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HAJŁASZ–SOBOLEV EXTENSION AND IMBEDDING

YUAN ZHOU

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Jyväskylä, March 2010

Yuan Zhou

List of Included Articles

This dissertation consists of an introductory part and the following publications:

[KYZ-1] P. Koskela, D. Yang and Y. Zhou, A Jordan Sobolev extension domain, Ann. Acad. Sci. Fenn. Math. 35 (2010), 309-320.

[Z-1] Y. Zhou, Criteria for optimal global integrability of Hajłasz-Sobolev functions, Illinois J. Math. (to appear). "`http://arxiv.org/abs/1004.5304`"

[Z-2] Y. Zhou, Hajłasz-Sobolev imbedding and extension, submitted. "`http://arxiv.org/abs/1004.5307`"

[KYZ-2] P. Koskela, D. Yang and Y. Zhou, A characterization of Hajłasz-Sobolev and Triebel-Lizorkin spaces via grand Littlewood-Paley functions, J. Funct. Anal. 258 (2010), 2637-2661.

[KYZ-3] P. Koskela, D. Yang and Y. Zhou, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, submitted. "`http://arxiv.org/abs/1004.5507`"

The author of this dissertation has actively taken part in the research of the joint papers [KYZ-1], [KYZ-2] and [KYZ-3].

Introduction

1. Sobolev extension and imbedding

In this introduction, we always let $n \geq 2$ unless we specify it, and $\Omega \subset \mathbb{R}^n$ be a domain, namely, a connected open subset. Let $X(\Omega)$ and $Y(\Omega)$ be function spaces defined on Ω. Then Ω is called an *X-extension domain* if $X(\Omega) = X(\mathbb{R}^n)|_\Omega$ with equivalent norms, where $X(\mathbb{R}^n)|_\Omega \equiv \{u|_\Omega : \ u \in X(\mathbb{R}^n)\}$ and for all $v \in X(\mathbb{R}^n)|_\Omega$, $\|v\|_{X(\mathbb{R}^n)|_\Omega} \equiv \inf \|u\|_{X(\mathbb{R}^n)}$ with the infimum taken over all $u \in X(\mathbb{R}^n)$ such that $u|_\Omega = v$. Also Ω is said to *support an imbedding from $X(\Omega)$ to $Y(\Omega)$* if $X(\Omega)$ is a subset of $Y(\Omega)$ and for all $u \in X(\Omega)$, $\|u\|_{Y(\Omega)} \leq C\|u\|_{X(\Omega)}$ with a constant C independent of u. For several other geometric notions of domains appearing below, such as John domain, uniform domain and so on, see Appendix.

For $p \in [1, \infty]$, we always denote the *homogeneous first order Sobolev space* by $\dot{W}^{1,p}(\Omega)$. Namely, $\dot{W}^{1,p}(\Omega)$ is the set of all measurable functions u satisfying $\nabla u \in L^p(\Omega)$, where ∇u is the distributional gradient of u. Actually, it easily follows that $u \in L^p_{\mathrm{loc}}(\Omega)$ for each $u \in \dot{W}^{1,p}(\Omega)$. For $u \in \dot{W}^{1,p}(\Omega)$, its norm is defined by $\|u\|_{\dot{W}^{1,p}(\Omega)} \equiv \|\nabla u\|_{L^p(\Omega)}$. Define the *inhomogeneous Sobolev space* $W^{1,p}(\Omega) \equiv \dot{W}^{1,p}(\Omega) \cap L^p(\Omega)$ with $\|u\|_{W^{1,p}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \|u\|_{\dot{W}^{1,p}(\Omega)}$ for every $u \in W^{1,p}(\Omega)$.

It is well-known that the possibility of $W^{1,p}$-extension for a domain depends not only on its geometric structure but also on the exponent p. Indeed, a domain having smooth boundary is a $W^{1,p}$-extension domain for all $p \in [1, \infty]$, while the planar domain $B(0, 1) \setminus \{(x, 0) : \ x \geq 0\}$ is not a $W^{1,p}$-extension domain for any $p \in [1, \infty]$. Moreover, Maz'ya [19] constructed a planar Jordan domain Ω such that it is a $W^{1,p}$-extension domain for all $p \in [1, 2)$ but not a $W^{1,p}$-extension domain for any $p \in [2, \infty]$, while $\mathbb{R}^2 \setminus \overline{\Omega}$ is a $W^{1,p}$-extension domain exactly when $p \in (2, \infty]$. Motivated by this, for each $q \in (1, 2)$, Romanov [22] constructed a planar domain G_q, whose boundary contains a curve generated by a certain Cantor set, such that G_q is a $W^{1,p}$-extension domain if and only if $p \in [1, q)$. As an extension of this, we [KYZ-1] simplify the construction of [22] and establish the following conclusion.

