UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
AND STATISTICS

REPORT 126

UNIVERSITÄT JYVÄSKYLÄ
INSTITUT FÜR MATHEMATIK
UND STATISTIK

BERICHT 126

RECURSIVE SET CONSTRUCTIONS AND ITERATED FUNCTION SYSTEMS: SEPARATION CONDITIONS AND DIMENSION

MARKKU VILPPOLAINEN

JYVÄSKYLÄ
2010

UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
AND STATISTICS

REPORT 126

UNIVERSITÄT JYVÄSKYLÄ
INSTITUT FÜR MATHEMATIK
UND STATISTIK

BERICHT 126

RECURSIVE SET CONSTRUCTIONS AND ITERATED FUNCTION SYSTEMS: SEPARATION CONDITIONS AND DIMENSION

MARKKU VILPPOLAINEN

To be presented, with the permission of the Faculty of Mathematics and Science
of the University of Jyväskylä, for public criticism in Auditorium S212,
on August 5th, 2010, at 12 o'clock noon.

JYVÄSKYLÄ
2010

Editor: Pekka Koskela
Department of Mathematics and Statistics
P.O. Box 35 (MaD)
FI–40014 University of Jyväskylä
Finland

ISBN 978-951-39-3927-4
ISSN 1457-8905

University Printing House
Jyväskylä 2010

Acknowledgements

I would like to thank all the people who made this work possible. First and foremost, I wish to express my sincere gratitude to my supervisor and co-author, Antti Käenmäki, for the excellent and inspiring guidance I received throughout the work. To my second co-author, Tapio Rajala, I am deeply grateful for the fruitful and enjoyable collaboration. I extend my gratitude to Christoph Bandt and Jeremy Tyson, who pre-examined this work, for their valuable comments. I would also like to thank Pertti Mattila for kindly accepting to be the opponent during the public examination of my thesis.

I am also grateful to the Department of Mathematics and Statistics at the University of Jyväskylä for the excellent conditions it has provided for my graduate studies.

Finally, to my wife Pirjo and to my parents I give my deepest appreciation for their continuous patience and support.

Jyväskylä, June 2010
Markku Vilppolainen

List of included articles

This dissertation consists of an introductory part and the following three articles:

- [**A**] A. Käenmäki and M. Vilppolainen, *Separation conditions on controlled Moran constructions*, Fund. Math. **200** (2008), 69–100.
- [**B**] A. Käenmäki and M. Vilppolainen, *Dimension and measures on sub-self-affine sets*, Monatsh. Math., to appear.
- [**C**] T. Rajala and M. Vilppolainen, *Weakly controlled Moran constructions and iterated function systems in metric spaces*, Preprint 2010. (arXiv:1003.0349)

The work presented in this dissertation has been carried out at the Department of Mathematics and Statistics of University of Jyväskylä during the period 2007–2010. The author of this dissertation has actively taken part in the research of the articles.

1. Introduction

This thesis examines fractal properties of sets and measures. The origin of fractal mathematics goes back to the early works of Cantor [4]. He showed that a nonempty perfect subset of the real line is uncountable. At that time, fractal type behavior were seen in many examples, which, however, were considered to be only pathological counterexamples for some property. For example, the Weierstrass function is an example of a continuous and nondifferentiable function. The later development of geometric measure theory gave the necessary tools for studying these kinds of objects. A nice overview for the beginning of fractal mathematics can be found in the book of Edgar [5].

Mainly because of Mandelbrot's intuition [12], fractals started to be seen as models of real world phenomena instead of pathological examples. Although there is no generally accepted definition for the term "fractal", the fundamental idea behind this notion is self-similarity: small pieces of a set appear to be similar to the whole set. The mathematical class of (strictly) self-similar sets was introduced by Hutchinson [9]. The idea goes back to Moran [15] who studied similar constructions but without mappings. A mapping $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *similitude* if there is $s > 0$, the *ratio* of f, such that $|f(x) - f(y)| = s|x - y|$ whenever $x, y \in \mathbb{R}^d$. If the ratios of the similitude mappings $f_1, \ldots, f_k$ are all strictly less than one, then a nonempty compact set $E \subset \mathbb{R}^d$ is called *self-similar* provided that it satisfies

$$E = f_1(E) \cup \cdots \cup f_k(E).$$

Then $E = \bigcup_{i,j=1}^{k} f_i \circ f_j(E) = \bigcup_{h,i,j=1}^{k} f_h \circ f_i \circ f_j(E)$ and so on, showing that a self-similar set E consists of smaller and smaller pieces which are geometrically similar to E.

The self-similar sets provide a natural starting point for various generalizations. One possibility is to look at self-affine sets which are defined using affine mappings in place of similitudes. As a phenomenon, self-affinity often occurs in nature and this is one reason for the large interest in their mathematical properties. Another possibility to generalize self-similar sets is to break the rigidity of iterated function systems and look at nested classes of sets allowing more flexibility for the sizes and shapes of the offsprings. These kind of constructions are termed Moran constructions.

The main objective of this dissertation is to study different separation conditions on various Moran constructions in order to specify circumstances under which the dimension of a generalized self-similar set can be determined. Inseparably, there is the question whether the constructed set carries a natural measure. Hausdorff measures are the traditional candidates for such measures: in ideal situations, they serve as suitable substitutes for the natural volume function given by the Lebesgue measure on a Euclidean space. Thus we study mainly the Hausdorff dimension and give results concerning the positivity and finiteness of the Hausdorff measure of the constructed set in the appropriate dimension. Our main tool is a simplified version of the so-called thermodynamic formalism, which associates the concept of *pressure* with the dimension of a dynamically constructed set.

By the fundamental result of Schief [17], the open set condition and the positivity of the t-dimensional Hausdorff measure are equivalent on self-similar sets, where t is the zero of the topological pressure. In [A, Corollary 3.10] and [C, Theorem 4.9], we prove an analogous result for a class of Moran constructions. We also remark that [C, Theorem 4.9] gives a positive answer to the question of Balogh and Rohner [1, Remark 6.2].

In [B] we define and study, applying symbolic dynamics, certain dynamically defined subsets of a self-affine set (without assuming any separation conditions) and show that for a typical sub-self-affine set, the Hausdorff and the Minkowski dimensions coincide and their common value is the zero of an appropriate topological pressure (provided that the affine mappings used in the construction have suitably small contracting ratios). See [B, Theorem 5.2] in particular. This gives a partial positive answer to the question of Falconer [8].

As a final step, adapting the theory developed for Moran constructions, we construct in [C, Theorem 5.4] sets that give an answer to the open questions proposed by Balogh, Tyson, and Warhurst [2, Remarks 4.9 and 4.10] concerning the comparison of the Riemannian and sub-Riemannian Hausdorff dimension in Carnot groups.

