B] we prove the uniqueness of the Dirichlet problem. In the latter the significant point is that the proof does not use the theory of viscosity solutions at all.

1. Infinity harmonic functions

As an introduction, let us first consider a model case where the functional (0.1) has the form

$$F(u,\Omega) = \|Du\|_{L^\infty(\Omega)}. \tag{1.1}$$

If $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is an absolute minimizer of (1.1), we say that it is *infinity harmonic in Ω*. This model case is important because of its simple structure and because all the techniques that are developed to study (1.1) can be applied to more general functionals of the form

$$F(u,\Omega) = \|H(Du)\|_{L^\infty(\Omega)}. \tag{1.2}$$

We will discuss these general functionals in the next section.

The model problem (1.1) was first studied by Aronsson [4]. He approached it in the following way: for a given bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in \mathrm{Lip}(\partial\Omega)$ find "the best" Lipschitz extension of g to Ω. By the best extension we mean that the function should satisfy the condition

$$L_u(V) = L_u(\partial V) \quad \text{for all } V \subset \Omega, \tag{1.3}$$

where $L_u(E) = \sup_{x,y\in E} \frac{u(y)-u(x)}{|y-x|}$ denotes the smallest Lipschitz constant of u in a set E. Notice that for any function v in any bounded domain V we have that $L_v(V) \geq$

$L_v(\partial V)$. We can surely find a Lipschitz extension of g by the McShane-Whitney formulas

$$\Psi(x) = \inf_{y \in \partial\Omega} (\, g(y) + L_g(\partial\Omega)|y - x|\,)$$
$$\Lambda(x) = \sup_{y \in \partial\Omega} (\, g(y) - L_g(\partial\Omega)|y - x|\,),$$

but these functions do not satisfy the condition (1.3) except in some trivial cases. To be precise, Ψ and Λ satisfy (1.3) if and only if $\Psi \equiv \Lambda$, see [4]. It turns out that the functions which satisfy condition (1.3) are exactly the ones that are infinity harmonic, see [5].

Aronsson proved [4] that there always exists a solution for the minimization problem (1.1) for any Lipschitz Dirichlet boundary data. He also formally derived the infinity Laplace equation

$$\triangle_\infty u(x) = (D^2u(x)Du(x)) \cdot Du(x) = 0 \tag{1.4}$$

where $D^2u(x)$ denotes the Hessian of u. Aronsson proved that $u \in C^2(\Omega)$ is infinity harmonic if and only if it satisfies (1.4), but he did not yet have the tools in the 1960's to interpret the equation (1.4) for non-smooth functions. This was a major problem since he knew that there are non-smooth infinity harmonic functions, such as $u(x,y) = y^{\frac{4}{3}} - x^{\frac{4}{3}}$ in the plane.

After the development of the viscosity solution theory by Crandall and P.L. Lions in the 1980's, Jensen [22] proved that infinity harmonic functions are exactly the ones which satisfy (1.4) in the viscosity sense. He also proved the existence and uniqueness of infinity harmonic functions for continuous Dirichlet boundary data. There are more recent proofs for the uniqueness by Crandall, Gunnarsson and P. Y. Wang [13], Barles and Busca [6], and Armstrong and Smart [1]. We recall that u is *a viscosity subsolution of (1.4) in Ω* if it is upper semicontinuous, and whenever $\varphi \in C^2(\Omega)$ is such that $\varphi - u$ has a local minimum at some point $x \in \Omega$, then we have that

$$\triangle_\infty \varphi(x) \geq 0.$$

A function u is a viscosity supersolution of (2.2) in Ω if $-u$ is a viscosity subsolution and u is a viscosity solution if it is both a sub- and a supersolution. For more about the theory of viscosity solutions see [14].