Theorem 1. *For each $q \in (1, 2)$, there exists a Jordan domain $G_q \subset \mathbb{R}^2$ such that G_q is a $W^{1,p}$-extension domain if and only if $p \in [1, q)$, and $\mathbb{R}^2 \setminus \overline{G_q}$ is a $W^{1,s}$-extension domain if and only if $s \in (q/(q-1), \infty]$.*

A remarkable result regarding Sobolev extension domains was established by Jones [15]. He proved that a uniform domain is always a $W^{1,p}$-extension domain for all $p \in [1, \infty]$. Conversely, it was proved by Vodop'janov, Gol'dšteǐn and Latfullin [32] that a simply connected planar $W^{1,2}$-extension domain is a uniform domain; see also [15, 9]. Moreover, let Ω be a $W^{1,n}$-extension domain. Then Gehring and Martio [7] proved that Ω has the locally linear connectivity (for short, LLC) property, and if Ω is also quasiconformally equivalent to a uniform domain, then it is a uniform domain; see also [16, 8, 9, 10, 32].

On the other hand, for $\alpha \in (0, 1]$, let $\mathrm{loc\ Lip}_\alpha(\Omega)$ denote the (semi) local Lipschitz space of order α on Ω as in [6]. Notice that $W^{1,\infty}(\Omega) = \mathrm{loc\ Lip}_1(\Omega)$; see [6, 18]. Then

Gehring and Martio [6] proved that for $\alpha \in (0,\, 1]$, Ω is a weak α-cigar domain if and only if it is a loc Lip_α-extension domain. In particular, Ω is a $W^{1,\,\infty}$-extension domain if and only if it is a weak 1-cigar domain (namely, it is quasiconvex).

When $p \in (n,\, \infty)$, some geometric criteria have been established for a domain to support a $W^{1,\,p}$-extension or an imbedding from $W^{1,\,p}(\Omega)$ to $C^{1-n/p}(\overline{\Omega})$ (for short, $W^{1,\,p}$-imbedding). Indeed, a $W^{1,\,p}$-extension domain Ω always supports a $W^{1,\,p}$-imbedding; conversely, as proved by Koskela [17], a $W^{1,\,p}$-imbedding domain is a $W^{1,\,q}$-extension domain for all $q \in (p,\, \infty)$. Moreover, let $\alpha \in (0,\, 1)$, $p = (n-\alpha)/(1-\alpha)$ and Ω be a weak α-cigar domain. Then it was proved by Buckley and Koskela [3] that Ω supports a $\dot{W}^{1,\,q}$-imbedding for all $q \in [p,\, \infty)$; and by Koskela [17] that Ω supports a $W^{1,\,q}$-extension for all $q \in (p,\, \infty)$, which was further improved by Shvartsman [28] to all $q \in (p^*,\, \infty)$ with some $p^* \in (n,\, p)$. Conversely, with the additional assumption that Ω has the slice property, Buckley and Koskela [3] proved that if Ω supports a $\dot{W}^{1,\,p}$-imbedding, then it is a weak α-cigar domain. In [4], the slice property was further reduced to some weak slice properties.

When $p \in [1,\, n]$, some geometric criteria were also established in [1, 2, 3] for a bounded domain to support a $(pn/(n-p),\, p)$-Sobolev-Poincaré imbedding for $p \in [1,\, n)$ or a Trudinger imbedding for $p = n$. More precisely, Bojarski [1] proved that a John domain always supports a $(pn/(n-p),\, p)$-Sobolev-Poincaré imbedding for all $p \in [1,\, n)$. Smith and Stegenga [29] proved that a weak carrot domain always supports the Trudinger imbedding. Conversely, let Ω be a bounded simply connected planar domain, or a bounded domain satisfying an additional separation property when $p \in [1,\, n)$ and a slice property when $p = n$. Then Buckley and Koskela [2, 3] proved that if Ω supports a $(pn/(n-p),\, p)$-Sobolev-Poincaré imbedding for some/all $p \in [1,\, n)$, then it is a John domain, and if Ω supports the Trudinger imbedding, then it is a weak carrot domain.