2. Setting and basic notions

We use the following notation: Let I be a finite set with cardinality $\kappa := \#I \geq 2$. Put $I^* = \bigcup_{n=1}^{\infty} I^n$ and $I^\infty = I^{\mathbb{N}}$. For each $\mathtt{i} \in I^*$, there is $n \in \mathbb{N}$ such that $\mathtt{i} = (i_1, \ldots, i_n) \in I^n$. We call this n the *length* of $\mathtt{i}$ and we set $|\mathtt{i}| = n$. The length of elements in I^∞ is infinite. Moreover, if $\mathtt{i} \in I^*$ and $\mathtt{j} \in I^* \cup I^\infty$, then by the notation $\mathtt{ij}$ we mean the element obtained by juxtaposing the terms of $\mathtt{i}$ and $\mathtt{j}$. For $\mathtt{i} \in I^*$, we call the set $[\mathtt{i}] := \{\mathtt{ij} : \mathtt{j} \in I^\infty\}$ a *cylinder set* of level $|\mathtt{i}|$. If $\mathtt{j} \in I^* \cup I^\infty$ and $1 \leq n < |\mathtt{j}|$, we define $\mathtt{j}|_n$ to be the unique element $\mathtt{i} \in I^n$ for which $\mathtt{j} \in [\mathtt{i}]$. If $\mathtt{j} \in I^*$ and $n \geq |\mathtt{j}|$ then $\mathtt{j}|_n = \mathtt{j}$. We also set $\mathtt{i}^- = \mathtt{i}|_{|\mathtt{i}|-1}$. We say that the elements $\mathtt{i}, \mathtt{j} \in I^*$ are *incomparable* if $[\mathtt{i}] \cap [\mathtt{j}] = \emptyset$.

Defining

$$|\mathtt{i} - \mathtt{j}| = \begin{cases} 2^{-\min\{k-1 : \mathtt{i}|_k \neq \mathtt{j}|_k\}}, & \mathtt{i} \neq \mathtt{j} \\ 0, & \mathtt{i} = \mathtt{j} \end{cases}$$

whenever $\mathtt{i}, \mathtt{j} \in I^\infty$, the couple $(I^\infty, |\cdot|)$ is a compact metric space. We call $(I^\infty, |\cdot|)$ a *symbol space* and an element $\mathtt{i} = (i_1, i_2, \ldots) \in I^\infty$ a *symbol.* If there is no danger of misunderstanding, let us also call an element $\mathtt{i} \in I^*$ a symbol. Define the *left shift* σ by setting

$$\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots).$$

It is easy to see that σ is a continous transformation on the symbol space. By the notation $\sigma(i_1,\dots,i_n)$, we mean the symbol $(i_2,\dots,i_n)\in I^{n-1}$. Observe that to be precise in our definitions, we need an "empty symbol", that is, a symbol with zero length.

Definition 2.1. Let M be a metric space. A collection $\{X_{\mathtt{i}} : \mathtt{i}\in I^*\}$ of compact subsets of M is a *weakly controlled Moran construction (WCMC)* provided that there exist a constant $D\ge 1$ so that for every $\mathtt{i},\mathtt{j}\in I^*$ the following four conditions hold:

(W1) $X_{\mathtt{i}}\subset X_{\mathtt{i}^-}$,
(W2) $\max_{\mathtt{i}\in I^n}\operatorname{diam}(X_{\mathtt{i}}) < D^{-1}$ for some $n\in\mathbb{N}$,
(W3) $\operatorname{diam}(X_{\mathtt{ij}})\le D\operatorname{diam}(X_{\mathtt{i}})\operatorname{diam}(X_{\mathtt{j}})$,
(W4) $\operatorname{diam}(X_{\mathtt{i}})\ge D^{-1}\operatorname{diam}(X_{\mathtt{i}^-})$.

WCMC is a generalization of the notion termed *controlled Moran construction (CMC)* in [A]. In the definition of a controlled Moran construction we likewise use compact sets and require that (W1) and (W2) are satisfied, but instead of conditions (W3) and (W4) we assume the following stronger condition:

(C1) for every $\mathtt{i},\mathtt{j}\in I^*$ we have

$$D^{-1}\le\frac{\operatorname{diam}(X_{\mathtt{ij}})}{\operatorname{diam}(X_{\mathtt{i}})\operatorname{diam}(X_{\mathtt{j}})}\le D.$$

We define a *projection mapping* $\pi\colon I^\infty\to X$ by the relation

$$\{\pi(\mathtt{i})\}=\bigcap_{n=1}^{\infty}X_{\mathtt{i}|_n}$$

for $\mathtt{i}\in I^\infty$. It is clear that π is continuous. The compact set $E=\pi(I^\infty)$ is called the *limit set* (of the WCMC).

Now assume that the metric space M is complete and that for each $i\in I$ there is a contractive injection $f_i\colon M\to M$. By the *contractivity* of a function f we mean that there is a constant $0<s<1$ such that $|f(x)-fy)|\le s|x-y|$ whenever $x,y\in M$. The collection $\{f_i : i\in I\}$ is then called an *iterated function system (IFS)*. As shown in [9, §3], an elegant application of the Banach fixed point theorem implies the existence of a unique compact and nonempty set $E\subset X$ for which

$$E=\bigcup_{i\in I}f_i(E).$$

Such a set E is called an *invariant set* (for the corresponding IFS).

Given an IFS $\{f_i : i\in I\}$, we set $f_{\mathtt{i}}=f_{i_1}\circ\cdots\circ f_{i_n}$ for each $\mathtt{i}=(i_1,\dots,i_n)\in I^n$ and $n\in\mathbb{N}$. We are interested in situations where the set collection $\{f_{\mathtt{i}}(F) : \mathtt{i}\in I^*\}$ is a WCMC for some $F\subset M$. Let us call an IFS *weakly tractable* if there exists a compact set $F\subset M$ and a constant $C>0$ such that the collection $\{f_{\mathtt{i}}(F) : \mathtt{i}\in I^*\}$

is a WCMC and for each $\mathtt{i} \in I^*$ we have

$$|f_\mathtt{i}(x) - f_\mathtt{i}(y)| \le C \operatorname{diam}\bigl(f_\mathtt{i}(F)\bigr)|x-y|$$

whenever $x, y \in F$. If the collection $\{f_\mathtt{i}(F)\}_{\mathtt{i}\in I^*}$ can be chosen to be a CMC, we call the weakly tractable IFS simply *tractable*.