We will introduce yet another characterization of infinity harmonic functions called *comparison with cones*, which was first properly stated by Crandall, Evans and Gariepy [12]. Before stating the definition, we observe that for all $a > 0$ the cone function $C(x) = a|x|$ is a smooth solution of the infinity Laplace equation (1.4) in $\mathbb{R}^n \backslash \{0\}$. This can be easily seen by noticing that

$$|DC(x)| = a \quad \text{for all } x \neq 0. \tag{1.5}$$

By differentiating (1.5) we get

$$D(|DC(x)|) = \frac{1}{a} D^2C(x)DC(x) = 0 \qquad \text{for all } x \neq 0,$$

and therefore $\triangle_\infty C(x) = 0$ in $\mathbb{R}^n \backslash \{0\}$. The cone functions can thus be regarded as *fundamental solutions* of the infinity Laplace equation. We will proceed and say that $u \in C(\Omega)$ satisfies *the comparison with cones in Ω* if whenever $V \subset\subset \Omega$ and $x_0 \in \mathbb{R}^n \backslash V$ we have for every $a \geq 0$ that

$$\max_{x \in \bar{V}} (u(x) - a|x - x_0|) = \max_{x \in \partial V} (u(x) - a|x - x_0|) \tag{1.6}$$

and

$$\min_{x\in\bar{V}}(u(x)+a|x-x_0|)=\min_{x\in\partial V}(u(x)+a|x-x_0|). \tag{1.7}$$

This is a sort of weak comparison principle since we have the comparison principle with fundamental solutions only. In their paper Crandall, Evans and Gariepy gave an elegant proof for the following equivalencies:

(i) u is infinity harmonic in Ω
(ii) u is a viscosity solution of (1.4) in Ω,
(iii) u satisfies comparison with cones in Ω.

The comparison with cones has turned out to be a fruitful point of view, and, for example Savin's [28] proof for C^1 regularity of infinity harmonic functions in the plane is entirely based on cones. At this point we note that the regularity of infinity harmonic functions is an extremely difficult issue and is completely open in higher dimensions. This is mainly due to the fact that the infinity Laplace equation (1.4) is badly degenerate, and therefore, every known PDE method for regularity seems to fail. In the plane, infinity harmonic functions are known to be $C^{1,\alpha}$ regular for some $\alpha\in(0,\frac{1}{3}]$ according to Evans and Savin [18]. The upper bound for α can be seen by considering the Aronsson function $u(x,y)=y^{\frac{4}{3}}-x^{\frac{4}{3}}$.

Finally, we note that infinity harmonic functions are the limit of p-harmonic functions as p approaches infinity. This can be seen in the following way. Suppose that u_p solves the p-Laplace equation

$$\begin{aligned}\triangle_p u &:= \operatorname{div}(|Du|^{p-2}Du)\\ &= (p-2)|Du|^{p-4}\triangle_\infty u+|Du|^{p-2}\triangle u=0 \qquad \text{in } \Omega,\end{aligned} \tag{1.8}$$

where $\triangle u(x)=\sum_{i=1}^n \partial_{x_{ii}}u(x)$ is the Laplacian. Dividing (1.8) by $(p-2)|Du|^{p-4}$, we obtain that u_p solves

$$\triangle_\infty u_p+\frac{1}{p-2}|Du_p|^2\triangle u_p=0.$$

If $u_p\to u$ as $p\to\infty$ locally uniformly in Ω, then standard theorems about viscosity solutions imply that $\triangle_\infty u=0$, and therefore it is infinity harmonic in Ω.

2. L^∞-variational problem of the form $H(p)$ and papers [A] and [B]

Let us consider a somewhat more general L^∞-variational problem of the form (1.2), where the integrand $H:\mathbb{R}^n\to\mathbb{R}$ is called the Hamiltonian and it is assumed to be convex and coersive i.e. $H(p)\to\infty$ as $|p|\to\infty$. As we mentioned earlier, the techniques introduced in the previous section can be applied in the case of the functional (1.2). A cone with a slope $k\geq 0$ is defined as

$$C_k(x):=\max_{H(p)=k} p\cdot x.$$

It is easy to see that C_k is convex and 1-homogeneous, which implies that it is Lipschitz continuous in $\mathbb{R}^n$. In fact, C_k has all the properties of a norm except that it is not necessarily symmetric, that is, $C_k(x)\neq C_k(-x)$. Moreover, we have that

$$H(DC_k(x))=k \tag{2.1}$$

at the points where C_k is differentiable (compare (2.1) to (1.5)). The concept of comparison with cones can be defined as in the previous section by replacing $a|x-x_0|$

with $C_k(x - x_0)$ in (1.6) and with $C_k(x_0 - x)$ in (1.7). The equation corresponding to (1.4) has the form

$$\mathcal{A}_H[u](x) := D^2u(x)H_p(Du(x)) \cdot H_p(Du(x)) = 0 \tag{2.2}$$

where H_p denotes the gradient of H and is called the Aronsson equation. It is interpreted in the viscosity sense as before. As in the case of infinity harmonic functions, we have the following theorem.