2. Hajłasz and Hajłasz-Sobolev spaces

It was noticed by Hajłasz [11] that the simple pointwise inequality

$$|u(x) - u(y)| \le |x-y|^s[g(x) + g(y)] \tag{1}$$

can be used to characterize Sobolev functions u when $s = 1$. More generally, for $s \in (0,\, 1]$ and measurable function u, denote by $\mathcal{D}^s(u)$ the *collection of all nonnegative measurable functions* g such that (1) holds for all $x,\ y \in \Omega \setminus E$, where $E \subset \Omega$ with $|E| = 0$. We also denote by $\mathcal{D}^s_{\mathrm{ball}}(u)$ the *collection of all nonnegative measurable functions* g such that (1) holds for all $x,\ y \in \Omega \setminus E$ (with $|E| = 0$) satisfying $|x-y| < \frac{1}{2}\,\mathrm{dist}\,(x,\, \partial\Omega)$.

Definition 1. Let $s \in (0, 1]$ and $p \in (0,\, \infty]$.

(i) The *homogeneous Hajłasz space* $\dot{M}^{s,\,p}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,\,p}(\Omega)} \equiv \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(\Omega)} < \infty,$$

(ii) The *Sobolev-type Hajłasz space* $\dot{M}^{s,\,p}_{\mathrm{ball}}(\Omega)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,\,p}_{\mathrm{ball}}(\Omega)} \equiv \inf_{g \in \mathcal{D}^s_{\mathrm{ball}}(u)} \|g\|_{L^p(\Omega)} < \infty.$$

Moreover, we set $M^{s,p}(\Omega) \equiv L^p(\Omega) \cap \dot{M}^{s,p}(\Omega)$ with $\|u\|_{M^{s,p}(\Omega)} \equiv \|u\|_{\dot{M}^{s,p}(\Omega)} + \|u\|_{L^p(\Omega)}$ for all $u \in M^{s,p}(\Omega)$, and similarly define $M^{s,p}_{\mathrm{ball}}(\Omega)$.

Hajłasz(-Sobolev) spaces are closely related to Sobolev spaces. Indeed, as proved by Hajłasz [11] and Koskela and Saksman [18], $\dot{W}^{1,p}(\Omega) = \dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p \in (1, \infty]$. Moreover, for all $p \in (1, \infty)$, it was proved by Hajłasz, Koskela and Tuominen [13] that Ω is a $W^{1,p}$-extension domain if and only if it is regular and $W^{1,p}(\Omega) = M^{1,p}(\Omega)$, while Ω is regular if and only if Ω is an $M^{1,p}$-extension domain. With the aid of a metric measure space version of this (see [13]), one further concludes that for $s \in (0, 1]$ and $p \in (1, \infty)$, Ω is regular if and only if Ω is an $M^{s,p}$-extension domain; and Ω is an $M^{s,p}_{\mathrm{ball}}$-extension domain if and only if Ω is regular and $M^{s,p}_{\mathrm{ball}}(\Omega) = M^{s,p}(\Omega)$; see [Z-2, Lemma 4.1]. So, for a regular domain Ω, the possibility of $W^{1,p}$-extension or $M^{s,p}_{\mathrm{ball}}$-extension is equivalent to $W^{1,p}(\Omega) = M^{1,p}(\Omega)$ or $M^{s,p}_{\mathrm{ball}}(\Omega) = M^{s,p}(\Omega)$, respectively.

On the other hand, Hajłasz-Sobolev spaces are closely related to Hardy-Sobolev spaces and Triebel-Lizorkin spaces. More precisely, when $s = 1$, it was proved by Koskela and Saksman [18] that $\dot{H}^{1,p}(\Omega) = \dot{M}^{1,p}_{\mathrm{ball}}(\Omega)$ for $p \in (n/(n+1), 1]$, where $\dot{H}^{1,p}(\Omega)$ denotes the Hardy-Sobolev space as in [21]. Recall that $\dot{W}^{1,p}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$ for $p \in (1, \infty]$ and $\dot{H}^{1,p}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$ for $p \in (0, 1]$; see [30]. Then as a corollary of this and [11, 18], we have that $\dot{M}^{1,p}_{\mathrm{ball}}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$ for all $p \in (n/(n+1), \infty]$. Here and in what follows, $\dot{F}^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $p, q \in (0, \infty]$ denotes the homogeneous Triebel-Lizorkin spaces as in [30]. Moreover, when $s \in (0, 1)$, as observed by Yang [33], $\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n) = \dot{F}^s_{p,\infty}(\mathbb{R}^n)$ for $p \in (1, \infty]$; we also recall that DeVore and Sharpley [5] characterized $\dot{F}^s_{p,\infty}(\mathbb{R}^n)$ via a kind of fractional sharp maximal function. In [KYZ-2], with the aid of grand Littlewood-Paley functions, we further extend the equivalence in [33] to $p \in (n/(n+s), \infty]$ as follows.