The basic example of a weakly tractable (but not necessarily tractable) IFS on a Euclidean space is the *affine IFS*. By definition, an IFS $\{f_i\colon \mathbb{R}^d \to \mathbb{R}^d\}_{i\in I}$ affine if for each $i \in I$ there is a non-singular contracting linear transformation T_i and a translation vector a_i such that $f_i = T_i + a_i$. The invariant set E of an affine IFS is called *self-affine*. Observe that the composed mappings $f_\mathtt{i}$ are also affine: $f_\mathtt{i} = T_\mathtt{i} + a_\mathtt{i}$ where $T_\mathtt{i} = T_{i_1} \circ \cdots \circ T_{i_{|\mathtt{i}|}}$ and $a_\mathtt{i} \in \mathbb{R}^d$. We say that $T_\mathtt{i}$ is the linear part of $f_\mathtt{i}$. The image $T_\mathtt{i}(B)$ of the closed unit ball $B = \overline{B}(0,1) \subset \mathbb{R}^d$ is a d-dimensional ellipsoid. The lengths of the principal semiaxes of this ellipsoid are the *singular values* of $T_\mathtt{i}$. Equivalently, the singular values, denoted henceforth by $\alpha_1(T_\mathtt{i}), \ldots, \alpha_d(T_\mathtt{i})$, are the square roots of the eigenvalues of $T_\mathtt{i}^* T_\mathtt{i}$, where $T_\mathtt{i}^*$ is the transpose of $T_\mathtt{i}$. We adopt the usual convention that $\alpha_1(T_\mathtt{i}) \ge \alpha_2(T_\mathtt{i}) \ge \cdots \ge \alpha_d(T_\mathtt{i})$. Note that $\alpha_1(T_\mathtt{i})$ is the operator norm of $T_\mathtt{i}$, which is more commonly denoted by $\|T_\mathtt{i}\|$. For $0 \le t < d$, we set

$$\varphi^t(T_\mathtt{i}) = \alpha_1(T_\mathtt{i}) \cdots \alpha_l(T_\mathtt{i}) \alpha_{l+1}(T_\mathtt{i})^{t-l}, \tag{2.1}$$

where l is the unique integer such that $l \le t < l+1$. For completeness, we let $\varphi^t(T_\mathtt{i}) = |\det(T_\mathtt{i})|^{t/d}$ for $t \ge d$. The function $t \mapsto \varphi^t(T_\mathtt{i})$ defined on nonnegative reals is then called the *singular value function* of the linear transform $T_\mathtt{i}$.

To see that an affine IFS $\{f_i\}_{i\in I}$ is weakly tractable, choose a constant $R > 0$ so large that the ball $F := \overline{B}(0, \frac{R}{2}) \subset \mathbb{R}^d$ contains $f_i(F)$ as a subset with every $i \in I$. Then we have $\operatorname{diam}(f_\mathtt{i}(F)) = R\alpha_1(T_\mathtt{i})$ and thus

$$|f_\mathtt{i}(x) - f_\mathtt{i}(y)| = |T_\mathtt{i}(x-y)| \le \alpha_1(T_\mathtt{i})|x-y| = R^{-1} \operatorname{diam}(f_\mathtt{i}(F))|x-y|$$

for any $x, y \in \mathbb{R}^d$ and $\mathtt{i} \in I^*$. Also notice that

$$\operatorname{diam}(f_\mathtt{i}(F)) = R\|T_{\mathtt{i}j} \circ T_j^{-1}\| \le R\|T_{\mathtt{i}j}\|\|T_j^{-1}\| \le \max_{k\in I} \|T_k^{-1}\| \operatorname{diam}(f_{\mathtt{i}j}(F))$$

for each $\mathtt{i} \in I^*$ and $j \in I$.

Finally, we introduce a principal class of tractable IFSs in a general metric space M. We say that the IFS is *semiconformal* if the corresponding invariant set has positive diameter and there are constants $D \ge 1$ and $0 < \underline{s}_\mathtt{i} \le \overline{s}_\mathtt{i} < 1$, $\mathtt{i} \in I^*$, such that $\overline{s}_\mathtt{i} \le D\underline{s}_\mathtt{i}$ and

$$\underline{s}_\mathtt{i} d(x,y) \le d(f_\mathtt{i}(x), f_\mathtt{i}(y)) \le \overline{s}_\mathtt{i} d(x,y) \tag{2.2}$$

for any $x, y \in M$ and $\mathtt{i} \in I^*$. If we can choose $\underline{s}_i = \overline{s}_i$ for each $i \in I$ (implying that the mappings of the IFS are contractive similitudes), we call the invariant set *self-similar*. Note that in this special case we can choose (and assume that) $\underline{s}_\mathtt{i} = \overline{s}_\mathtt{i} = \Pi_{k=1}^n \overline{s}_{i_k}$ for each $\mathtt{i} = (i_1, \ldots, i_n) \in I^n$ with $n \in \mathbb{N}$.

In the Euclidean setting, we get a standard semiconformal IFS by choosing each f_i to be a contractive $C^{1+\varepsilon}$ conformal mapping defined on an open set $\Omega \subset \mathbb{R}^d$, provided that the mappings do not share a common fixed point and there exists a closed and nonempty $X \subset \Omega$ satisfying

$$\bigcup_{i \in I} f_i(X) \subset X.$$

In this case we say that the restrictions of the mappings f_i to X form a *conformal IFS*. It can be deduced from the well known bounded distortion principle that each conformal IFS is semiconformal. See, for example, [13, Remark 2.3]. Observe that the converse does not necessarily hold. For example, the semiconformal IFS constructed in [11, Example 2.1] is not conformal. Moreover, there are many metric spaces with no usable differentiable structure to be linked with the metric (and hence having no direct analogue of a conformal mapping) but for which semiconformal IFSs make perfect sense. For an easy example, consider the snowflaked Euclidean spaces $(\mathbb{R}^d, d^s)$ where $0 < s < 1$ and $d(x,y) = |x-y|^s$ for $x, y \in \mathbb{R}^d$. On these spaces, an IFS is semiconformal if and only if it is semiconformal with respect to the standard Euclidean metric.

We call a CMC $\{X_{\mathtt{i}}\}_{\mathtt{i} \in I^*}$ semiconformal if the relative positions of the construction pieces vary within uniform bounds, by which we mean that there exists a constant $C^* \geq 1$ such that

$$\frac{\operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}})}{\operatorname{diam}(X_{\mathtt{h}})} \leq C^* \frac{\operatorname{dist}(X_{\mathtt{ki}}, X_{\mathtt{kj}})}{\operatorname{diam}(X_{\mathtt{k}})}$$

for all $\mathtt{h}, \mathtt{i}, \mathtt{j}, \mathtt{k} \in I^*$. It is obvious that if E is a self-similar set determined by a similitude IFS $\{f_i\}_{i \in I}$, then $\{f_{\mathtt{i}}(E)\}_{\mathtt{i} \in I^*}$ is a semiconformal CMC with $C^* = 1$. More importantly, we have the following result.

Lemma 2.2 ([A, Lemma 5.2]). *If $\{f_i : i \in I\}$ is a semiconformal IFS and a compact set A with positive diameter satisfies $f_i(A) \subset A$ for every $i \in I$ then $\{f_{\mathtt{i}}(A) : \mathtt{i} \in I^*\}$ is a semiconformal CMC. In particular, a semiconformal IFS is tractable.*

We get most of our results concerning the semiconformal IFSs by studying the metric features of the corresponding semiconformal Moran constructions.

3. Pressure

Assume that $\psi(\mathtt{i}) > 0$ for each $\mathtt{i} \in I^*$. We say that these numbers are *submultiplicative weights* if the following two assumptions are satisfied:

(S1) $\psi(\mathtt{ij}) \leq \psi(\mathtt{i})\psi(\mathtt{j})$ for all $\mathtt{i}, \mathtt{j} \in I^*$,
(S2) $\max_{\mathtt{i} \in I^n} \psi(\mathtt{i}) \to 0$ as $n \to \infty$.