Theorem 2.1. *The following are equivalent for smooth H,*

(i) *u is an absolute minimizer of (1.2),*
(ii) *u is a viscosity solution of (2.2) in Ω,*
(iii) *u satisfies the comparison with cones in Ω.*

For the exact assumptions on H and for the proofs, see Gariepy, Wang and Yu [19]. For the proof of the implication $(ii) \Rightarrow (iii)$, see Crandall, Wang and Yu [15].

2.1. **Paper [A].** In this paper we study the behavior of absolute minimizers of (1.2) near an isolated singular point. The Hamiltonian is assumed to satisfy the following conditions,

(A1) $H \in C^2(\mathbb{R}^n)$
(A2) $H(0) = \min_{\mathbb{R}^n} H = 0$,
(A3) H is uniformly convex,

$$\alpha|\xi|^2 \le H_{pp}\xi \cdot \xi \le \beta|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

The function $H(p) = \frac{1}{2}|p|^2$ satisfies the assumptions $(A1) - (A3)$. In this case the Aronsson equation (2.2) is the infinity Laplace equation (1.4). Notice also that the uniform convexity of H guarantees that H is coersive, that is $H(p) \to \infty$ as $|p| \to \infty$.

We will employ Theorem 2.1 and treat u as a viscosity solution of (2.2) rather than an absolute minimizer of (1.2). The goal is to show that such a function behaves asymptotically like a cone C_k near an isolated singular point, and the cone C_k can be regarded as *a fundamental solution* of (2.2). The fact that C_k solves (2.2) in $\mathbb{R}^n \setminus \{0\}$ follows from the fact that under the assumptions $(A1) - (A3)$ we have that $C_k \in C^2(\mathbb{R}^n \setminus \{0\})$. Differentiating (2.1) yields

$$D^2C_k(x)H_p(Du(x)) = 0 \qquad \text{for all } x \neq 0,$$

and therefore $\mathcal{A}_H[C_k](x) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

The following classical result about isolated singularities is due to Bôcher from 1903 [10]. Recall that $u \in C^2(\Omega)$ is *harmonic in Ω* if it is a (classical) solution of the Laplace equation

$$\triangle u(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x) = 0 \qquad \text{in } \Omega.$$

Theorem 2.2 (Bôcher)**.** *Suppose $n \ge 2$. If a nonnegative harmonic function $u \in C(B^n(0,1) \setminus \{0\})$ becomes infinite at the origin, then u has the form*

$$u(x) = C\Psi(x) + v(x) \quad \textit{where} \quad \Psi(x) = \begin{cases} -\log|x| & \textit{when } n = 2 \\ |x|^{n-2} & \textit{when } n \ge 3, \end{cases}$$

and v is harmonic in $B(0,1)$ and $C \ge 0$ is a constant. Function Ψ is called the fundamental solution *of the Laplace equation.*

The assumption of nonnegativity cannot be removed. For example, a function $u(x) = \partial_{x_1}\Psi(x)$, where Ψ is the fundamental solution of the Laplace equation, is harmonic in $B(0,1)\backslash\{0\}$ but cannot be of the form $C\Psi(x)+v(x,y)$ for some harmonic v.