Theorem 2. *If $s \in (0, 1)$ and $p \in (n/(n+s), \infty]$, then $\dot{M}^{s,p}_{\mathrm{ball}}(\mathbb{R}^n) = \dot{F}^s_{p,\infty}(\mathbb{R}^n)$.*

Motivated by Theorem 2, it is natural to inquire if $\dot{F}^s_{p,q}(\mathbb{R}^n)$ with full scales can be characterized in a pointwise manner. To some extent, this is the case as showed in [KYZ-3]; see Section 5 below for an introduction.

Moreover, observe that by Theorem 2, Ω is an $\dot{M}^{s,p}_{\mathrm{ball}}$-extension domain if and only if $\dot{M}^{s,p}_{\mathrm{ball}}(\Omega) = \dot{F}^s_{p,\infty}(\mathbb{R}^n)|_\Omega$. This further motivated us to study the geometric structure of $\dot{M}^{s,p}_{\mathrm{ball}}$-extension domains in [Z-1, Z-2]; see Sections 3 and 4 below for an introduction. Recall that it is an interesting subject to establish some intrinsic characterizations of $\dot{F}^s_{p,q}(\mathbb{R}^n)|_\Omega$, the restriction of the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ on the domain Ω; see [23, 24, 30, 31] for more discussions. In particular, some intrinsic characterizations of the restriction of Triebel-Lizorkin spaces to Lipschitz domains were established by Rychkov [23, 24] and Triebel [31], to uniform domains by Seeger [25], and to regular sets of $\mathbb{R}^n$ by Shvartsman [26].

3. Hajłasz-Sobolev extension and imbedding

In [Z-2], we establish the following geometric criteria for a bounded domain to support an $\dot{M}^{s,p}_{\mathrm{ball}}$-extension with $s \in (0, 1]$ and $p \in [n/s, \infty]$ or support an imbedding from

$\dot{M}^{s,p}_{\rm ball}(\Omega)$ to $\dot{M}^{s-n/p,\infty}_{\rm ball}(\Omega)$ with $s \in (0, 1]$ and $p \in (n/s, \infty]$ (for short, $\dot{M}^{s,p}_{\rm ball}$-imbedding). These generalize the corresponding results of [15, 6, 3, 28] as mentioned in Section 1.

Theorem 3. *If $\Omega \subset \mathbb{R}^n$ is a bounded $\dot{M}^{s,n/s}_{\rm ball}$-extension domain for some $s \in (0, 1]$, then Ω has the LLC property.*

Theorem 4. *(i) Let $\alpha \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded weak α-cigar domain. Then for all $s \in (\alpha, 1]$ and $p \in [(n-\alpha)/(s-\alpha), \infty)$, Ω is an $\dot{M}^{s,p}_{\rm ball}$-extension domain and, especially, an $\dot{M}^{s,p}_{\rm ball}$-imbedding domain.*

(ii) Let $s \in (0, 1]$, $p \in (n/s, \infty)$ and $\alpha \in [(ps-n)/(p-1), 1]$. If $\Omega \subset \mathbb{R}^n$ is a bounded $\dot{M}^{s,p}_{\rm ball}$-extension or $\dot{M}^{s,p}_{\rm ball}$-imbedding domain, that has the slice property, then Ω is a weak α-cigar domain.

Also, in the case $p = \infty$, observe that for every $\alpha \in (0, 1]$, $\dot{M}^{\alpha,\infty}(\Omega)$ and $\dot{M}^{\alpha,\infty}_{\rm ball}(\Omega)$ coincide with $\mathrm{Lip}_\alpha(\Omega)$ and $\mathrm{loc\,Lip}_\alpha(\Omega)$ as in [6], respectively. So, as proved by Gehring and Martio [6], a bounded domain Ω is a weak α-cigar domain with $\alpha \in (0, 1]$ if and only if it is an $\dot{M}^{\alpha,\infty}$-extension domain, and if and only if it is an $\dot{M}^{\alpha,\infty}_{\rm ball}$-imbedding domain.

Moreover, let Ω be a bounded simply connected planar domain, or a bounded domain that is quasiconformally equivalent to a uniform domain. Then it always has the slice property (see [3]), and if it also satisfies the LLC property, then it is a uniform domain (see [16]). So, as a corollary to Theorem 3, Theorem 4 and [6], we have the following conclusion, which gives an intrinsic characterization of the restriction of the Triebel-Lizorkin space $\dot{F}^s_{p,\infty}(\mathbb{R}^n)|_\Omega$ for a class of domains Ω.