Assume further that for each fixed $t > 0$ we are given submultiplicative weights $\psi_t(\mathtt{i})$, $\mathtt{i} \in I^*$, and $\psi_0(\mathtt{i}) = c_0 > 0$ for each $\mathtt{i} \in I^*$. Then we call the indexed collection $\psi = \{\psi_t(\mathtt{i}) : \mathtt{i} \in I^*\}_{t \ge 0}$ a *submultiplicative scheme* provided that there furthermore exist constants $0 < \underline{a} \le \overline{a} < 1 \le b$ such that

$$b^{-\delta}\underline{a}^{|\mathtt{i}|\delta}\psi_t(\mathtt{i}) \le \psi_{t+\delta}(\mathtt{i}) \le b^{\delta}\overline{a}^{|\mathtt{i}|\delta}\psi_t(\mathtt{i}) \tag{3.1}$$

for all $t, \delta \ge 0$ and $\mathtt{i} \in I^*$. See also [10, §2]. We define the *pressure* of a submultiplicative scheme ψ to be the function $P_\psi \colon [0,\infty) \to \mathbb{R}$ given by

$$P_\psi(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \psi_t(\mathtt{i}) = \inf_{n\in\mathbb{N}} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \psi_t(\mathtt{i}).$$

The limit appearing above exists and satisfies the latter equality by the standard theory of subadditive sequences. Condition (3.1) implies that

$$0 < -\delta \log \overline{a} \le P_\psi(t) - P_\psi(t+\delta) \le -\delta \log \underline{a}$$

for $t \ge 0$ and $\delta > 0$, indicating that P_ψ is Lipschitz continuous and strictly decreasing with $\lim_{t\to\infty} P_\psi(t) = -\infty$. In particular, noting that $P_\psi(0) = c_0 > 0$, the pressure has a unique zero $P_\psi^{-1}(0) > 0$.

For a WCMC (or a CMC) $\{X_\mathtt{i}\}_{\mathtt{i}\in I^*}$ we define the *topological pressure* P by choosing $\psi_t(\mathtt{i}) = D\,\mathrm{diam}(X_\mathtt{i})^t$ for $\mathtt{i} \in I^*$ in the definition of $P_\psi(t)$ above, with D as in (W2)–(W4). The constant D can obviously be replaced by any other constant, so we have

$$P(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \mathrm{diam}(X_\mathtt{i})^t$$

for all $t \ge 0$. A simple application of Hölder's inequality shows that the topological pressure is a convex function. Note that (W4) is an essential condition for the continuity of P at 0 and for bounding the topological pressure from below. Without (W4) it may happen that $P(t) = -\infty$ for all $t > 0$ (see [C, Remark 2.4]).

In the CMC case, the topological pressure is very closely linked with the growth rate of the sum $\sum_{\mathtt{i}\in I^n} \mathrm{diam}(X_\mathtt{i})^t$ as $n \to \infty$.

Lemma 3.1 ([A, Lemma 2.1])**.** *Given a CMC, we have*

$$D^{-t}e^{nP(t)} \le \sum_{\mathtt{i}\in I^n} \mathrm{diam}(X_\mathtt{i})^t \le D^t e^{nP(t)}$$

for all $n \in \mathbb{N}$ and $t \ge 0$.

Now assume that $\{T_i + a_i : i \in I\}$ is an affine IFS on $\mathbb{R}^d$ with associated singular value functions $\varphi(T_\mathtt{i})$, $\mathtt{i} \in I^*$, as defined in (2.1). According to [19, Corollary V.1.1] and [7, Lemma 2.1], we have

$$\varphi^t(T_{\mathtt{ij}}) \le \varphi^t(T_\mathtt{i})\varphi^t(T_\mathtt{j})$$

for all $t \ge 0$ and $\mathtt{i}, \mathtt{j} \in I^*$. Furthermore, by taking $\underline{a} = \min_{i\in I} \alpha_d(T_i)$ and $\overline{a} = \max_{i\in I} \alpha_1(T_i)$ we get

$$\varphi^t(T_\mathtt{i})\underline{a}^{\delta|\mathtt{i}|} \le \varphi^{t+\delta}(T_\mathtt{i}) \le \varphi^t(T_\mathtt{i})\overline{a}^{\delta|\mathtt{i}|}$$

for all $t,\delta \geq 0$ and $\mathtt{i} \in I^*$. Thus $\{\varphi^t(T_\mathtt{i})\}_{t\geq 0}$ is a submultiplicative scheme. We denote the corresponding pressure by P_φ, so that

$$P_\varphi(t) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in I^n} \varphi^t(T_\mathtt{i}).$$

Recalling the definition of the singular value function, the following lemma is easy to believe.

Lemma 3.2 ([B, Lemma 2.1])**.** *Given an affine IFS, the pressure P_φ is convex on the connected components of* $[0,\infty) \setminus \{1,\ldots,d\}$.

Lastly, we note that for a semiconformal IFS, the pressure is naturally defined as the topological pressure for the corresponding CMC, see Lemma 2.2.

4. Measures on the symbol space and derivative of the pressure

We denote the collection of all Borel probability measures on I^∞ by $\mathcal{M}(I^\infty)$. We set $\mathcal{M}_\sigma(I^\infty) = \{\mu \in \mathcal{M}(I^\infty) : \mu \text{ is } \sigma\text{-invariant}\}$, where the *$\sigma$-invariance of μ* means that $\mu([\mathtt{i}]) = \mu\bigl(\sigma^{-1}([\mathtt{i}])\bigr) = \sum_{i\in I} \mu([i\mathtt{i}])$ for all $\mathtt{i} \in I^*$. Observe that if $\mu \in \mathcal{M}_\sigma(I^\infty)$, then $\mu(A) = \mu\bigl(\sigma^{-1}(A)\bigr)$ for all Borel sets $A \subset I^\infty$ by [3, Theorem 5.4]. Furthermore, we set $\mathcal{E}_\sigma(I^\infty) = \{\mu \in \mathcal{M}_\sigma(I^\infty) : \mu \text{ is ergodic}\}$, where the *ergodicity of μ* means that $\mu(A) = 0$ or $\mu(A) = 1$ for every Borel set $A \subset I^\infty$ with $A = \sigma^{-1}(A)$. Recall from [20, Theorem 6.10] that the set $\mathcal{M}_\sigma(I^\infty)$ is compact and convex with ergodic measures as its extreme points.