Similar problems were studied by Serrin in [30], [31] for singular solutions of a quasilinear elliptic equation of type

$$\operatorname{div}(\mathcal{A}(x,u,Du)) = \mathcal{B}(x,u,Du),$$

where p-Laplacian (1.8) is the model case. Singular p-harmonic functions are further investigated, for example, in [25], where the main result states roughly that near an isolated singular point a nonnegative p-harmonic function behaves asymptotically like $|x|^{\frac{p-n}{p-1}}$ when $p < n$ and $-\log|x|$ when $p = n$, which are the fundamental solutions of the p-Laplace equation. As discussed in [25], a similar result holds for the p-Laplace equation when $p > n$ and the fundamental solution is $|x|^{\frac{p-n}{p-1}}$. See also the work by Mandfredi [27], where he considers the case $2 < p < \infty$ in the plane.

Savin, Wang and Yu [29] proved that near a singular point an infinity harmonic function behaves asymptotically like a cone $|x|$. In my paper I generalize this result for the Aronsson equation (2.2). Here is the main result [A, Theorem 1.2].

Theorem 2.3. *Suppose $n \geq 2$ and that H satisfies the conditions $(A1)-(A3)$. Let a function $u \geq 0$ be a viscosity solution of the Aronsson equation (2.2) in $B(x_0,r)\backslash\{x_0\}$. Then one of the two alternatives holds:*

(i) *u is a viscosity solution of (2.2) in the whole ball $B(x_0,r)$*
(ii) *either*

$$u(x) = b + C_k(x-x_0) + o(|x-x_0|)$$

or

$$u(x) = b - C_k(x_0-x) + o(|x-x_0|),$$

for some $k > 0$ and $b \geq 0$.

Using Theorem 2.3 and some regularity estimates from [32], we derive an interesting corollary [A, Corollary 1.3], which is a generalization of the Corollary 1.2 from [29].

Corollary 2.4. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain and $x_0 \in \Omega$. Let a function $u \in C(\overline{\Omega})$ be a solution of the Aronsson equation (2.2) in $\Omega\backslash\{x_0\}$ which takes boundary values $u = 1$ in $\partial\Omega$ and $u(x_0) = 0$. Then $u \in C^2(\Omega\backslash\{x_0\})$ if and only if*

$$u(x) = C_k^H(x-x_0)$$

and $\Omega = \{x \in \mathbb{R}^2 \mid C_k^H(x-x_0) < 1\}$ for some $k > 0$.

One consequence of this corollary is that, unlike p-harmonic functions, infinity harmonic functions are in general not smooth near a singular point.

2.2. **Paper [B].** In this section we still consider the minimizing problem (1.2) and assume the Hamiltonian H to satisfy the following conditions,

(B1) H is convex,
(B2) $H(0) = \min_{\mathbb{R}^n} H = 0$,
(B3) the set $\{p \mid H(p) = 0\}$ is bounded and has empty interior.

The most notable difference to the previous section is that now H is not assumed to be smooth, which causes a problem with the Aronsson equation (2.2). Notice that the assumptions $(B1)$ and $(B3)$ guarantee that H is coersive.

It turns out to be useful to split the definition of absolute minimizer into two halves.

Definition 2.5. A function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is *an absolute subminimizer of (1.2) in Ω* if the inequality (0.2) holds for all $v \in \mathrm{Lip}_{\mathrm{loc}}(V) \cap C(\bar{V})$ such that $v \leq u$ and $v = u$ on ∂V. Similarly $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is *an absolute superminimizer of (1.2) in Ω* if (0.2) holds for all v such that $v \geq u$ and $v = u$ on ∂V.

It is easy to verify that a function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is an absolute minimizer of (1.2) if and only if it is both absolute sub- and superminimizer.

The main result in the paper is the following comparison principle [B, Theorem 2.1].

Theorem 2.6. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $u, v \in C(\bar{\Omega})$ are absolutely sub- and superminimizing in Ω, respectively. Then*

$$\max_{\bar{\Omega}}(u - v) = \max_{\partial\Omega}(u - v). \tag{2.3}$$

In particular with given Dirichlet boundary values $g \in C(\partial\Omega)$, the absolute minimizer $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ for which $u = g$ on $\partial\Omega$ is unique.

This comparison principle was first proved by Jensen, C. Wang and Y. Yu in [23], where they assumed H to be smooth. This assumption allows the authors to use Theorem 2.1, that is, absolute minimizers are solutions of the Aronsson equation (2.2). By using this the comparison principle (2.3) can be proven with the help of the machinery of viscosity solution theory. The main point in Theorem 2.6 is that here H is only assumed to be convex, a fact that requires completely new arguments.