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded simply connected planar domain, or a bounded domain that is quasiconformally equivalent to a uniform domain.*
(I) For every $\alpha \in (0, 1)$, the following are equivalent:
(i) Ω is a weak α-cigar domain;
(ii) $\dot{F}^s_{p,\infty}(\mathbb{R}^n)|_\Omega = \dot{M}^{s,p}_{\rm ball}(\Omega)$ for some/all $s \in [\alpha, 1)$ and $p = (n-\alpha)/(s-\alpha)$;
(iii) Ω is an $\dot{M}^{s,p}_{\rm ball}$-extension domain for some/all $s \in [\alpha, 1]$ and $p = (n-\alpha)/(s-\alpha)$;
(iv) Ω is an $\dot{M}^{s,p}_{\rm ball}$-imbedding domain for some/all $s \in [\alpha, 1]$ and $p = (n-\alpha)/(s-\alpha)$.
(II) The following are equivalent:
(i) Ω is a uniform domain;
(ii) Ω is an $\dot{M}^{s,n/s}_{\rm ball}$-extension domain for some/all $s \in (0, 1]$;
(iii) $\dot{F}^s_{n/s,\infty}(\mathbb{R}^n)|_\Omega = \dot{M}^{s,n/s}_{\rm ball}(\Omega)$ for some/all $s \in (0, 1)$.

4. Optimal global integrability of Hajłasz-Sobolev functions

In [Z-1], for all $s \in (0, 1]$, we establish some geometric criteria for a bounded domain to support a $(pn/(n-ps), p)_s$-Hajłasz-Sobolev-Poincaré (for short, $(pn/(n-ps), p)_s$-HSP) imbedding when $p \in (n/(n+s), n/s)$ or an s-Hajłasz-Trudinger (for short, s-HT) imbedding when $p = n/s$. These extend the corresponding results of [1, 2, 3, 29] as mentioned in Section 1.

We first recall that for $p \in (n/(n+s),\, n/s)$, a bounded domain Ω is said to support a $(pn/(n-ps),\, p)_s$-*HSP imbedding* if there exists a constant $C > 0$ such that for all $u \in \dot M^{s,\,p}_{\rm ball}(\Omega)$,

$$\|u - u_\Omega\|_{L^{pn/(n-ps)}(\Omega)} \le C\|u\|_{\dot M^{s,\,p}_{\rm ball}(\Omega)},$$

where $u_\Omega \equiv \frac{1}{|\Omega|}\int_\Omega u(z)\,dz$. Similarly, Ω is said to support an s-*HT imbedding* if there exists a constant $C > 0$ such that for all $u \in \dot M^{s,\,n/s}_{\rm ball}(\Omega)$,

$$\|u - u_\Omega\|_{\phi_s(L)(\Omega)} \le C\|u\|_{\dot M^{s,\,n/s}_{\rm ball}(\Omega)},$$

where $\phi_s(t) \equiv \exp(t^{n/(n-s)}) - 1$ and

$$\|u\|_{\phi_s(L)(\Omega)} \equiv \inf\left\{t > 0,\ \int_\Omega \phi_s\left(\frac{|u(x)|}{t}\right)\,dx \le 1\right\}.$$

It should be pointed out that since $\dot M^{1,\,p}_{\rm ball}(\Omega) = \dot W^{1,\,p}(\Omega)$ for all $p \in (1,\,\infty)$, $(pn/(n-ps),\, p)_1$-HSP imbedding with $p \in (1,\, n)$ coincides with the classical $(pn/(n-p),\, p)$-Sobolev-Poincaré imbedding as in [3, (1.1)], and 1-HT imbedding coincides with the classical Trudinger imbedding as in [3, (1.2)]. Moreover, an $\dot M^{s,\,p}_{\rm ball}$-extension domain always supports a $(pn/(n-ps),\, p)_s$-HSP imbedding when $p \in (n/(n+s),\, n/s)$ and an s-HT imbedding when $p = n/s$.

Theorem 5. *(i) A John domain of* $\mathbb{R}^n$ *always supports a* $(pn/(n-ps),\, p)_s$-HSP *imbedding for all* $s \in (0,\, 1]$ *and* $p \in (n/(n+s),\, n/s)$.

(ii) Assume that $\Omega \subset \mathbb{R}^n$ *is a bounded domain and satisfies the separation property. If* Ω *supports a* $(pn/(n-ps),\, p)_s$-HSP *imbedding for some* $s \in (0,\, 1]$ *and* $p \in (n/(n+s),\, n/s)$, *then* Ω *is a John domain.*

Theorem 6. *(i) A weak carrot domain of* $\mathbb{R}^n$ *always supports an* s-HT *imbedding for all* $s \in (0,\, 1]$.