If $\mu \in \mathcal{M}_\sigma(I^\infty)$, then we define the *entropy* of μ by setting

$$h(\mu) = -\lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \mu([\mathtt{i}]).$$

Here we postulate $0\log 0 = \lim_{x\to 0+} x\log x = 0$ for convenience. In addition, if $\{\psi_t(\mathtt{i}) : \mathtt{i} \in I^*\}_{t\geq 0}$ is a submultiplicative scheme, then for every $t \geq 0$ we define the *t-energy* of μ by setting

$$\Lambda_\psi^t(\mu) = \lim_{n\to\infty} \tfrac{1}{n} \sum_{\mathtt{i}\in I^n} \mu([\mathtt{i}]) \log \psi_t(\mathtt{i}).$$

The existence of the above limits is guaranteed by [10, Lemmas 2.3 and 2.2].

Given a pressure P_ψ corresponding to a submultiplicative scheme ψ, it follows easily that

$$P_\psi(t) \geq h(\mu) + \Lambda_\psi^t(\mu)$$

for all $\mu \in \mathcal{M}_\sigma(I^\infty)$ and $t \geq 0$. We call a measure $\mu \in \mathcal{M}_\sigma(I^\infty)$ a *t-equilibrium measure* (for the scheme $\{\psi_t(\mathtt{i})\}$) if

$$P_\psi(t) = h(\mu) + \Lambda_\psi^t(\mu).$$

The existence of an ergodic t-equilibrium measure is shown in [10, Theorem 4.1]. Observe also that the proof of [B, Theorem 3.3] easily generalizes into more general submultiplicative schemes than just the one we obtain from the singular value function.

We call a measure $\mu \in \mathcal{M}_\sigma(I^\infty)$ a *t-Gibbs measure* (for the scheme $\{\psi_t(\mathtt{i})\}$) if there exists a constant $c \ge 1$ such that

$$c^{-1}e^{-|\mathtt{i}|P_\psi(t)}\psi_t(\mathtt{i}) \le \mu([\mathtt{i}]) \le ce^{-|\mathtt{i}|P_\psi(t)}\psi_t(\mathtt{i}) \tag{4.1}$$

for all $\mathtt{i} \in I^*$. It follows immediately from the right-hand side inequality of (4.1) that a t-Gibbs measure is always a t-equilibrium measure. Note that in some situations it is more natural to call t-Gibbs measures t-semiconformal.

The following theorem shows that if a t-Gibbs measure is ergodic, then it is the only t-equilibrium measure. Observe that in [B] the result is proved in the case when the submultiplicative scheme is the one obtained from the singular value function, but the same proof works in the general case.

Theorem 4.1 ([B, Theorem 3.6]). *Suppose that $\{\psi_t(\mathtt{i}) : \mathtt{i} \in I^*\}_{t\ge 0}$ is a submultiplicative scheme. If $t \ge 0$, $c \ge 1$ and $\mu \in \mathcal{M}_\sigma(I^\infty)$ satisfies $\mu([\mathtt{i}]) \ge c^{-1}e^{-|\mathtt{i}|P_\psi(t)}\psi_t(\mathtt{i})$ for all $\mathtt{i} \in I^*$, then any t-equilibrium measure is absolutely continuous with respect to μ. Moreover, if μ lies in the convex hull of a countable family of ergodic t-equilibrium measures, then the closure of the convex hull is precisely the set of all t-equilibrium measures. In particular, if μ is itself an ergodic t-equilibrium measure, then it is the only t-equilibrium measure.*

In [B, Example 6.2], we show that a t-equilibrium measure is not necessarily unique. The same example also shows that there exists a non-ergodic t-Gibbs measure. In [B, Example 6.4], we show that if a t-equilibrium measure is unique, it is not necessarily t-Gibbs.

We may construct a related measure M_ψ^t by applying the familiar Carathéodory method of producing Borel measures for various metric spaces. Given a submultiplicative scheme $\{\psi_t(\mathtt{i}) : \mathtt{i} \in I^*\}_{t\ge 0}$ and $A \subset I^\infty$, we set

$$M_{\psi,n}^t(A) = \inf\Big\{\sum_{\mathtt{i}\in C}\psi_t(\mathtt{i}) : C \subset I^*, A \subset \bigcup_{\mathtt{i}\in C}[\mathtt{i}], |\mathtt{i}| \ge n\Big\}$$

for all $n \in \mathbb{N}$. Then we define

$$M_\psi^t(A) = \lim_{n\to\infty} M_{\psi,n}^t(A).$$

Theorem 4.2 ([C, Proposition 2.8]). *Suppose that $\{\psi_t(\mathtt{i}) : \mathtt{i} \in I^*\}_{t\ge 0}$ is a submultiplicative scheme and $t \ge 0$. Given $A \subset I^\infty$ with $M_\psi^t(A) > 0$, there exists a measure $\mu \in \mathcal{M}(I^\infty)$ so that $0 < \mu(A) < \infty$ and*

$$\mu([\mathtt{i}]) \le \psi_t(\mathtt{i})$$

for all $\mathtt{i} \in I^*$.

In the WCMC case, with ψ as the associated topological pressure, we denote the Borel measure M_ψ^t by M^t. In [C, Lemma 2.7], we show that $M^t(I^\infty) > 0$ whenever

$P(t) \geq 0$. Observe that as we do not know whether the measure μ of Theorem 4.2 is σ-invariant, it is not necessarily a t-equilibrium measure. In the CMC case, the situation is quite a bit more satisfactory.

Proposition 4.3 ([A, Proposition 3.2]). *Given a CMC, there exists a unique ergodic t-Gibbs measure μ_t for all $t \geq 0$. In particular, if $P(t) = 0$, then there exists a constant $c \geq 1$ so that*

$$c^{-1} \operatorname{diam}(X_{\mathtt{i}})^t \leq \mu_t([\mathtt{i}]) \leq c \operatorname{diam}(X_{\mathtt{i}})^t$$

for all $\mathtt{i} \in I^*$.

The existence of a t-equilibrium measure allows us to study the differentiability of the pressure. If the ergodic t-Gibbs measure exists, then we are able to determine the derivative. Observe that we consider only the submultiplicative scheme given by the singular value function.

Theorem 4.4 ([B, Lemma 4.2 and Theorem 4.4]). *Given an affine IFS, the pressure P_φ is differentiable except for at most countably many points of $(0, d)$. Furthermore, if $t \in (0, d) \setminus \mathbb{N}$ and there exists a measure $\mu \in \mathcal{E}_\sigma(I^\infty)$ so that $\mu([\mathtt{i}]) \geq c^{-1} e^{-|\mathtt{i}| P_\varphi(t)} \varphi^t(T_{\mathtt{i}})$ for all $\mathtt{i} \in I^*$, where $c \geq 1$ is a constant, then the derivative $P'_\varphi(t)$ exists.*

In [B, Example 6.5], we exhibit a non-differentiable pressure.

5. Separation conditions and dimension results

We will focus mainly on the Hausdorff dimension and measures of compact sets. Let M be a separable metric space. Let $0 < s < \infty$ and $E \subset M$. The *s-dimensional Hausdorff measure* of E is defined as

$$\mathcal{H}^s(E) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s : E \subset \bigcup_{i=1}^{\infty} A_i \text{ and } \sup_{i \in \mathbb{N}} \operatorname{diam}(A_i) \leq \delta \right\}.$$

The 0-dimensional Hausdorff measure $\mathcal{H}^0$ is defined to be the counting measure. The *Hausdorff dimension* of a set $E \subset M$ is $\dim_{\mathrm{H}}(E) := \inf\{s : \mathcal{H}^s(E) = 0\}$. It is a basic fact that $\mathcal{H}^t(E) = \infty$ for $0 \leq t < \dim_{\mathrm{H}}(E)$ and $\mathcal{H}^t(E) = 0$ for $t > \dim_{\mathrm{H}}(E)$.