Armstrong and Smart [1] proved the comparison principle (2.3) in the case $H(p) = |p|$ using basically only the fact that for infinity harmonic function u the map

$$t \mapsto u^t(x) = \max_{y \in \bar{B}(x,t)} u(y) \tag{2.4}$$

is convex and

$$t \mapsto u_t(x) = \min_{y \in \bar{B}(x,t)} u(y) \tag{2.5}$$

is concave for every x. Our idea is to prove Theorem 2.6 by using similar arguments and therefore avoid the use of (2.2). The first thing we need to know is the right convexity and concavity criterion similar to (2.4) and (2.5) when u is an absolute minimizer of (1.2).

It was conjectured by Barron, Evans and Jensen [7] that in the case of general Hamiltonian H one should replace $u^t(x)$ in (2.4) by $v(t, x)$ which solves the evolutionary Hamilton-Jacobi equation

$$\partial_t v = H(Dv) \quad \text{and} \quad v(0, x) = u(x) \tag{2.6}$$

and $u_t(x)$ in (2.5) by $w(t, x)$ which solves

$$\partial_t w = -H(Dw) \quad \text{and} \quad w(0, x) = u(x), \tag{2.7}$$

where u is an absolute minimizer of (1.2). Then $t \mapsto v(t, x)$ should be convex and $t \mapsto w(t, x)$ concave for every x. The formal argument for this is the following. Suppose that the absolute minimizer u and the Hamiltonian H are both smooth. By Theorem 2.1 u is a solution of (2.2). Differentiating (2.6) with respect to t and x_j yields that

$$\partial_{tt} v = \sum_{j=1}^{n} H_{p_j}(Dv) \partial_{tx_j} v$$

and

$$\partial_{x_j t} v = \sum_{i=1}^{n} H_{p_i}(Dv)\partial_{x_j x_i} v.$$

By inserting the second equation to the first, we get

$$\partial_{tt} v = \sum_{j,i=1}^{n} H_{p_j}(Dv)H_{p_i}(Dv)\partial_{x_j x_i} v = \mathcal{A}_H[v].$$

Next we use the fact that the flow (2.6) preserves subsolutions, which is not entirely obvious and we skip the argument in order not to make these formal calculations too long. Since u is a solution of (2.2), then $v(t,\cdot)$ is a subsolution of (2.2) for every $t > 0$, which implies that $\mathcal{A}_H[v(t,\cdot)] \geq 0$ for every $t > 0$, if v is assumed to be smooth. Hence,

$$\partial_{tt} v \geq 0 \quad \text{for every } t > 0,$$

and the map $t \mapsto v(t,x)$ is therefore convex for every x. The formal argument for the concavity of $t \mapsto w(t,x)$ is analogous.

This argument was entirely formal. Absolute minimizers are in general not smooth, and since the Hamiltonian H was not assumed to be smooth, we are not allowed to use the equation (2.2). However, since H is convex, we can at least solve (2.6) and (2.7) by using the Hopf-Lax formulas

$$T^t u(x) := \sup_{y\in\Omega}\Big(u(y) - tL\Big(\frac{y-x}{t}\Big)\Big)$$
$$T_t u(x) := \inf_{y\in\Omega}\Big(u(y) + tL\Big(\frac{x-y}{t}\Big)\Big),$$

where $L(q) := \sup_{p\in\mathbb{R}^n}(p\cdot q - H(p))$ is the Lagrangian. Then $v(t,x) = T^t u(x)$ solves (2.6) and $w(t,x) = T_t u(x)$ solves (2.7) in the viscosity sense. Encouraged by the formal calculations we make the following definition.

Definition 2.7. We say that $u \in \mathrm{Lip_{loc}}(\Omega)$ satisfies *the convexity criterion in Ω* if for every $V \subset\subset \Omega$ there is $\delta > 0$ such that the map

$$t \mapsto T^t u(x) \quad \text{is convex on the interval } [0,\delta]$$

for every $x \in V$. Similarly, $u \in \mathrm{Lip_{loc}}(\Omega)$ satisfies *the concavity criterion in Ω* if for every $V \subset\subset \Omega$ there is $\delta > 0$ such that the map

$$t \mapsto T_t u(x) \quad \text{is concave on the interval } [0,\delta]$$

for every $x \in V$.