(ii) Assume that $\Omega \subset \mathbb{R}^n$ *is a bounded domain and satisfies the slice property. If* Ω *supports an* s-HT *imbedding for some* $s \in (0,\, 1]$, *then* Ω *is a weak carrot domain.*

Notice that, as proved in [2, 3], every simply connected planar domain or every domain that is quasiconformally equivalent to a uniform domain satisfies the slice property and the separation property. So, as a corollary to Theorems 5 and 6, we have the following conclusion.

Corollary 2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded simply connected planar domain or a bounded domain that is quasiconformally equivalent to a uniform domain. Then*

(i) Ω *is a John domain if and only if it supports a* $(pn/(n-ps),\, p)_s$-HSP *imbedding for some/all* $s \in (0,\, 1]$ *and* $p \in (n/(n+s),\, n/s)$;

(ii) Ω *is a weak carrot domain if and only if it supports an* s-HT *imbedding for some/all* $s \in (0,\, 1]$.

5. Hajłasz-Besov and Hajłasz-Triebel-Lizorkin spaces

In [KYZ-3], we characterize, in terms of pointwise inequalities, the classical Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ for all $s \in (0, 1)$ and $p, q \in (n/(n+s), \infty]$. More precisely, by developing the notion of the gradient of Hajłasz [11], we introduce the following fractional Hajłasz gradient.

Definition 2. Let $s \in (0, \infty)$, $n \in \mathbb{N}$ and u be a measurable function on $\mathbb{R}^n$. A sequence of nonnegative measurable functions, $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$, is called a *fractional s-Hajłasz gradient* of u if there exists $E \subset \mathbb{R}^n$ with $|E| = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus E$ satisfying $2^{-k-1} \le |x-y| < 2^{-k}$,

$$|u(x) - u(y)| \le |x-y|^s [g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^s(u)$ the *collection of all fractional s-Hajłasz gradients of u.*

Relying on this concept we now introduce counterparts of Besov and Triebel-Lizorkin spaces. For simplicity, we only deal here with the case $p \in (0, \infty)$; the remaining case $p = \infty$ is given in [KYZ-3]. In what follows, for $p, q \in (0, \infty]$, we always write $\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^q} \equiv \{\sum_{j\in\mathbb{Z}} |g_j|^q\}^{1/q}$ when $q < \infty$ and $\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty} \equiv \sup_{j\in\mathbb{Z}} |g_j|$,

$$\|\{g_j\}_{j\in\mathbb{Z}}\|_{L^p(\mathbb{R}^n,\, \ell^q)} \equiv \|\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}$$

and

$$\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^q(L^p(\mathbb{R}^n))} \equiv \|\{\|g_j\|_{L^p(\mathbb{R}^n)}\}_{j\in\mathbb{Z}}\|_{\ell^q}.$$

Definition 3. Let $n \in \mathbb{N}$, $s, p \in (0,\infty)$ and $q \in (0, \infty]$.

(i) The *homogeneous Hajłasz-Triebel-Lizorkin space* $\dot{M}^s_{p,q}(\mathbb{R}^n)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^s_{p,q}(\mathbb{R}^n)} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \|\vec{g}\|_{L^p(\mathbb{R}^n,\, \ell^q)} < \infty.$$

(ii) The *homogeneous Hajłasz-Besov space* $\dot{N}^s_{p,q}(\mathbb{R}^n)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{N}^s_{p,q}(\mathbb{R}^n)} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \|\vec{g}\|_{\ell^q(L^p(\mathbb{R}^n))} < \infty.$$

Then we have the following results from [KYZ-3].

Theorem 7. *Let $n \in \mathbb{N}$.*

(i) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (n/(n+s), \infty]$, then $\dot{M}^s_{p,q}(\mathbb{R}^n) = \dot{F}^s_{p,q}(\mathbb{R}^n)$.

(ii) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (0, \infty]$, then $\dot{N}^s_{p,q}(\mathbb{R}^n) = \dot{B}^s_{p,q}(\mathbb{R}^n)$.

The proof of Theorem 7 is based on a characterization of the Triebel-Lizorkin and Besov spaces in terms of grand Littlewood-Paley functions [KYZ-2, KYZ-3]. In [KYZ-3], we also give a metric measure space generalization of Theorem 7 and establish the quasiconformal invariance of $\dot{M}^s_{n/s,q}(\mathbb{R}^n)$ for all $s \in (0, 1)$, $q \in (0, \infty]$ and $n \ge 2$.