Another dimension we consider is the *(upper) Minkowski dimension*, which is defined for $E \subset M$ as

$$\dim_{\mathrm{M}}(E) = \limsup_{r \downarrow 0} \frac{-\log N(E, r)}{\log r},$$

where $N(E, r) = \min \left\{ k \in \mathbb{N} \cup \{\infty\} : A \subset \bigcup_{i=1}^{k} B(x_i, r) \right\}$. The basic relation between these two dimensions is that $\dim_{\mathrm{M}}(E) \geq \dim_{\mathrm{H}}(E)$ for all $E \subset M$.

In the following results, the main theme is to find how the pressure and/or the separation conditions are related to the size (i.e. dimension or measure) of the limit set or the invariant set.

Proposition 5.1 ([C, Proposition 2.6]). *Given a WCMC, if $P(t) \le 0$ for some $t \ge 0$, then* $\dim_{\mathrm{M}}(E) \le t$.

Recall that given an affine IFS, it follows from [7, Theorem 5.4] that $P_\varphi(t) \le 0$ implies $\dim_{\mathrm{M}}(E) \le t$.

Given contractive affine mappings $f_i = T_i + a_i$ with linear parts $T_i \colon \mathbb{R}^d \to \mathbb{R}^d$, $i \in I = \{1, \ldots, \kappa\}$, we call any nonempty compact set $E \subset \mathbb{R}^d$ satisfying

$$E \subset \bigcup_{i \in I} f_i(E) \tag{5.1}$$

sub-self-affine. This gives a generalization of a self-affine set. Sub-self-affine sets include many interesting examples, such as sub-self-similar sets, graph directed self-affine sets, unions of self-affine sets, and topological boundaries of self-affine sets. The general form of a sub-self-affine set $E \subset \mathbb{R}^d$ satisfying (5.1) is

$$E = E_{K,a} = \pi_a(K)$$

where $K \neq \emptyset$ is any compact subset of I^∞ satisfying $\sigma(K) \subset K$, and the projection $\pi_a \colon I^\infty \to \mathbb{R}^d$ with $a = (a_1, \ldots, a_\kappa)$ is given by $\pi_a(\mathtt{i}) = \sum_{n=1}^{\infty} T_{\mathtt{i}|_{n-1}} a_{i_n}$ for $\mathtt{i} \in I^\infty$. See [B, §2].

Theorem 5.2 ([B, Theorem 5.2]). *Suppose $T_1, \ldots, T_\kappa \colon \mathbb{R}^d \to \mathbb{R}^d$ are invertible linear mappings with $\|T_i\| < \frac{1}{2}$ for each $i \in I = \{1, \ldots, \kappa\}$, and $K \subset I^\infty$ is as in the definition of $E_{K,a}$. Then for some $t \in [0, d]$ we have*

$$\dim_{\mathrm{H}}(E_{K,a}) = \dim_{\mathrm{M}}(E_{K,a}) = t$$

for $\mathcal{H}^{d\kappa}$-almost every $a \in \mathbb{R}^{d\kappa}$.

Observe that [B, Theorem 5.2] also shows the existence of an invariant Borel probability measure μ with $\mathrm{spt}(\mu) \subset E_{K,a}$ of full dimension. Moreover, we show in [B] that there exists a natural pressure for sub-self-affine sets such that in the situation of Theorem 5.2 above, the dimension of the sub-self-affine set is the zero of this pressure for a typical choice of translation vectors $a_1, \ldots, a_\kappa$.

Falconer [8] asked if $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{M}}(E)$ for all sub-self-similar sets E. Theorem 5.2 gives a partial positive answer to this question. The question remains open for sub-self-similar sets which do not satisfy the open set condition and are constructed by using exceptional (in the sense of Theorem 5.2) translation vectors.

For a CMC (with E as the limit set), we have the following related result.

Theorem 5.3 ([A, Theorem 4.3]). *If a CMC is semiconformal and $t = \dim_{\mathrm{H}}(E)$, then $\dim_{\mathrm{M}}(E) = t$ and $\mathcal{H}^t(E) < \infty$.*

Simple examples of self-affine carpets (see e.g. [6, Example 9.11]) show that in the WCMC case $\dim_{\mathrm{H}}(E) < \dim_{\mathrm{M}}(E)$ might happen. Observe that these kind of carpets are typical examples of exceptional cases in Theorem 5.2. Given a WCMC, it is also possible that $\mathcal{H}^t(E) = \infty$ when $t = \dim_{\mathrm{H}}(E)$, see [C, Example 3.2] and [B, Example 6.4].

It is natural to expect more precise results if we know more about the relative positions of the construction pieces. Given a WCMC, define for $r > 0$

$$Z(r) = \{\mathtt{i} \in I^* : \operatorname{diam}(X_\mathtt{i}) \le r < \operatorname{diam}(X_{\mathtt{i}^-})\}$$

and if in addition $x \in E$, set

$$Z(x,r) = \{\mathtt{i} \in Z(r) : X_\mathtt{i} \cap B(x,r) \ne \emptyset\}.$$

It is often useful to know the cardinality of the set $Z(x,r)$. We say that a WCMC satisfies the *finite clustering property* if $\sup_{x\in E} \limsup_{r\downarrow 0} \#Z(x,r) < \infty$. Moreover, if $\sup_{x\in E}\sup_{r>0} \#Z(x,r) < \infty$ then the WCMC is said to satisfy the *uniform finite clustering property.*

Definition 5.4. We say that a WCMC $\{X_\mathtt{i} : \mathtt{i} \in I^*\}$ satisfies the *ball condition* if the following holds for some $0 < \delta < 1$: For each $x \in E$ there is $r_x > 0$ such that, given any positive $r < r_x$, we can find a collection $\{B(x_\mathtt{i}, \delta r) : \mathtt{i} \in Z(x,r)\}$ of mutually disjoint balls with $\max_{\mathtt{i}\in Z(x,r)} \operatorname{dist}(x_\mathtt{i}, X_\mathtt{i}) < r$.

We note that the ball condition implies the finite clustering property if the WCMC is defined on a doubling metric space, see [C, Remark 3.6(iv)]. Recall that a metric space is *doubling* if there exists a constant $N \in \mathbb{N}$ such that every ball $B(x, 2r)$ can be covered with N balls of radius r. If the underlying doubling space is connected, these two separation conditions are in fact equivalent. Since connected spaces are uniformly perfect, this follows by the next proposition. Recall that a metric space M is *uniformly perfect* if it contains at least two points and there exists a constant $C > 1$ such that for each $x \in M$ and for each $r > 0$ the set $B(x,r) \setminus B(x, r/C)$ is nonempty whenever the set $M \setminus B(x,r)$ is nonempty.