We prove that the conjecture made by Barron, Evans and Jensen is true, even when H is only convex [B, Theorem 4.8].

Theorem 2.8. *Suppose that H satisfies $(H1)-(H3)$ and $u \in \mathrm{Lip_{loc}}(\Omega)$. Then*

(i) *u is absolutely subminimizing in Ω if and only if it satisfies the convexity criterion in Ω,*
(ii) *u is absolutely superminimizing in Ω if and only if it satisfies the concavity criterion in Ω.*

This theorem was first proved by Juutinen and Saksman [24] in the case when $H \in C^2(\mathbb{R}^n\backslash\{0\}) \cap C^1(\mathbb{R}^n)$ and is locally uniformly convex in $\mathbb{R}^n\backslash\{0\}$. Our proof is more straightforward and does not use the Aronsson equation (2.2).

Theorem 2.6 is proved by using the convexity and the concavity criteria. The proof uses the idea of passing to a *finite difference equation* as in [1] [B, Lemma 2.7], which originates from Le Gruyer and Archer [26]. In our case the exact structure of the Hamiltonian H is unknown, which makes the final conclusion much harder. To overcome this obstacle we use the so-called *patching argument* [B, Lemma 5.1], which was previously used by Crandal, Gunnarson and P. Wang in [13] in the case when H is a norm and by Gariepy, Jensen and C. Wang [23] in the case of general smooth H. Roughly speaking, this argument allows us to slightly change the initial absolute subminimizer for another which has the property that its gradient is bounded away from 0. This argument permits us to carry out the schema of [1] and finish the proof. We also prove a similar comparison result for unbounded exterior domains Ω, that is, for unbounded domains with compact boundary.

3. Paper [C]

The last paper deals with the L^∞ functional of the form (1.2) in its full generality. This differs from the previous section since the minimizing problem is no longer similar to the model case (1.1). For instance, there is no comparison principle such as Theorem 2.6, and the solution of the Dirichlet problem is no longer unique, see [23]. We are interested in the existence of absolute minimizer of (1.2) with Dirichlet boundary values. The integrand H is assumed to satisfy the following conditions:

(C1) H is measurable for almost all $x \in \Omega$ and the map $H(x,\cdot,\cdot)$ is lower semicontinuous in $\mathbb{R}\times\mathbb{R}^n$,

(C2) $p \mapsto H(x,s,p)$ is uniformly coercive, that is, for all $c \in \mathbb{R}$ there is $R \geq 0$ such that for every $(x,s) \in \Omega\times\mathbb{R}$ it holds that $\{p \in \mathbb{R}^n \mid H(x,s,p) \leq c\} \subset B(0,R)$,

(C3) for all $(x,s) \in \Omega\times\mathbb{R}$ the map $H(x,s,\cdot)$ is quasiconvex in $\mathbb{R}^n$.

For the last assumption we note that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex* if for every $p, q \in \mathbb{R}^n$ and $t \in [0,1]$ it holds

$$f(tp + (1-t)q) \leq \max\{f(p), f(q)\}.$$

Notice that quasiconvexity is a weaker condition than convexity. A quasiconvex function need not be even continuous.

We are obligated to work on the Sobolev space $W^{1,\infty}(\Omega)$ in order to state our main theorem. However, we do not want to bring any extra burden to the reader with the exact definitions of Sobolev spaces. We are satisfied just by noting that $u \in W^{1,\infty}(\Omega)$ implies that u is locally Lipschitz in Ω. For fixed $g \in W^{1,\infty}(\Omega)$, $W^{1,\infty}_g(\Omega)$ denotes the functions u for which $u - g \in W^{1,\infty}_0(\Omega)$, which means that u satisfies boundary conditions g *in the Sobolev sense*. For more about Sobolev spaces see chapter 5 in [17]. Our main theorem is the following [C, Theorem 2.4].