Appendix

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a domain. See, for example, [15, 6, 1, 16, 2, 3], for the following notions.

(I) Ω is a *John domain* with respect to $x_0 \in \Omega$ and $C > 0$ if it is bounded and for every $x \in \Omega$, there exists a rectifiable curve $\gamma : [0, T] \to \Omega$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(T) = x_0$ and $d(\gamma(t), \Omega^{\complement}) \geq Ct$.

(II) Ω is a *weak carrot domain* with respect to $x_0 \in \Omega$ and $C \geq 1$ if for all $x \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x and x_0 such that

$$\int_\gamma \frac{1}{d(z, \Omega^{\complement})} \leq C \log\left(\frac{C}{d(x, \Omega^{\complement})}\right).$$

(III) Ω is a *weak α-cigar domain* with $\alpha \in (0, 1]$ if there exists a positive constant C such that for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x and y, and satisfying

$$\int_\gamma [d(z, \Omega^{\complement})]^{\alpha-1}\,|dz| \leq C|x-y|^\alpha.$$

(IV) Ω is a *uniform domain* if there exists a positive constant C such that for all $x, y \in \Omega$, there exists a rectifiable curve $\gamma :\ [0, T] \to \Omega$, parameterized by arclength, with $\gamma(0) = x$ and $\gamma(T) = y$, and satisfying that $T \leq C|x-y|$ and

$$\bigcup_{t\in[0,\,T]} B\left(\gamma(t),\, \frac{1}{C}\min\{t,\, T-t\}\right) \subset \Omega.$$

(V) Ω is *linearly locally connected* (for short, LLC) if there exists a constant $b \in (0, 1]$ such that for all $z \in \mathbb{R}^n$ and $r > 0$,

LLC(1) points in $\Omega \cap B(z, r)$ can be joined in $\Omega \cap B(z, r/b)$;

LLC(2) points in $\Omega \setminus B(z, r)$ can be joined in $\Omega \setminus B(z, br)$.

(VI) Ω is *regular* if there exist positive constants θ and C such that for all $x \in \overline{\Omega}$ and $r \in (0, \theta)$, $|B(x, r) \cap \Omega| \geq C|B(x, r)|$.

(VII) Ω has a *separation property* with respect to $x_0 \in \Omega$ and $C > 1$ if for every $x \in \Omega$, there exists a curve $\gamma :\ [0, 1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$, and such that for each $t \in (0, 1]$, either $\gamma([0, t]) \subset B \equiv B(\gamma(t), Cd(\gamma(t), \Omega^{\complement}))$ or each $y \in \gamma([0, t]) \setminus B$ belongs to a different component of $\Omega \setminus \partial B$ than x_0.

(VIII) Ω has a *slice property* with respect to $C > 1$ if for every pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$, and pairwise disjoint collection of open subsets $\{S_i\}_{i=0}^j$, $j \geq 0$, of Ω such that

(i) $x \in S_0$, $y \in S_j$ and x and y are in different components of $\Omega \setminus \overline{S_i}$ for $0 < i < j$;

(ii) if $F \subset\subset \Omega$ is a curve containing both x and y, and $0 < i < j$, then $\operatorname{diam}(S_i) \leq C\ell(F \cap S_i)$;

(iii) for $0 \leq t \leq 1$, $B(\gamma(t), C^{-1}d(\gamma(t), \Omega^{\complement})) \subset \cup_{i=0}^j S_i$;

(iv) if $0 \leq i \leq j$, then $\operatorname{diam} S_i \leq Cd(z, \Omega^{\complement})$ for all $z \in \gamma_i \equiv \gamma \cap S_i$; also, there exists $x_i \in S_i$ such that $x_0 = x$, $x_j = y$ and $B(x_i, C^{-1}d(x_i, \Omega^{\complement})) \subset S_i$.

References

[1] B. Bojarski, Remarks on Sobolev imbedding inequalities, Complex analysis, Joensuu 1987, 52-68, Lecture Notes in Math., 1351, Springer, Berlin, 1988.

[2] S. M. Buckley and P. Koskela, Sobolev-Poincaré implies John, Math. Res. Lett. 2 (1995), 577-594.

[3] S. M. Buckley and P. Koskela, Criteria for imbeddings of Sobolev-Poincaré type, Internat. Math. Res. Notices 18 (1996), 881-901.

[4] S. M. Buckley and A. Stanoyevitch, Weak slice conditions and Hölder imbeddings, J. London Math. Soc. (2) 64 (2001), 690-706.