Proposition 5.5 ([C, Proposition 3.5])**.** *Given a WCMC, if the underlying metric space is doubling and uniformly perfect, then the finite clustering property is satisfied if and only if the ball condition holds.*

We will see that the ball condition is a natural replacement for the familiar open set condition used in self-similar constructions. The following proposition together with Proposition 5.1 shows that finite clustering is a sufficient condition to guarantee $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{M}}(E)$ in the setting of weakly controlled Moran constructions.

Proposition 5.6 ([C, Proposition 3.1])**.** *Assume that for a WCMC the finite clustering property holds and $P(t) = 0$. Then $\mathcal{H}^t(E) > 0$. Moreover, $\mathcal{H}^t(E) < \infty$ if and only if $M^t(I^\infty) < \infty$.*

For certain controlled Moran constructions, we have a nice reverse implication of Proposition 5.6. The following theorem generalizes a fundamental result of Schief [17, Theorem 2.1] into a natural setting. For the purposes of this exposition, we say that a CMC $\{X_\mathtt{i}\}_{\mathtt{i}\in I^*}$ is *tractable* if with some $C > 0$ we have

$$\operatorname{dist}(X_{\mathtt{hi}}, X_{\mathtt{hj}}) \le C \operatorname{diam}(X_\mathtt{h}) \operatorname{dist}(X_\mathtt{i}, X_\mathtt{j})$$

for all $\mathtt{h}, \mathtt{i}, \mathtt{j} \in I^*$. See [A, p. 80] for a more general definition of this notion. Most notably, a tractable IFS defines a tractable CMC (see [A, Lemma 5.1]).

Theorem 5.7 ([A, Corollary 3.10]). *For a tractable CMC, the following conditions are equivalent:*

(1) *The finite clustering property.*
(2) *The uniform finite clustering property.*
(3) $\mathcal{H}^t(E) > 0$, *when* $P(t) = 0$.
(4) *With* $t = P^{-1}(0)$, *there exist constants* $r_0 > 0$ *and* $K \geq 1$ *such that*

$$K^{-1} r^t \leq \mathcal{H}^t|_E\big(B(x, r)\big) \leq K r^t$$

whenever $x \in E$ *and* $0 < r < r_0$.

Note that although Theorem 5.7 is proved in [A] only in the Euclidean setting, the arguments used in the proof are valid in a general metric space.

We say that an IFS satisfies an *open set condition (OSC)*, if there exists a nonempty open set $U \subset \Omega$ such that

$$f_i(U) \subset U$$

for all $i \in I$ and

$$f_i(U) \cap f_j(U) = \emptyset$$

as $i \neq j$. Any such open set U is called *feasible*. Furthermore, if a feasible set intersects E, then we say that the IFS satisfies a *strong open set condition (SOSC)*.

At the first glance, it seems that for a semiconformal IFS the OSC is easier to check than the ball condition. However, there are cases when it is much more convenient to consider the ball condition rather than the OSC. See [A, Example 6.5]. Observe that by [17, Corollary 2.3] (or more generally [16, Corollary 1.2] or [A, Proposition 5.6]) a feasible set in [A, Example 6.5] is quite intricate.

Proposition 5.8 ([A, Proposition 5.6]). *If a semiconformal IFS in* $\mathbb{R}^d$ *satisfies the OSC and* $\dim_{\mathrm{H}}(E) = d$, *then the invariant set* E *is the closure of its interior. Moreover, in this case any feasible open set is a subset of* E.

We say that a semiconformal IFS is *properly semiconformal* if there is an essential open set $W \subsetneq M$ such that for each $x \in W$ there is $r_x > 0$ so that

$$B\big(f_{\mathtt{i}}(x), \underline{s}_{\mathtt{i}} r\big) \subset f_{\mathtt{i}}\big(B(x, r)\big)$$

for all $x \in W$, $0 < r \leq r_x$, and $\mathtt{i} \in I^*$. Here W is an *essential open set* if the closure of W is the whole space M and $W \cap E \neq \emptyset$. Observe that this condition is automatically satisfied if $M = \mathbb{R}^d$ or the mappings of the IFS are bijective.

The following theorem summarizes the main implications shown for a semiconformal IFS. It also gives a positive answer to the question posed in [1, Remark 6.2]. Note that by the ball condition and the finite clustering property of a semiconformal IFS $\{f_i\}_{i \in I}$ we mean the respective conditions for the semiconformal CMC $\{f_{\mathtt{i}}(E)\}_{\mathtt{i} \in I^*}$ determined by the invariant set E.

Theorem 5.9 ([C, Theorem 4.9]). *For a properly semiconformal IFS in a complete doubling metric space, the following are equivalent:*

(1) *The ball condition.*
(2) *The finite clustering property.*
(3) *The open set condition.*
(4) *The strong open set condition.*
(5) $\mathcal{H}^t(E) > 0$ *for* $t = P^{-1}(0)$.

In [C, Example 4.5], we show that the assumption of proper semiconformality is essential. Namely, we exhibit a similitude IFS in a complete doubling metric space satisfying the OSC, but which does not satisfy the SOSC nor the ball condition, and $P(\dim_{\mathrm{H}}(E)) > 0$. This example apparently gives a counterexample to [1, Theorem 3.1] as the theorem is lacking the required assumption of bijectivity.

It remains an open question whether or not the strong open set condition and the ball condition are equivalent for a semiconformal IFS defined in a complete doubling metric space (without the assumption of proper semiconformality).

In a general complete metric space, the OSC ceases to imply any bounds for the size of the invariant set. As shown in [18, Example 3.1], the invariant set of a similitude IFS in a complete metric space might consist of a single point, even when the OSC is satisfied. The SOSC, however, continues to be relevant. The following result generalizes [14, Theorem 3.3] from a self-similar case to a more general setting. See also [A, Proposition 4.9].

Proposition 5.10 ([C, Proposition 4.12]). *If a semiconformal IFS* $\{f_i\}_{i\in I}$ *defined on a complete metric space satisfies the SOSC, then with respect to the invariant set* E *we have* $\dim_{\mathrm{H}}(E) = P^{-1}(0)$ *and*

$$\dim_{\mathrm{H}}\big(f_{\mathtt{i}}(E) \cap f_{\mathtt{j}}(E)\big) < \dim_{\mathrm{H}}(E)$$

whenever $\mathtt{i}\perp\mathtt{j}$.

We end this introduction by giving a natural topological prerequisite for the validity of the dimension formula $\dim_{\mathrm{H}}(E) = P^{-1}(0)$. Together with Proposition 5.10, the following result implies that the SOSC, unlike the OSC, well deserves to be called a separation condition also in the general setting.