Theorem 3.1. *Suppose that Ω is a bounded domain and assume that H satisfies $(C1)-(C3)$. Then for any $g \in W^{1,\infty}(\Omega)$ there exists an absolute minimizer of (1.2) in $W^{1,\infty}_g(\Omega)$.*

Let us look briefly into the history of this problem. In a classic problem of calculus of variation we are asked to minimize a functional of the form

$$I(u) = \int_\Omega H(x,u(x),Du(x))\,dx \tag{3.1}$$

subject to given boundary values. Here Du denotes the weak gradient of $u : \Omega \to \mathbb{R}$. It is well known that the uniform coercivity and the *convexity* of the function $p \to H(x,s,p)$ for every (x,s) are the main assumptions which guarantee the existence of a minimizer of I with given boundary values, see [16]. When it comes to miniming the supremum functional (0.1), Barron, Jensen and Wang [8] noticed that the right kind of convexity condition for $p \to H(x,s,p)$ is the quasiconvexity. They gave the first proof for the existence of the absolute minimizer of (0.1). They assumed that H is continuous, satisfies the condition $(C3)$ and has growth and coercivity conditions stronger than $(C2)$. In the proof they used an L^p-approximation argument to approximate (0.1) by functionals I_p of the following form,

$$I_p(u) = \left(\int_\Omega (H(x,u(x),Du(x)))^p \, dx\right)^p \to F(u,\Omega) \qquad \text{as } p \to \infty$$

assuming that $H \geq 0$. The philosophy is roughly that for a fixed $g \in W^{1,\infty}(\Omega)$ a sequence $(u_p)_p$, where u_p is a minimizer of I_p with $u_p = g$ in the Sobolev sense, converges to some function $u \in W^{1,\infty}(\Omega)$. With a lot of work one can show that u is in fact an absolute minimizer of (0.1). Champion, De Pascale and Prinari [11] improved this result by using the same argument and verified the existence under the assumptions $(C1)-(H3)$ and that the integrand $H(x,s,p)$ is continuous with respect to the second variable.

Aronsson [4] proved the existence of the Lipschitz extension problem (1.1), that is the case when $H(x,s,p) = |p|$, by attacking the problem directly and used the Perron's method to prove the existence. Perron's method is somewhat more direct and natural than L^p-approximation, and by using it Champion, De Pascale and Prinari [11] gave another proof for the existence. However, also in this proof they needed an additional assumption on the integrand H. Roughly speaking, this assumption, (H_2) in section 4.2 in [11], clarifies the structure of the set of the minimizers

$$\mathcal{A}(g,\Omega) := \{u \in W^{1,\infty}_g(\Omega) \mid u \text{ is a minimizer of (0.1)}\}, \tag{3.2}$$

where u is a minimizer of (0.1) if

$$F(u,\Omega) = \inf\{F(v,\Omega) \mid v \in W^{1,\infty}_g(\Omega)\}.$$

The point of Theorem 3.1 is that this assumption can be removed.

The proof of Theorem 3.1 is based on Perron's method and uses some lower semicontinuity results from [11]. The key is to define a class of functions

$$\mathcal{A}_{sub}(g,\Omega) = \{u \in W^{1,\infty}_g(\Omega) \mid u \text{ is an absolute subminimizer of } F(\cdot,\Omega)\},$$

where $u \in W^{1,\infty}(\Omega)$ is an absolute subminimizer of $F(\cdot,\Omega)$ if for every $V \subset\subset \Omega$ and $v \in W^{1,\infty}(V) \cap C(\bar V)$ such that $v \leq u$ and $v = u$ on ∂V it holds that

$$F(v,V) \geq F(u,V).$$

We show that the set $\mathcal{A}_{sub}(g,\Omega)$ is non-empty and an absolute minimizer can be found by a formula

$$u(x) = \sup_{v \in \mathcal{A}(g,\Omega) \cap \mathcal{A}_{sub}(g,\Omega)} v(x),$$

where $\mathcal{A}(g,\Omega)$ is the set (3.2). Moreover, u is the largest of the absolute minimizers in $W^{1,\infty}_g(\Omega)$.

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