[5] R. A. DeVore and R. C. Sharpley, Maximal functions measuring smoothness, Mem. Amer. Math. Soc. 47 (1984), no. 293, 1-115.

[6] F. W. Gehring and O. Martio, Lipschitz classes and quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 203-219.

[7] F. W. Gehring and O. Martio, Quasiextremal distance domains and extension of quasiconformal mappings, J. Analyse Math. 45 (1985), 181-206.

[8] V. M. Gol'dšteĭn, Quasiconformal, quasi-isometric mappings and the Sobolev spaces, Complex analysis and applications '81 (Varna, 1981), 202-212, Publ. House Bulgar. Acad. Sci., Sofia, 1984.

[9] V. M. Gol'dšteĭn and Yu. G. Reshetnyak, Introduction to the theory of functions with generalized derivatives, and quasiconformal mappings (Russian), Moscow, 1983. 285 pp.

[10] V. M. Gol'dšteĭn and S. K. Vodop'anov, Prolongement de fonctions diffrentiables hors de domaines plans, C. R. Acad. Sci. Paris Sr. I Math. 293 (1981), 581-584.

[11] P. Hajłasz, Sobolev spaces on an arbitrary metric spaces, Potential Anal. 5 (1996), 403-415.

[12] P. Hajłasz, P. Koskela and H. Tuominen, Sobolev imbeddings, extensions and measure density condition, J. Funct. Anal. 254 (2008), 1217-1234.

[13] P. Hajłasz, P. Koskela and H. Tuominen, Measure density and extendability of Sobolev functions, Rev. Mat. Iberoam. 24 (2008), 645-669.

[14] P. Hajłasz and O. Martio, Traces of Sobolev functions on fractal type sets and characterization of extension domains, J. Funct. Anal. 143 (1997), 221-246.

[15] P. W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147 (1981), 71-88.

[16] P. Koskela, Capacity extension domains, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes No. 73 (1990), 42 pp.

[17] P. Koskela, Extensions and imbeddings, J. Funct. Anal. 159 (1998), 369-384.

[18] P. Koskela and E. Saksman, Pointwise characterizations of Hardy-Sobolev functions. Math. Res. Lett. 15 (2008), 727-744.

[19] V. G. Maz'ya, On the extension of functions belonging to S. L. Sobolev spaces, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113 (1981), 231-236.

[20] V. G. Maz'ya, Sobolev Spaces, Translated from the Russian by T. O. Shaposhnikova, Springer-Verlag, Berlin, 1985.

[21] A. Miyachi, Hardy-Sobolev spaces and maximal functions, J. Math. Soc. Japan 42 (1990), 73-90.

[22] A. S. Romanov, On the extension of functions that belong to Sobolev spaces, Sibirsk. Mat. Zh. 34 (1993), 149-152, translation in Siberian Math. J. 34 (1993), 723-726.

[23] V. S. Rychkov, On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. (2) 60 (1999), 237-257.

[24] V. S. Rychkov, Linear extension operators for restrictions of function spaces to irregular open sets, Studia Math. 140 (2000), 141-162.

[25] A. Seeger, A note on Triebel-Lizorkin spaces, Approximation and function spaces (Warsaw, 1986), 391-400, Banach Center Publ., 22, PWN, Warsaw, 1989.

[26] P. Shvartsman, Local approximations and intrinsic characterizations of spaces of smooth functions on regular subsets of $\mathbb{R}^n$, Math. Nachr. 279 (2006), 1212-1241.

[27] P. Shvartsman, On extension of Sobolev functions defined on regular subsets of metric measure spaces, J. Approx. Theory 144 (2007), 139-161.

[28] P. Shvartsman, On Sobolev extension domains in R^n, J. Funct. Anal. 258 (2010), 2205-2245.

[29] W. Smith and D. A. Stegenga, Sobolev imbeddings and integrability of harmonic functions on Hölder domains, Potential theory (Nagoya, 1990), 303-313, de Gruyter, Berlin, 1992.

[30] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.

[31] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15 (2002), 475-524.

[32] S. K. Vodop'janov, V. M. Gol'dšteĭn and T. G. Latfullin, A criterion for the extension of functions of the class L_2^1 from unbounded plane domains, Sibirsk. Mat. Zh.(Russian) 20 (1979), 416-419; Siberian Math. J.(English transl.) 20 (1979), 298-301.

[33] D. Yang, New characterizations of Hajłasz-Sobolev spaces on metric spaces, Sci. China Ser. A 46 (2003), 675-689.

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