Proposition 5.11 ([C, Proposition 4.13]). *Assume that* E *is the invariant set of a semiconformal IFS* $\{f_i\}_{i\in I}$ *such that* $P(\dim_{\mathrm{H}}(E)) = 0$. *Then* $f_{\mathtt{i}}(E) \cap f_{\mathtt{j}}(E)$ *is nowhere dense in* E *whenever* $\mathtt{i}\perp\mathtt{j}$.

References

[1] Z. M. Balogh and H. Rohner. Self-similar sets in doubling spaces. *Illinois J. Math.*, 51(4):1275–1297, 2007.

[2] Z. M. Balogh, J. T. Tyson, and B. Warhurst. Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups. *Adv. Math.*, 220(2):560–619, 2009.

[3] H. Bauer. *Measure and integration theory*, volume 26 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2001.
[4] G. Cantor. Über unendliche, lineare Punktmannichfaltigkeiten. *Math. Ann.*, 21(4):545–591, 1883.
[5] G. A. Edgar. *Classics on Fractals*. Addison-Wesley Publishing Company, New York, 1993.
[6] K. Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.
[7] K. J. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.*, 103(2):339–350, 1988.
[8] K. J. Falconer. Sub-self-similar sets. *Trans. Amer. Math. Soc.*, 347(8):3121–3129, 1995.
[9] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.
[10] A. Käenmäki. On natural invariant measures on generalised iterated function systems. *Ann. Acad. Sci. Fenn. Math.*, 29(2):419–458, 2004.
[11] A. Käenmäki. Geometric rigidity of a class of fractal sets. *Math. Nachr.*, 279(1):179–187, 2006.
[12] B. B. Mandelbrot. *Fractal Geometry of Nature*. Freeman, San Francisco, 1982.
[13] R. D. Mauldin and M. Urbański. Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc.*, 73(3):105–154, 1996.
[14] M. Morán. Dynamical boundary of a self-similar set. *Fund. Math.*, 160(1):1–14, 1999.
[15] P. A. P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.*, 42:15–23, 1946.
[16] Y. Peres, M. Rams, K. Simon, and B. Solomyak. Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets. *Proc. Amer. Math. Soc.*, 129(9):2689–2699, 2001.
[17] A. Schief. Separation properties for self-similar sets. *Proc. Amer. Math. Soc.*, 122(1):111–115, 1994.
[18] A. Schief. Self-similar sets in complete metric spaces. *Proc. Amer. Math. Soc.*, 124(2):481–490, 1996.
[19] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*. Springer-Verlag, New York, 1988.
[20] P. Walters. *An Introduction to Ergodic Theory*. Springer-Verlag, New York-Berlin, 1982.

89. KÄRKKÄINEN, SALME, Orientation analysis of stochastic fibre systems with an application to paper research. (53 pp.) 2003
90. HEINONEN, JUHA, Geometric embeddings of metric spaces. (44 pp.) 2003
91. RAJALA, KAI, Mappings of finite distortion: Removable singularities. (23 pp.) 2003
92. FUTURE TRENDS IN GEOMETRIC FUNCTION THEORY. RNC WORKSHOP JYVÄSKYLÄ 2003. Edited by D. Herron. (262 pp.) 2003
93. KÄENMÄKI, ANTTI, Iterated function systems: Natural measure and local structure. (14 pp.) 2003
94. TASKINEN, SARA, On nonparametric tests of independence and robust canonical correlation analysis. (44 pp.) 2003
95. KOKKI, ESA, Spatial small area analyses of disease risk around sources of environmental pollution: Modelling tools for a system using high resolution register data. (72 pp.) 2004
96. HITCZENKO, PAWEŁ, Probabillistic analysis of sorting algorithms. (71 pp.) 2004
97. NIEMINEN, TOMI, Growth of the quasihyperbolic metric and size of the boundary. (16 pp.) 2005
98. HAHLOMAA, IMMO, Menger curvature and Lipschitz parametrizations in metric spaces. (8 pp.) 2005
99. MOLTCHANOVA, ELENA, Application of Bayesian spatial methods in health and population studies using registry data. (55 pp.) 2005
100. HEINONEN, JUHA, Lectures on Lipschitz analysis. (77 pp.) 2005
101. HUJO, MIKA, On the approximation of stochastic integrals. (19 pp.) 2005
102. LINDQVIST, PETER, Notes on the p-Laplace equation. (80 pp.) 2006
103. HUKKANEN, TONI, Renormalized solutions on quasi open sets with nonhomogeneous boundary values. (41 pp.) 2006
104. HÄHKIÖNIEMI, MARKUS, The role of representations in learning the derivative. (101 pp.) 2006
105. HEIKKINEN, TONI, Self-improving properties of generalized Orlicz–Poincaré inequalities. (15 pp.) 2006
106. TOLONEN, TAPANI, On different ways of constructing relevant invariant measures. (13 pp.) 2007
107. HORPPU, ISMO, Analysis and evaluation of cell imputation. (248 pp.) 2008
108. SIRKIÄ, SEIJA, Spatial sign and rank based scatter matrices with applications. (29 pp.) 2007
109. LEIKAS, MIKA, Projected measures on manifolds and incomplete projection families. (16 pp.) 2007
110. TAKKINEN, JUHANI, Mappings of finite distortion: Formation of cusps. (10 pp.) 2007
111. TOLVANEN, ASKO, Latent growth mixture modeling: A simulation study. (201 pp.) 2007
112. VARPANEN, HARRI, Gradient estimates and a failure of the mean value principle for p-harmonic functions. (66 pp.) 2008
113. MÄKÄLÄINEN, TERO, Nonlinear potential theory on metric spaces. (16 pp.) 2008
114. LUIRO, HANNES, Regularity properties of maximal operators. (11 pp.) 2008
115. VIHOLAINEN, ANTTI, Prospective mathematics teachers' informal and formal reasoning about the concepts of derivative and differentiability. (86 pp.) 2008
116. LEHRBÄCK, JUHA, Weighted Hardy inequalities and the boundary size. (21 pp.) 2008
117. NISSINEN, KARI, Small area estimation with linear mixed models from unit-level panel and rotating panel data. (230 pp.) 2009
118. BOJARSKI, B.V., Generalized solutions of a system of differential equations of the first order and elliptic type with discontinuous coefficients. (64 pp.) 2009
119. RAJALA, TAPIO, Porosity and dimension of sets and measures. (22 pp.) 2009
120. MYLLYMÄKI, MARI, Statistical models and inference for spatial point patterns with intensity-dependent marks. (115 pp.) 2009
121. AVIKAINEN, RAINER, On generalized bounded variation and approximation of SDEs. (18 pp.) 2009
122. ZÜRCHER, THOMAS, Regularity of Sobolev–Lorentz mappings on null sets. (13 pp.) 2009
123. TOIVOLA, ANNI, On fractional smoothness and approximations of stochastic integrals. (19 pp.) 2009
124. VIHOLA, MATTI, On the convergence of unconstrained adaptive Markov chain Monte Carlo algorithms. (29 pp.) 2010
125. ZHOU, YUAN, Hajłasz–Sobolev extension and imbedding. (13 pp.) 2010

ISBN 978-951-39-3927-4
ISSN 1457-8905